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Robust Stability for Singularly Perturbed Systems with Structured State Space Uncertainties

HIROAKI MUKAIDANI
Hiroshima City University, Japan
KOICHI MIZUKAMI
Hiroshima Kokusai Gakuin University, Japan

SUMMARY

This paper considers the robust stability of singularly perturbed systems with structured state space uncertainties. By making use of the Lyapunov stability criterion and combining it with the Lyapunov equations, a new approach for deciding a robust stability for uncertain linear singularly perturbed systems is presented. Based on the assumption that the reduced nominal system is stable, we also derive some sufficient conditions for robust stability. Some analytical methods and the Bellman–Gronwall inequality are used to investigate such sufficient conditions. In this paper, it is worth pointing out that we do not need to investigate both the slow system and the fast system by means of the singular perturbation methods because the proposed method is very direct. Furthermore, we only assume that the uncertainties are norm-bounded. Therefore, the robust stability condition derived here is less conservative than those reported in the control literature. A numerical example is given to demonstrate the validity of our new results. © 2000 Scripta Technica, Electr Eng Jpn, 132(4): 62–72, 2000

Key words: Singularly perturbed systems; structured state space uncertainties; Bellman–Gronwall lemma; robust stability.

1. Introduction

It is well known that the singular perturbation methods [1] on the basis of the two-time-scale approach are very powerful analysis and design tools for singularly perturbed systems. A feature of this theory is to decompose the full design problem for the full-order dynamics into two design problems for the slow and fast dynamics. A basic design procedure is as follows. First, we set the singular perturbation parameter $\varepsilon = 0$. Then, by using both the standard nonsingularity assumption on A_{22} and the new time variable $\tau = t/\varepsilon$, we can get two subsystems, that is, a slow and a fast subsystem. Second, analysis and design problems are solved in two stages, first for the fast mode and then for the slow mode independently. Finally, the two subsystem designs are combined to give a design for the full system. It is quite natural that the robust stability problem for a singularly perturbed uncertain system is also discussed by making use of the above method [1, 3–7].

Two main stability problems for a singularly perturbed system have been studied. The first one is the asymptotic stability of singularly perturbed linear time-variant systems [1, 3–5]. In this case, it is assumed that the coefficient matrices $A_{ij}(t)$ $(i, j = 1, 2)$ are bounded with the nonlinear function, and that the upper bounds of $A_{ij}(t)$ are known. In Kokotović and colleagues [1] and Javid [3], the largest positive ε such that the singularly perturbed system is asymptotically stable has been derived. O'Reilly [4] proposed a composite controller that guarantees an asymptotic stability. The second one is the robust stability and the stabilization problems for the singularly perturbed uncertain systems [6, 7]. In this case, it is assumed that the coefficient matrices $A_{ij}(t)$ $(i, j = 1, 2)$ can be decomposed into the time-invariant nominal part and the time-invariant or time-variant uncertainty part, respectively. It is also assumed that the upper and lower bound of uncertainty are known, while the forms of uncertainty are unknown. For the robust stability problem, in Ref. 6 the upper bound of the parameter ε such that the singularly perturbed system is stable has been presented by means of the H_∞ sense. In Ref. 7, the controller such that the singularly perturbed uncertain system is quadratically stable is proposed by using Lyapunov's direct method.

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The following fact [subsequently termed the modified Klimushev–Krasovskii (MKK) condition] is used in the proof of theorem in Ref. 15 and in Ref. 16:

Fact 1 (MKK condition) Consider the singularly perturbed uncertain system

$$\dot{y}_1 = [M_{11} + O_{11}(\varepsilon)]y_1 + [M_{12} + O_{12}(\varepsilon)]y_2, \quad (1a)$$

$$\varepsilon\dot{y}_2 = [M_{21} + O_{21}(\varepsilon)]y_1 + [M_{22} + O_{22}(\varepsilon)]y_2, \quad (1b)$$

$$y_1(t_0) = y_1^0, \quad y_2(t_0) = y_2^0. \quad (1c)$$

Let M_{22} be nonsingular. If the matrices $M_0 := M_{11} - M_{12}M_{22}^{-1}M_{21}$ and M_{22} are stable, then there exists a small perturbation parameter $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$ the singularly perturbed uncertain system (1) *is asymptotically stable.*

However, compared with Ref. 1, there is at present no exact proof for the asymptotic stability of the singularly perturbed system (1) such that the time-invariant uncertainty $O_{ij}(\varepsilon)$ $(i, j = 1, 2)$ is included in the coefficient matrices M_{ij} $(i, j = 1, 2)$. That is, there is no exact proof of the MKK condition.

Kokotović and colleagues [1] and Shao and Sawan [6] investigated the stability problem such that the singularly perturbed autonomous system without the control input is asymptotically stable. In recent years, in Ref. 20, a new result on stability/performance of linear time-invariant singularly perturbed systems with an uncertain parameter has been derived using guardian map theory. However, in Ref. 1, the coefficient matrices M_{ij} have not been taking the uncertainty $O(\varepsilon)$ into consideration. Furthermore, for such a problem given by both Example 1 in Ref. 6 and Saydy [20], only the time-invariant uncertainty has been considered. Thus, it is very interesting to study the robust stability problem such that the singularly perturbed system with time-variant bounded uncertainty is asymptotically stable.

In this paper, we study the robust stability of singularly perturbed systems with time-variant norm-bounded state space uncertainties [8, 12] which is an extension of the uncertain $O_{ij}(e)$ dependent on the parameter ε. By using the transformation given in Ref. 1, we transform the state matrix of the nominal system (i.e., the system in the absence of uncertainty) into the block diagonal matrix. Then, since it looks like a class of large-scale interconnected dynamical systems, we derive a new robust stability condition such that the uncertain linear singularly perturbed system is asymptotically stable by making use of the Lyapunov stability criterion proposed by Zhang and colleagues [13]. Some analytical methods and the Bellman–Gronwall inequality are used to investigate such sufficient conditions. It is shown that if such conditions are met, then the singularly perturbed uncertain system is asymptotically stable for very small ε. Furthermore, by applying our new theorem, it is also shown that system (1) with uncertainty $O(\varepsilon)$ is asymptotically stable for small ε. That is, we find that the corollary introduced in the proof of theorem both in Ref. 15 and in Ref. 16 is satisfied.

The notations used in this paper are fairly standard. A superscript T denotes matrix transpose. I_j denotes the $j \times j$ identity matrix. For any matrix X let $\|X\|_S$ denotes its maximum singular value, that is, $\|X\|_S \equiv [\lambda_{\max}(X^TX)]^{1/2}$. $\|G(s)\|_\infty$ denotes its H_∞ norm for a transfer matrix function $G(s)$, that is, $\|G(s)\|_\infty \equiv \sup_s[\lambda_{\max}(G^*(s)G(s))]^{1/2}$ $(s = j\omega, \omega \in \mathbf{R})$. $\|x\|_E$ is the Euclidean norm of a vector x, that is, $\|x\|_E \equiv [x^Tx]^{1/2}$. $\lambda_i(X)$ is the i-th eigenvalue for any matrix X. $\mathrm{Re}\lambda_i(X)$ is the real part of the i-th eigenvalue for any matrix X. Z^+ is modulus matrix of the matrix $Z = [z_{ij}]$, that is, replacing the entries of Z by their absolute values—$Z^+ = [|z_{ij}|]$—and for some matrixes $Z = [z_{ij}]$ and $Y = [y_{ij}]$, let $Z [\le] Y$ denote $z_{ij} \le y_{ij}$, for all entries i and j.

2. Problem Formulation

Let us consider linear time-invariant singularly perturbed systems

$$\dot{x}_1 = [A_{11} + \Delta A_{11}]x_1 + [A_{12} + \Delta A_{12}]x_2 \quad (2a)$$

$$\varepsilon\dot{x}_2 = [A_{21} + \Delta A_{21}]x_1(t) + [A_{22} + \Delta A_{22}]x_2 \quad (2b)$$

where ε is a small positive parameter, $x^T := (x_1^T, x_2^T)$ is the n-dimensional state vector, with x_1 of dimension n_1 and x_2 of dimension $n_2 := n - n_1$. The initial condition for (1) is given by $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, respectively. $\Delta A_{ij} = \Delta A_{ij}(t)$ is a Lebesgue measurable matrix of uncertain parameters. All matrices above are of appropriate dimensions.

We now consider the stability of such a singularly perturbed system under the following basic assumption [8, 12].

Assumption 1 *For system* (2), *the bounds are available on the absolute values of the maximum variations in the element of $\Delta A_{ij}(t)$. That is,*

$$|\Delta a_{kl}^{ij}(t)| \le \bar{a}_{kl}^{ij}, \; i = 1, 2 \; j = 1, 2 \quad (3)$$

where $\Delta a_{kl}^{ij}(t)$ denote the elements of $\Delta A_{ij}(t)$ with entry (k, l) and $\bar{a}_{kl}^{ij}$ denote the upper bound of $|\Delta a_{kl}^{ij}(t)|$. Thus, the uncertainty of $\Delta A_{ij}(t)$ has the upper bound given by (3). *Namely, we can change the form of* (3) *as follows*:

$$\Delta A_{ij}(t)^+ [\le] \Delta\bar{A}_{ij}, \; i = 1, 2 \; j = 1, 2 \quad (4)$$

3. Robust Stabilization for Singularly Perturbed Systems

Without loss of generality, we shall make the following assumptions for system (2) [1, 20].

Assumption 2 *The matrix A_{22} is invertible. The matrices $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and A_{22} are both stable.*

The problem considered in this paper is to find a sufficient condition under which the uncertain system (1) is quadratically stable. The following is the main result of this paper.

Theorem 1 *Let Q_1 and Q_2 be any positive definite symmetric matrices. Assume that the parameter ε is very small, and define as follows*:

$$\alpha_1 < \min_i\{|\mathrm{Re}\lambda_i(A_s)|, i = 1, 2, \cdots\} \tag{5a}$$

$$\alpha_2 < \min_i\{|\mathrm{Re}\lambda_i(A_f)|, i = 1, 2, \cdots\} \tag{5b}$$

If it satisfies the following conditions:

$$\alpha_1 > \beta_{11}K_1^2 \tag{6a}$$

$$\alpha_2 > \beta_{22}K_2^2 \tag{6b}$$

$$(\alpha_1 - \beta_{11}K_1^2)(\alpha_2 - \beta_{22}K_2^2) > \beta_{12}\beta_{21}K_1^2K_2^2 \tag{6c}$$

then the uncertain system (1) is quadratically stable, where

$$T^{-1}\begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix} T = \begin{bmatrix} A_s & 0 \\ 0 & \varepsilon^{-1}A_f \end{bmatrix}$$

$$T = \begin{bmatrix} I_{n_1} & \varepsilon H \\ -L & I_{n_2} - \varepsilon LH \end{bmatrix}$$

$$A_s = A_{11} - A_{12}L = A_0 + O(\varepsilon)$$

$$A_f = A_{22} + \varepsilon LA_{12} = A_{22} + O(\varepsilon)$$

$$||\Delta\bar{A}_{11} + \Delta\bar{A}_{12}L^+ + H^+\{\Delta\bar{A}_{21} + \Delta\bar{A}_{22}L^+\} + \varepsilon(HL)^+\{\Delta\bar{A}_{11} + \Delta\bar{A}_{12}L^+\}||_S = \beta_{11}$$

$$||\Delta\bar{A}_{12} + H^+\Delta\bar{A}_{22} + \varepsilon\{\Delta\bar{A}_{11}H^+ + \Delta\bar{A}_{12}(LH)^+\} + \varepsilon\{(HL)^+\Delta\bar{A}_{12} + H^+\Delta\bar{A}_{21}H^+ + H^+\Delta\bar{A}_{22}(LH)^+\} + \varepsilon^2(HL)^+\{\Delta\bar{A}_{11}H^+ + \Delta\bar{A}_{12}(LH)^+\}||_S = \beta_{12}$$

$$||\Delta\bar{A}_{21} + \Delta\bar{A}_{22}L^+ + \varepsilon L^+\{\Delta\bar{A}_{11} + \Delta\bar{A}_{12}L^+\}||_S = \beta_{21}$$

$$||\Delta\bar{A}_{22} + \varepsilon\{L^+\Delta\bar{A}_{12} + \Delta\bar{A}_{21}H^+ + \Delta\bar{A}_{22}(LH)^+\} + \varepsilon^2L^+\{\Delta\bar{A}_{11}H^+ + \Delta\bar{A}_{12}(LH)^+\}||_S = \beta_{22}$$

$$K_1 = \sqrt{\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}},\ K_2 = \sqrt{\frac{\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}}$$

$$(A_s + \alpha_1 I)^T P_1 + P_1(A_s + \alpha_1 I) + Q_1 = 0$$

$$(A_f + \alpha_2 I)^T P_2 + P_2(A_f + \alpha_2 I) + Q_2 = 0$$

and

$$A_{22}L - A_{21} - \varepsilon L(A_{11} - A_{12}L) = 0 \tag{7a}$$

$$H(A_{22} + \varepsilon LA_{12}) - A_{12} - \varepsilon(A_{11} - A_{12}L)H = 0 \tag{7b}$$

$$(A_s + \alpha_1 I_{n_1})^T P_1 + P_1(A_s + \alpha_1 I_{n_1}) + Q_1 = 0 \tag{7c}$$

$$(A_f + \alpha_2 I_{n_2})^T P_2 + P_2(A_f + \alpha_2 I_{n_2}) + Q_2 = 0 \tag{7d}$$

Then the matrices $L = L(\varepsilon)$ and $H = H(\varepsilon)$ satisfy (7a) and (7b), respectively. Furthermore, the symmetric positive-definite matrices P_1 and P_2 are the solutions of the Lyapunov equations (7c) and (7d), respectively.

Remark 1 *It is well known that the unique conditions of* (7a) *and* (7b) *exist under Assumption 2* [1].

Remark 2 *Comparing this paper with Refs.* 6 *and* 7, *we do not use the singular perturbation methods. The reason is as follows. For such an uncertain system* (2), *by making use of the singular perturbation methods, we consider the following fast subsystem*:

$$\varepsilon\dot{\hat{x}}_2(t) = [A_{22} + \Delta A_{22}(t)]\hat{x}_2(t) \tag{8}$$

Then, it is easy to check the quadratic stability condition using several approaches [*e.g.,* 8–11, 17, 18]. *On the other hand, we consider the following slow subsystem*:

$$\dot{\bar{x}}_1(t) = [(A_{11} + \Delta A_{11}(t)) - (A_{12} + \Delta A_{12}(t))(A_{22} + \Delta A_{22}(t))^{-1}(A_{21} + \Delta A_{21}(t))]\bar{x}_1(t) \tag{9}$$

Since it is hard to separate the uncertainty $\Delta A_{22}(t)$ from $[A_{22} + \Delta A_{22}(t)]^{-1}$, we cannot easily obtain the robust stability conditions. In Ref. 7, *for example, the robust stability conditions of singularly perturbed systems are investigated by using the assumption of uncertain matrix such that $||(A_{22} + \Delta A_{22}(t))^{-1}|| \le \bar{\delta}_{22}$. However, there is no taking the upper bound of matrix $A_{22} + \Delta A_{22}(t)$ into account directly. In Ref.* 6, *how to find the largest positive ε_0 such that the singularly perturbed system is robustly stable has been presented by using the condition of the H_∞-norm such that $||\Delta A_{22}||_\infty \le h||(sI - A_{22})^{-1}||_\infty^{-1}$, $h < 1$. However, in the above paper, it is found that the upper bound of matrix $A_{22} + \Delta A_{22}(t)$ is not completely considered because of h.*

Remark 3 *For such construction parameters $\alpha_1 > 0$, $\alpha_2 > 0$ given by* (5), *we need to choose these parameters as large as possible, since if α_1, α_2 are too small, then the conditions* (6) *are not satisfied.*

In the proofs we will use the following lemma.

Lemma 1 *The Bellman–Gronwall inequality* [2, 12]

Let $p(t)$ and $q(t)$ be positive real continuous functions of t, and let N be a positive real constant. If a continuous function $p(t)$ has the property that

$$p(t) \le q(t) + N\int_0^t p(\tau)d\tau \tag{10}$$

then we have the following two results:

$$p(t) \le q(0)\exp(Nt) + \int_0^t \dot{q}(\tau)\exp\{N(t-\tau)\}d\tau \tag{11}$$

$$p(t) \le q(t) + N\int_0^t q(\sigma)\exp\{N(t-\sigma)\}d\sigma \tag{12}$$

Lemma 2 [12] *Given any vector w and any real matrix X, then $w^TXw \le ||w||_E^2||X||_S$.*

Lemma 3 [8] *Given any real matrix X, $||X||_S \le ||X^+||_S$.*

Proof: Since A_{22} is nonsingular from Assumption 1, there exists a transformation† $x(t) = Ty(t)$, that is,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = T \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & \varepsilon H \\ -L & I_{n_2} - \varepsilon LH \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Ty(t)$$

For y, we get

$$\dot{y}_1 = A_s y_1 + \Delta\tilde{A}_{11} y_1 + \Delta\tilde{A}_{12} y_2 \quad (13a)$$

$$\varepsilon\dot{y}_2 = A_f y_2 + \Delta\tilde{A}_{21} y_1 + \Delta\tilde{A}_{22} y_2 \quad (13b)$$

where

$$T^{-1} \begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \varepsilon^{-1}\Delta A_{21} & \varepsilon^{-1}\Delta A_{22} \end{bmatrix} T = \begin{bmatrix} \Delta\tilde{A}_{11} & \Delta\tilde{A}_{12} \\ \varepsilon^{-1}\Delta\tilde{A}_{21} & \varepsilon^{-1}\Delta\tilde{A}_{22} \end{bmatrix} \quad (13c)$$

Then

$$\|x(t)\|_E \le \|T\|_S \cdot \|y(t)\|_E \quad (14)$$

Since $\|T\|_S < \infty$, it follows that $\|y(t)\|_E \to 0$, $(t \to \infty)$ implies $\|x(t)\|_E \to 0$, $(t \to \infty)$. Thus, it is sufficient to consider $\|y(t)\|_E$. Set

$$V(y,t) = V_1(y_1,t) + \varepsilon V_2(y_2,t) = y_1^T(t) P_1 y_1(t) + \varepsilon y_2^T(t) P_2 y_2(t) \quad (15)$$

By Rayleigh's principle [12], we have

$$\lambda_{\min}(P_i)\|y_i\|_E^2 \le V_i(y_i,t) \le \lambda_{\max}(P_i)\|y_i\|_E^2, \quad (i = 1,\ 2) \quad (16)$$

By making use of Lemma 2, we obtain the time derivative of $V(y, t)$ along the trajectory of the system (13) as follows:

$$\frac{dV(y,t)}{dt} \le -2\sum_{i=1}^{2} \alpha_i V_i(y_i,t) + 2\sum_{i=1}^{2}\sum_{j=1}^{2} \|y_i^T P_i\|_E \|\Delta\tilde{A}_{ij}\|_S \cdot \|y_j\|_E \quad (17)$$

where the uncertain matrices $\Delta\tilde{A}_{ij}$ $(i = 1, 2, j = 1, 2)$ satisfy (18):

$$\Delta\tilde{A}_{11} = \Delta A_{11} - \Delta A_{12}L - H\Delta A_{21} + H\Delta A_{22}L + \varepsilon HL(\Delta A_{11} + \Delta A_{12}L) \quad (18a)$$

$$\Delta\tilde{A}_{12} = \Delta A_{12} - H\Delta A_{22} + \varepsilon(\Delta A_{11} - \Delta A_{12}L)H - \varepsilon H(L\Delta A_{12} + \Delta A_{21}H - \Delta A_{22}LH) - \varepsilon^2 HL(\Delta A_{11} - \Delta A_{12}L)H \quad (18b)$$

$$\Delta\tilde{A}_{21} = \Delta A_{21} - \Delta A_{22}L + \varepsilon L(\Delta A_{11} - \Delta A_{12}L) \quad (18c)$$

$$\Delta\tilde{A}_{22} = \Delta A_{22} + \varepsilon(L\Delta A_{12} + \Delta A_{21}H - \Delta A_{22}LH) + \varepsilon^2(L\Delta A_{11} - L\Delta A_{12}L)H \quad (18d)$$

Furthermore, from Lemma 3 and Assumption 1, taking norms in (18), we have

$$\|\Delta\tilde{A}_{11}\|_S \le \|\Delta\bar{A}_{11} + \Delta\bar{A}_{12}L^+ + H^+\{\Delta\bar{A}_{21} + \Delta\bar{A}_{22}L^+\} + \varepsilon(HL)^+\{\Delta\bar{A}_{11} + \Delta\bar{A}_{12}L^+\}\|_S = \beta_{11} \quad (19a)$$

$$\|\Delta\tilde{A}_{12}\|_S \le \|\Delta\bar{A}_{12} + H^+\Delta\bar{A}_{22} + \varepsilon\{\Delta\bar{A}_{11}H^+ + \Delta\bar{A}_{12}(LH)^+\} + \varepsilon\{(HL)^+\Delta\bar{A}_{12} + H^+\Delta\bar{A}_{21}H^+ + H^+\Delta\bar{A}_{22}(LH)^+\} + \varepsilon^2(HL)^+\{\Delta\bar{A}_{11}H^+ + \Delta\bar{A}_{12}(LH)^+\}\|_S = \beta_{12} \quad (19b)$$

$$\|\Delta\tilde{A}_{21}\|_S \le \|\Delta\bar{A}_{21} + \Delta\bar{A}_{22}L^+ + \varepsilon L^+\{\Delta\bar{A}_{11} + \Delta\bar{A}_{12}L^+\}\|_S = \beta_{21} \quad (19c)$$

$$\|\Delta\tilde{A}_{22}\|_S \le \|\Delta\bar{A}_{22} + \varepsilon\{L^+\Delta\bar{A}_{12} + \Delta\bar{A}_{21}H^+ + \Delta\bar{A}_{22}(LH)^+\} + \varepsilon^2 L^+\{\Delta\bar{A}_{11}H^+ + \Delta\bar{A}_{12}(LH)^+\}\|_S = \beta_{22} \quad (19d)$$

Thus, from (17) and (19) we have the inequality on $dV(y, t)/dt$ as follows:

$$\frac{dV(y,\ t)}{dt} \le -2\sum_{i=1}^{2} \alpha_i V_i(y_i,t) + 2\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\lambda_{\max}(P_i)}{\sqrt{\lambda_{\min}(P_i)}} \cdot \sqrt{V_i(y_i,t)}\beta_{ij}\|y_j\|_E \quad (20)$$

It is obvious from the definition of $V(y, t)$ that, if an inequality of the form

$$\frac{dV_1(y_1,t)}{dt} \le -2\alpha_1 V_1(y_1,t) + 2\frac{\lambda_{\max}(P_1)}{\sqrt{\lambda_{\min}(P_1)}} \cdot \sqrt{V_1(y_1,t)} \sum_{j=1}^{2} \beta_{1j}\|y_j\|_E \quad (21a)$$

$$\varepsilon\frac{dV_2(y_2,t)}{dt} \le -2\alpha_2 V_2(y_2,t) + 2\frac{\lambda_{\max}(P_2)}{\sqrt{\lambda_{\min}(P_2)}} \cdot \sqrt{V_2(y_2,t)} \sum_{j=1}^{2} \beta_{2j}\|y_j\|_E \quad (21b)$$

†Here the considered ε satisfies the inequality, that is, $0 < \varepsilon \le \varepsilon^*$ for the largest ε^* such that there exists a transformation T. Thus, we study the robust stability problem under the previous inequality.

holds, inequality (20) is also satisfied. Therefore, it is sufficient to consider inequalities (21) in the rest of this proof. First, from (21a), we obtain the inequality on $V_1(t)$ as follows:

$$\begin{aligned} V_1(t) \;\leq\; & \lambda_{\max}(P_1)\|y_1^0\|_E^2 \exp(-2\alpha_1 t) + \\ & 2\int_0^t \exp\{-2\alpha_1(t-\tau)\}\frac{\lambda_{\max}(P_1)}{\sqrt{\lambda_{\min}(P_1)}} \\ & \cdot\sqrt{V_1(\tau)}\sum_{j=1}^{2}\beta_{1j}\|y_j(\tau)\|_E d\tau \end{aligned} \tag{22}$$

Here, we define an auxiliary function $S_1(t)$ as follows:

$$\begin{aligned} S_1(t) \;=\; & \Big[\lambda_{\max}(P_1)\|y_1^0\|_E^2 \exp(-2\alpha_1 t) + \\ & 2\int_0^t \exp\{-2\alpha_1(t-\tau)\}\frac{\lambda_{\max}(P_1)}{\sqrt{\lambda_{\min}(P_1)}} \\ & \cdot\sqrt{V_1(\tau)}\sum_{j=1}^{2}\beta_{1j}\|y_j(\tau)\|_E d\tau\Big]^{\frac{1}{2}} \end{aligned} \tag{23}$$

Differentiating (23) yields

$$\begin{aligned} \frac{dS_1(t)}{dt} \;=\; & -\alpha_1 S_1(t) + \\ & \frac{\lambda_{\max}(P_1)}{\sqrt{\lambda_{\min}(P_1)}}\sum_{j=1}^{2}\beta_{1j}\|y_j\|_E \end{aligned} \tag{24}$$

Integrating for $dS_1(t)/dt$, it follows from (24) that

$$\begin{aligned} S_1(t) \;\leq\; & \sqrt{\lambda_{\max}(P_1)}\|y_1^0\|_E \exp(-\alpha_1 t) + \\ & \int_0^t \exp\{-\alpha_1(t-\tau)\} \\ & \frac{\lambda_{\max}(P_1)}{\sqrt{\lambda_{\min}(P_1)}}\cdot\sum_{j=1}^{2}\beta_{1j}\|y_j(\tau)\|_E d\tau \end{aligned} \tag{25}$$

Moreover, we obtain

$$\begin{aligned} \|y_1(t)\|_E \;\leq\; & \sqrt{\frac{V_1(t)}{\lambda_{\min}(P_1)}} \leq \frac{S_1(t)}{\sqrt{\lambda_{\min}(P_1)}} \\ \leq\; & \sqrt{\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\|y_1^0\|_E \exp(-\alpha_1 t) + \\ & \int_0^t \exp\{-\alpha_1(t-\tau)\}\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}\cdot \\ & \sum_{j=1}^{2}\beta_{1j}\|y_j(\tau)\|_E d\tau \end{aligned} \tag{26}$$

After multiplying both sides by $\exp(-\alpha_1 t)$, using the Bellman–Gronwall lemma (11) (see Refs. 2 and 12), from (26) we easily obtain the following inequality on $\|y_1(t)\|_E$:

$$\begin{aligned} \|y_1(t)\|_E \;\leq\; & K_1\|y_1^0\|_E \exp\{(\beta_{11}K_1^2-\alpha_1)t\} + \\ & \beta_{12}K_1^2\int_0^t \|y_2(\tau)\|_E\cdot\exp\{(\beta_{11}K_1^2-\alpha_1)(t-\tau)\}d\tau \end{aligned} \tag{27}$$

By means of a step similar to (27), we obtain from (21b)

$$\begin{aligned} \|y_2(t)\|_E \;\leq\; & K_2\|y_2^0\|_E \exp\{\varepsilon^{-1}(\beta_{22}K_2^2-\alpha_2)t\} + \\ & \frac{\beta_{21}K_2^2}{\varepsilon}\int_0^t \|y_1(\tau)\|_E \\ & \cdot\exp\{\varepsilon^{-1}(\beta_{22}K_2^2-\alpha_2)(t-\tau)\}d\tau \end{aligned} \tag{28}$$

Substituting (28) into (27) and integrating for the right-hand side of the inequality, we obtain

$$\begin{aligned} \|y_1(t)\|_E \;\leq\; & K_1\|y_1^0\|_E \exp(-\sigma_1 t) + \\ & \frac{\beta_{12}K_1^2K_2\|y_2^0\|_E}{\sigma_1-\sigma_2}[\exp(-\sigma_2 t)-\exp(-\sigma_1 t)] \\ & +\frac{\beta_{12}\beta_{21}K_1^2K_2^2}{\varepsilon\sigma_1}\int_0^t \|y_1(s)\|_E\cdot\exp\{-\sigma_2(t-s)\}ds \end{aligned} \tag{29}$$

where

$$\sigma_1 = \alpha_1 - \beta_{11}K_1^2,\; \sigma_2 = \varepsilon^{-1}(\alpha_2-\beta_{22}K_2^2)$$

Multiplying both sides of (29) by $\exp(\sigma_2 t)$ and applying the Bellman–Gronwall lemma (12) and integration, yields

$$\begin{aligned} & \|y_1(t)\|_E \;\leq \\ & \Big[K_1\|y_1^0\|_E + \frac{\beta_{12}K_1^2K_2\|y_2^0\|_E}{\bar{\sigma}-\sigma_1} - \frac{\bar{\sigma}-\sigma_2}{\bar{\sigma}-\sigma_1}K_1\|y_1^0\|_E\Big]\exp(-\sigma_1 t) \\ & +\Big[-\frac{\beta_{12}K_1^2K_2\|y_2^0\|_E}{\bar{\sigma}-\sigma_1} + \frac{\bar{\sigma}-\sigma_2}{\bar{\sigma}-\sigma_1}K_1\|y_1^0\|_E\Big]\exp(-\bar{\sigma}t) \end{aligned} \tag{30}$$

where

$$\bar{\sigma} = \sigma_2 - \frac{\beta_{12}\beta_{21}K_1^2K_2^2}{\varepsilon\sigma_1}$$

Substituting (27) into (28) and using a step similar to (30), we obtain

$$\begin{aligned} & \|y_2(t)\|_E \;\leq \\ & \Big[K_2\|y_2^0\|_E + \frac{\beta_{21}K_1K_2^2\|y_1^0\|_E}{\hat{\sigma}-\sigma_2} - \frac{\hat{\sigma}-\sigma_1}{\hat{\sigma}-\sigma_2}K_2\|y_2^0\|_E\Big]\exp(-\sigma_2 t) \\ & +\Big[-\frac{\beta_{21}K_1K_2^2\|y_1^0\|_E}{\hat{\sigma}-\sigma_2} + \frac{\hat{\sigma}-\sigma_2}{\hat{\sigma}-\sigma_1}K_2\|y_2^0\|_E\Big]\exp(-\hat{\sigma}t) \end{aligned} \tag{31}$$

where

$$\hat{\sigma} = \sigma_1 - \frac{\beta_{12}\beta_{21}K_1^2K_2^2}{\varepsilon\sigma_2}$$

Thus, for (13) to be uniformly asymptotically stable, we need inequalities $\sigma_1 > 0$, $\sigma_2 > 0$, $\bar{\sigma} > 0$, and $\hat{\sigma} > 0$ to be satisfied. Finally, solving these inequalities, we can get the conditions (6). Therefore, (13) is uniformly asymptotically stable which implies that (2) is uniformly asymptotically stable. □

The next corollary follows immediately from Theorem 1.

Corollary 1 *Suppose that the system uncertainties satisfy the norm conditions* $\|\Delta A_{ij}\|_S = \|O_{ij}(\varepsilon)\|_S \le m_k\varepsilon$ $(i, j = 1, 2, k = 1, \ldots, 4)$. *If Assumption 1 holds, then there exists a small perturbation parameter* $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon}]$ *system* (1) *is asymptotically stable.*

Proof: For the bounded uncertain $O_{ij}(\varepsilon)$ $(i, j = 1, 2)$ dependent on the parameter ε in the singularly perturbed systems (1), there exists a positive constant m_{ij} $(i, j = 1, 2)$ such that $\|O_{ij}(\varepsilon)\|_S \le m_{ij}\varepsilon$. Furthermore, by substituting $\|O_{ij}(\varepsilon)\|_S \le m_{ij}\varepsilon$ into (19), there exists a positive constant $k_{ij}(i, j = 1, 2)$ such that $\beta_{ij} = k_{ij}\varepsilon$. Thus, substituting $k_{ij}\varepsilon$ into β_{ij} of (6), we obtain

$$\alpha_1 > k_{11}\varepsilon K_1^2 \tag{32a}$$

$$\alpha_2 > k_{22}\varepsilon K_2^2 \tag{32b}$$

$$(\alpha_1 - k_{11}\varepsilon K_1^2)(\alpha_2 - k_{22}\varepsilon K_2^2) > k_{12}k_{21}\varepsilon^2 K_1^2 K_2^2 \tag{32c}$$

It is obvious that there exists a small parameter $\bar{\varepsilon}$ such that inequality (32) is satisfied. That is, the uncertain system (1) is asymptotically stable for all $\varepsilon \in (0, \bar{\varepsilon}]$ from Theorem 1. □

Remark 4 *From Corollary 1, it is guaranteed that for bounded uncertainty such that* $\|O_{ij}(\varepsilon)\|_S \le m_{ij}\varepsilon$ *depend on the parameter* ε *the uncertain system is asymptotically stable. Furthermore, by applying Theorem* 1, *it is also guaranteed that if the uncertainty* $\Delta A_{ij}(t)$ *satisfies condition* (6), *then system* (1) *is asymptotically stable for small* ε. *Thus, although* $\Delta A_{ij}(t)$ *is independent of* ε, *if the upper bound of* $\Delta A_{ij}(t)$ *is too small, for such an uncertainty the system* (1) *is robust*.

Now, we compare the results in this paper with those in Refs. 6 and 7. In Ref. 6, it is assumed that the matrices A_{12} and A_{21} of system (2) are not zero matrices. However, since these constrained conditions are unrealistic, for the obtained results, we cannot handle the wider class of singularly perturbed systems. The difference from the results of Ref. 6 is that there is no zero matrices assumption on A_{12} and A_{21}. Thus, our new results are less conservative than those in the literature. Furthermore, since we only assume that $\Delta A_{ij}(t)$ are structured state space uncertainties being time-variant and norm-bounded, our new results come from the relaxed assumption in comparison with Shao and Sawan [6]. However, we note that how to find the largest positive ε such that the singularly perturbed system is robustly stable has been presented in Ref. 6. That is, we need the assumption that ε is very small. Also, we cannot obtain the upper bound of the parameter such that the singularly perturbed uncertain system is robustly stable. On the other hand, in Ref. 6, although the constraint conditions for coefficient matrices are needed, such an upper bound of the parameter is given. In Ref. 7, the stability conditions such that the singularly perturbed uncertain system is quadratically stable are derived by using the singular perturbation methods. That is, it is shown that for singularly perturbed uncertain systems, if the slow and fast systems are stable, then the full-order system (2) is quadratically stable. However, we note that how to check the robust stability has not been presented for both slow and fast subsystems in Ref. 7. Thus, for example, in order to check for the robustness of both slow and fast subsystems, let us apply to the system the two main methods which are based on H_∞ control theory and on the Linear Fractional Transformation (LFT) framework by using the Structured Singular Value (SSV). Then it is easy to check the robustness of the fast subsystem by applying both methods. On the other hand, it is difficult to check the robustness of the slow subsystem by using similar methods. That is, when we use H_∞ control theory, we cannot hardly separate $\Delta A_{22}(t)$ from $A_{22\Delta}^{-1} = [A_{22} + \Delta A_{22}(t)]^{-1}$. Furthermore, when we use the LFT on the basis of the SSV, we must calculate the SSV at any risk. Consequently, if we take the upper and lower bound of SSV into account in Ref. 18, the robust stability condition becomes more conservative. Thus, it is hard to use the LFT on the basis of the SSV. The notable feature of the result obtained in this paper is that it is possible to check directly the asymptotic stability even though we do not use the singular perturbation methods. Furthermore, it is clear that the quadratic stability condition is considered positively because we take the upper bound of matrix $A_{22\Delta} = A_{22} + \Delta A_{22}(t)$ into account. Therefore, it is easy to check the robust stability directly in comparison with both H_∞ control theory and the LFT framework. Another important practical feature is that the bounds are available on the values of the maximum absolute variations in element of the uncertainty matrix by using the modulus matrix and that the differentiability and the continuity are not needed in comparison with Ref. 7. Therefore, it is worth pointing out that we extend the results given by Shao and Sawan [6] and Suzuki and colleagues [7] under the weak assumptions. It is also shown that we have proposed a new method for checking the quadratic stability without the singular perturbation methods. Note that the upper bound of norm for uncertainty becomes large because of transformation T given by Theorem 1. Moreover, since β_{ij} given by (19) is defined by the modulus matrix, the upper bound of norm for uncertainty also becomes large. Thus, if the uncertainty exists for all block matrices $A_{ij}(i, j = 1, 2)$, note that our new sufficient condition given by Theorem 1 yields conservative results because of those methods. Finally, if we can observe that the slow and fast subsystems have quadratic stability by means of any methods (see Ref. 9 or

18), we find that the method given in Ref. 7 is very useful practically.

4. Illustrative Example

In order to demonstrate the efficiency of the proposed algorithm, we have run some numerical examples.

4.1 Example 1

On the basis of Ref. 6, consider the singularly perturbed uncertain systems described by the following differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varepsilon \dot{z}_1 \\ \varepsilon \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & -1 & 0 \\ 0 & -0.5 & 0 & -1 \\ 1 & 0 & -2 & a(t) \\ 0 & 1 & b(t) & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} \quad (33)$$

The uncertainties $a(t)$ are $b(t)$ are characterized by $|a(t)| \le \bar{a}$ and $|b(t)| \le \bar{b}$, $(\bar{a} > 0, \bar{b} > 0)$. The problem addressed in this example is to find the sufficient condition such that system (33) is asymptotically stable. The singularly perturbed parameter is $\varepsilon = 0.1$. First we note that if the upper bound of uncertainties is known, then it is easy to check the robust stability for the fast subsystem. However, for the slow subsystem obtained by setting c = 0, that is,

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \left(\begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} + \begin{bmatrix} -2 & a(t) \\ b(t) & -2 \end{bmatrix}^{-1} \right) \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \quad (34)$$

because we cannot separate the $\Delta A_{22}(t)$ from $[A_{22} + \Delta A_{22}(t)]^{-1}$, that is, the second term on the right-hand side of (34), difficulties arise when we check the robust stability. That is, since

$$[A_{22} + \Delta A_{22}(t)]^{-1} = \begin{bmatrix} -2 & a(t) \\ b(t) & -2 \end{bmatrix}^{-1}$$

the uncertainties are not one block type structural uncertainty. Thus, it is difficult to find the matrices D and E such that $[A_{22} + \Delta A_{22}(t)]^{-1} = D\Delta(t)E$. Consequently, we cannot easily obtain the conditions of robust stability on the basis of Theorem 1 [7] because we cannot observe the robust stability of the slow subsystem. On the other hand, robust stability problems of uncertain dynamical systems have been formulated in the LFT framework and the SSV. However, in this example, since the purpose is to find the largest upper bounds of uncertainties, these methods may not apply for this example with unknown upper bounds of uncertainties. Actually, in case the upper bounds of both $\bar{a}$ and $\bar{b}$ are unknown, it is hard to use the inequalities on the basis of the sufficient conditions resulting from the LFT and SSV.

Let us use the result of Theorem 1 to find the upper bounds of uncertainties. From transformation (13c), we get the following matrix:

$$T = \begin{bmatrix} 1 & 0 & 0.0542 & 0 \\ 0 & 1 & 0 & 0.0542 \\ 0.5271 & 0 & 1.0286 & 0 \\ 0 & 0.5271 & 0 & 1.0286 \end{bmatrix} \quad (35)$$

Thus, we have

$$A_s = \begin{bmatrix} -1.0271 & 0 \\ 0 & -1.0271 \end{bmatrix} \quad (36a)$$

$$A_f = \begin{bmatrix} -1.9472 & 0 \\ 0 & -1.9472 \end{bmatrix} \quad (36b)$$

From (5), we choose

$$\min_i \{|\mathrm{Re}\lambda_i(A_s)|, i = 1, 2\} > \alpha_1 = 1.0 \quad (37a)$$

$$\min_i \{|\mathrm{Re}\lambda_i(A_f)|, i = 1, 2\} > \alpha_2 = 1.9 \quad (37b)$$

Further, from (19), we obtain

$$\|\Delta\tilde{A}_{11}\|_S \le \|H^+ \Delta\bar{A}_{22} L^+\|_S, \quad \|\Delta\tilde{A}_{12}\|_S \le \|H^+ \Delta\bar{A}_{22}\{I + \varepsilon(LH)^+\}\|_S$$

$$\|\Delta\tilde{A}_{21}\|_S \le \|\Delta\bar{A}_{22} L^+\|_S, \quad \|\Delta\tilde{A}_{22}\|_S \le \|\Delta\bar{A}_{22}\{I + \varepsilon(LH)^+\}\|_S$$

Therefore, taking norms of both sides of the above inequalities yields

$$\|H^+ \Delta\bar{A}_{22} L^+\|_S \le \|H^+\|_S \|L^+\|_S \|\Delta\bar{A}_{22}\|_S = 0.2856\sqrt{\max\{\bar{a}^2,\ \bar{b}^2\}} = \beta_{11}$$

$$\|H^+ \Delta\bar{A}_{22}\{I + \varepsilon(LH)^+\}\|_S \le \|H^+\|_S \|\Delta\bar{A}_{22}\|_S \|I + \varepsilon(LH)^+\|_S = 0.5575\sqrt{\max\{\bar{a}^2,\ \bar{b}^2\}} = \beta_{12}$$

$$\|\Delta\bar{A}_{22} L^+\|_S \le \|\Delta\bar{A}_{22}\|_S \|L^+\|_S = 0.5271\sqrt{\max\{\bar{a}^2,\ \bar{b}^2\}} = \beta_{21}$$

$$\|\Delta\bar{A}_{22}\{I + \varepsilon(LH)^+\}\|_S \le \|\Delta\bar{A}_{22}\|_S \|I + \varepsilon(LH)^+\|_S = 1.0286\sqrt{\max\{\bar{a}^2,\ \bar{b}^2\}} = \beta_{22}$$

Comparing A_s, A_f with (7c), (7d), respectively, we choose $K_1 = K_2 = 1.0$. By some trivial manipulations and solving inequality (6), we can get the sufficient conclusions

$$\max\{\bar{a}^2,\ \bar{b}^2\} < 1.4620 \quad (38)$$

Conversely, if inequality (38) is satisfied, then the robust stability condition (6) is also satisfied. Therefore, we conclude that the singularly perturbed uncertain system is asymptotically stable.

The results of simulation are depicted in Fig. 1 where $a(t)$, $b(t)$ represent the uncertainties characterized by $a(t) = 1.2\sin^2(\pi t)$, $b(t) = -2.4\sin(\pi t)\cos(\pi t)$.

For simulation, we give an initial condition as follows:

$$\begin{bmatrix} x_1 & x_2 & z_1 & z_2 \end{bmatrix} = \begin{bmatrix} 0.0 & 0.5 & 1.0 & 1.5 \end{bmatrix} \tag{39}$$

Since $\bar{a} = 1.2$, $\bar{b} = 1.2$, it is obvious that the constraint inequality (38) is satisfied. It is shown from Fig. 1 that the uncertain singularly perturbed system (33) is indeed asymptotically stable.

4.2 Example 2

We consider the R–L–C network in Fig. 2. In this network, L_0 and R are the inductance and the resistance, respectively. These capacitances are denoted by C_1, C_2. Suppose that L_0 is a very small positive parameter, that is, let $L_0 = \varepsilon$; the dynamics of this system is described by the singularly perturbed from

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varepsilon\dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1/RC_1 & 1/RC_1 & 1/C_1 \\ 1/RC_2 & -1/RC_2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{40}$$

where $x_i (i = 1, 2)$ denote the voltage across capacitances $C_i (i = 1, 2)$, respectively. x_3 denotes the electric current in the inductance. Moreover, u denotes the applied voltage that is, control input. The following electrical elements are defined: $L_0 = 1$ mH ($\varepsilon = 0.001$), $C_1 = 0.1$ F, $C_2 = 1000$ μF,

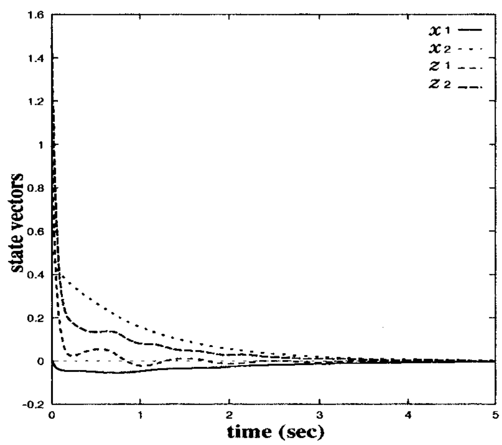

Fig. 1. Response of the state variables.

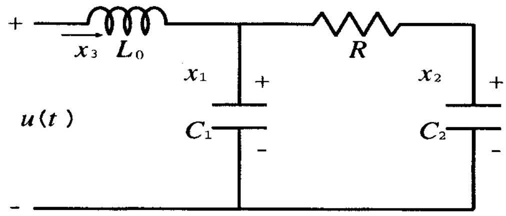

Fig. 2. Electric network of Example 2.

and $R = 1$ kΩ. It is well known that the resistance R is increased due to the environmental temperature. Thus, we suppose that the actual resistance R is within 10% of a nominal resistance taking the saturation into account. The objective is to design a robust state feedback controller such that system (40) is asymptotically stable when R is changed under the structured state space uncertainties.

Since R is within 10% of a nominal resistance, we assume that the uncertainties are bounded and satisfy

$$\begin{aligned} a(t) &= \frac{1}{RC_1} = \frac{1}{100 + 10 \times \delta(t)} = 0.0101 + 0.001 \times \Delta(t) \\ b(t) &= \frac{1}{RC_2} = \frac{1}{1 + 0.1 \times \delta(t)} = 1.01 + 0.1 \times \Delta(t) \\ & \quad 0 \le \delta(t) \le 1.0,\ |\Delta(t)| \le 1.0 \end{aligned}$$

Therefore, we can change system (40) into the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varepsilon\dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a(t) & a(t) & 10.0 \\ b(t) & -b(t) & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{41}$$

The nominal closed loop eigenvalues are given by -1.01, $-0.005 \pm 100.00i$. Hence, with zero control the nominal system is oscillative and close to instability. Furthermore, since system (41) includes the uncertainty, robust stability is not guaranteed. Thus, we need to construct the stabilizing controller. However, it is obvious that the existing method of finding the composite stabilizing controller in Ref. 7 is not valid for this example since system (41) is a nonstandard singularly perturbed system.

In this example, by making use of a standard Linear Quadratic Regulator (LQR) approach [17] which has robustness properties, that is, at least 60° phase margin and 6 dB gain margin, we present a linear state feedback controller for uncertain linear singularly perturbed systems. In order to construct the controller, the method in Ref. 14 and MATLAB are used. On the basis of the nominal system, the full state feedback controller is given by the following gain matrix F (see Ref. 17):

$$u = -Fx = -F \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$$

$$F = \begin{bmatrix} 0.4419 & 0.2902 & 1.4173 \end{bmatrix} \quad (42)$$

where the cost function for the LQR problem is written as

$$\min \left\{ \int_0^\infty (x_1^2 + x_2^2 + 2x_3^2 + u^2) dt \right\}$$

The nominal closed-loop eigenvalues of the resulting closed-loop system (43) described by (41) and (42) are given by –1.2, –10.0, –1407.1.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varepsilon \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a(t) & a(t) & 10.0 \\ b(t) & -b(t) & 0 \\ -1.4419 & -0.2902 & -1.4173 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (43)$$

Finally, by making use of Theorem 1, we verify that closed-loop system (43) is asymptotically stable. Now, we can get the matrices L and H by solving algebraic equations (7a) and (7b), respectively.

$$L = \begin{bmatrix} 1.0246 & 0.2064 \end{bmatrix}, \ H = \begin{bmatrix} -7.1592 \\ 0.0051 \end{bmatrix}$$

By using the transformation T, we obtain

$$A_s = \begin{bmatrix} -10.2563 & -2.0538 \\ 1.01 & -1.01 \end{bmatrix}, \ A_f = -1.4071$$

Since the eigenvalues of matrix A_s are $\lambda_1(A_s) = -10.0263$, $\lambda_2(A_s) = -1.2401$, we select $\alpha_1 = 0.5$, $\alpha_2 = 1.4$. Then, taking account of matrix A_f, we obtain $K_2 = 1.0$. On the other hand, we select $Q_1 = \mathrm{diag}\{5.0, 0.1\}$ for (7c). From (7c), we have

$$P_1 = \begin{bmatrix} 0.2532 & -0.0294 \\ -0.0294 & 0.2163 \end{bmatrix}$$

By computing eigenvalues of matrix P_1, we find that $\lambda_1(P_1) = 0.2001$, $\lambda_2(P_1) = 0.2694$. Thus, we obtain $K_1 = 1.1603$. Furthermore, by computing $\beta_{ij}(i, j = 1, 2)$, we obtain $\beta_{11} = 0.1414$, $\beta_{12} = 7.1465 \times 10^{-4}$, $\beta_{21} = 3.0637 \times 10^4$, $\beta_{22} = 1.552 \times 10^{-7}$. Substituting $\beta_{ij}(i, j = 1, 2)$ into (6), it is clear that the robust stability conditions given in Theorem 1 are satisfied. Then the uncertain system (43) with controller (42) is asymptotically stable. For simulation, we give an initial condition as follows:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 5.0 & 2.0 & 0.0 \end{bmatrix} \quad (44)$$

In fact, we have employed for simulation that the resistance R is given by $R = 1000.0 + 100.0 \times [1.0 - \exp(-t)]$.

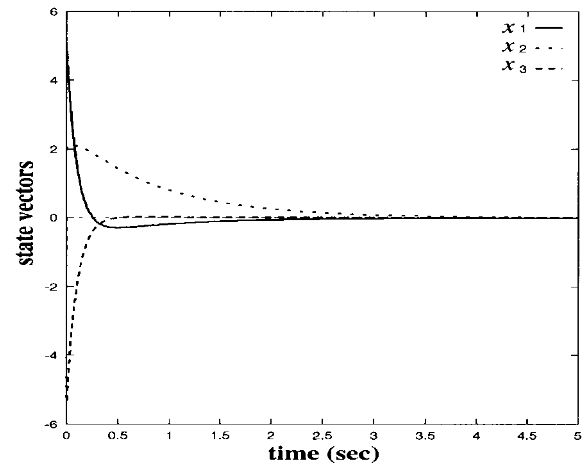

Fig. 3. Response of the state variables.

The results of the simulation of this example are depicted in Fig. 3, where it is seen that closed-loop system (43) is asymptotically stable.

5. Conclusions

The problem of robust stability of singularly perturbed systems with structured state space uncertainties has been studied. If the robust stability conditions (6) given in Theorem 1 are satisfied, it has been shown that the uncertain linear singularly perturbed system (2) is asymptotically stable. By applying our new results, we have proven the important corollary which is introduced in Refs. 15 and 16, that is, there exists a small perturbation parameter $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, system (1) with uncertainty $O(\varepsilon)$ is asymptotically stable. In addition, it should be remarked that the main difference between the results in Refs. 6 and 7 and the present paper is fold: (i) In Ref. 6 the assumptions that A_{12}, A_{21} are not equivalent to zero are made, but in this paper, these assumptions are not needed. Furthermore, in Ref. 6 the uncertainty is assumed to be time- invariant, while the assumption here is the time-variant structured uncertainty. Thus, comparing Ref. 6 with our new results, the assumption is less conservative. (ii) In Ref. 7 the standard reduced-order technique [1] is used to derive the quadratic stability conditions. However, there is no taking the upper bound of matrix $\Delta_{22\Delta} = A_{22} + \Delta A_{22}(t)$ into account, while since we do not use the standard reduced-order technique, it is clear that the quadratic stability condition positively includes the upper bound of matrix $\Delta_{22\Delta} = A_{22} + \Delta A_{22}(t)$. Therefore, it is easy to check the robust stability directly.

Numerical examples are given to demonstrate the validity of our new results in this paper. In these numerical examples, our robust stability condition is more useful than the result in the control literature [7]. We have considered the problem of robust stability for the standard singularly perturbed systems such that the matrix A_{22} is nonsingular. But in practice, the robust stabilization problem for the nonstandard singularly perturbed systems with control input such that the matrix A_{22} is singular is more realistic. This problem will also be investigated in the near future.

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AUTHORS (from left to right)

Hiroaki Mukaidani (member) received his B.S. degree in integrated arts and sciences from Hiroshima University in 1992 and his M.S. and Ph.D. degrees in information engineering from Hiroshima University in 1994 and 1997. Since 1998, he has been on the Faculty of Information Science at Hiroshima City university as a research associate. His research interests are in the area of robust control and its application to singularly perturbed systems. He is a member of the Institute of Electrical Engineers of Japan and the Society of Instrument and Control Engineers of Japan.

Koichi Mizukami (member) received his B.E. degree in electrical engineering from Hiroshima University in 1958 and his M.E. and D.Eng. degrees in electrical engineering from Kyoto University in 1960 and 1965. In 1963, he joined the Department of Electrical Engineering of Kyoto University as a research associate. He was a postdoctoral fellow of electrical engineering at the University of Toronto from 1966 to 1968. He then became an associate professor on the Faculty of Engineering of Hiroshima University, and served as a professor on the Faculty of Integrated Arts and Sciences at Hiroshima University from 1977 to 1999. Since then, he has been a professor on the Faculty of Engineering at Hiroshima Kokusai Gakuin University. His research interests are in the area of control and information sciences, especially function analysis, optimal control and information theory, fuzzy and neural network control, information retrieval, and computer graphics. He is on the editorial board of the *International Journal of Dynamics and Control* and the *Journal of Mathematical Analysis and Applications*. He is a member of the Institute of Electrical Engineers of Japan, the Information Processing Society of Japan, the Society of Instrument and Control Engineers of Japan, and the Institute of Systems, Control and Information Engineers of Japan. He is a Fellow of the Society of Instrument and Control Engineers of Japan.