CONTROL OF LINEAR MULTIPARAMETER SINGULARITY PERTUBED SYSTEMS

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Abstract: In this paper, the linear quadratic optimal control for multiparameter singularly perturbed systems (MSPS) is studied in a different approach from the existing methods. The attention is focused on the design of a near–optimal controller which does not depend on the values of the small unknown parameters. The resulting controller achieves $O(\|\mu\|^2)$ near–optimal cost compared with previously proposed result in the literature. To obtain such a controller, the existence of a unique and bounded solution of a multiparameter algebraic Riccati equation (MARE) is newly proven.

Keywords: Multiparameter singularly perturbed systems, Linear quadratic optimal control, Near–optimal control, Algebraic Riccati equation

1. INTRODUCTION

The deterministic and stochastic multimodeling stability, control, filtering and dynamic games have been investigated extensively by several researchers (see e.g., Khalil and Kokotović, 1978, 1979; Coumarbatch and Gajić, 2000; Gajić, 1988; Wang et al., 1994). The multimodeling problems arise in large scale dynamic systems. For example, these multimodel situations in practice are illustrated by the multiarea power system (Khalil and Kokotović, 1978) and the passenger car model (Coumarbatch and Gajić, 2000). In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE), which are parameterized by the small positive same order parameters $\varepsilon_j,\ j = 1,\ 2,\ \cdots$. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., Laub, 1979). One of the approaches is the invariant subspace approach based on the Hamiltonian matrix (Laub, 1979). However, when the ARE is known to be ill–conditioned (Laub, 1979) such an approach is not adequate to the multiparameter singularly perturbed systems (MSPS) since there is no guarantee of symmetry for the solution of the ARE. Note that it is very hard to solve directly the MARE due to high dimension and numerical stiffness (Coumarbatch and Gajić, 2000). More recently, the exact slow–fast decomposition method for solving the MARE has been proposed in Coumarbatch and Gajić (2000). However, a limitation of these approaches is that the small parameters are assumed to be known. In practice, the small perturbation parameters ε_j are often not known. Thus, it is not applicable to a large class of problems where the parameters represent small unknown perturbations whose values are not known exactly.

A popular approach to deal with the MSPS is the two–time–scale design method (see e.g., Khalil and Kokotović, 1978, 1979; Gajić, 1988; Wang et al., 1994). For example, optimal control of a class of MSPS, being only on the slow variable, has been studied by Khalil and Kokotović (1979), where the design of the ε_j–independent reduced–order controller has been suggested. However, in order to obtain the slow subsystem, the nonsingularity of the fast state matrices are needed. In Wang et el. (1994), by making use of the descriptor variable approach, the main results of Khalil and Kokotović (1979) have been extended to the nonstandard MSPS such that at least one of the fast state matrices is singular. However, the proposed controller only achieves $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $\|[\varepsilon_1 \ \cdots \ \varepsilon_N]\|$) approximation of the optimal cost.

In this paper, we study the linear quadratic optimal control problems for MSPS. We first investigate the unique and bounded solution of the MARE and establish its asymptotic structure. Thus, this paper presents an improvement on some of the results of Gajić (1988) in the sense that some assumptions are relaxed. Using the asymptotic structure, a new near–optimal controller which does not depend on the values of the small parameters is obtained. This is done by eliminating the parameters ε_j for the full–order controller. We emphasize that structure of the resulting controller achieves $O(\|\mu\|^2)$ near–optimal cost compared with the previously proposed in the literature. Even if the parameters are unknown, when the parameters are sufficiently small, the near–optimal controller can be used reliably for the MSPS.

2. MSPS

We consider a specific structure of N–lower level multi–fast subsystems interconnected through the dynamics of a higher level slow subsystem.

$$\dot{x}_0 = A_{00}x_0 + \sum_{j=1}^{N} A_{0j}x_j + \sum_{j=1}^{N} B_{0j}u_j, \tag{1a}$$

$$\varepsilon_j \dot{x}_j = A_{j0}x_0 + A_{jj}x_j + B_{jj}u_j, \tag{1b}$$

$$y_0 = C_{00}x_0, \tag{1c}$$

$$y_j = C_{j0}x_0 + C_{jj}x_j, \ j = 1, \ 2, \ \cdots, N, \tag{1d}$$

$$x_j(0) = x_j^0, \ j = 0, \ 1, \ 2, \ \cdots, N, \tag{1e}$$

where $x_j \in \mathbf{R}^{n_j}, \ j = 0, \ 1, \ \cdots, N$ are the state vectors, $u_j \in \mathbf{R}^{m_j}, \ j = 1, \ 2, \ \cdots, N$ are the control inputs, $y_j \in \mathbf{R}^{l_j}, \ j = 0, \ 1, \ \cdots, N$ are the outputs. We assume that the ratios of the small positive parameter $\varepsilon_j > 0, j = 1, \ 2, \ \cdots, N$ are bounded by some positive constants $\underline{k}_{ij}, \ \bar{k}_{ij}$ (see e.g., Khalil and Kokotović, 1978, 1979),

$$0 < \underline{k}_{ij} \le \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \le \bar{k}_{ij} < \infty. \tag{2}$$

Note that at least one of the fast state matrices $A_{jj}, \ j = 1, \ 2, \ \cdots, N$ may be singular. The performance criterion is given by

$$J = \frac{1}{2} \int_0^{\infty} \left(y^T y + \sum_{j=1}^{N} u_j^T R_j u_j \right) dt, \tag{3}$$

where $y^T = \left[\, y_0^T \ \cdots \ y_m^T \, \right]^T \in \mathbf{R}^{\bar{l}}, \ \bar{l} = \sum_{j=0}^{N} l_j$.

In order to find a near–optimal control without the knowledge of the small perturbation parameters ε_j, let the optimal control for the regulator problem (1) and (3) be

$$u_{\rm opt} = -R^{-1}B_e^T P_e x, \tag{4}$$

where P_e satisfies the MARE

$$A_e^T P_e + P_e A_e - P_e S_e P_e + Q = 0, \tag{5}$$

with

$$\begin{aligned}
A_e &:= \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1}A_{f0} & \Pi_e^{-1}A_f \end{bmatrix}, \\
\Pi_e &:= \mathrm{block-diag}\left(\, \varepsilon_1 I_{n_1} \ \cdots \ \varepsilon_N I_{n_N} \, \right), \\
A_{0f} &:= \left[\, A_{01} \ \cdots \ A_{0N} \, \right], \ A_{f0} := \left[\, A_{10}^T \ \cdots \ A_{N0}^T \, \right]^T, \\
A_f &:= \mathrm{block-diag}\left(\, A_{11} \ \cdots \ A_{NN} \, \right), \\
S_e &:= B_e R^{-1} B_e^T = \begin{bmatrix} S_{00} & \Pi_e^{-1}S_{0f} \\ S_{0f}^T\Pi_e^{-1} & \Pi_e^{-1}S_f\Pi_e^{-1} \end{bmatrix}, \\
S_{00} &:= \sum_{j=1}^{N} B_{0j}R_j^{-1}B_{0j}^T, \ S_{0f} \ = \ \left[\, S_{01} \ \cdots \ S_{0N} \, \right] \\
&= \left[\, B_{01}R_1^{-1}B_{11}^T \ \cdots \ B_{0N}R^{-1}B_{NN}^T \, \right], \\
S_f &:= \mathrm{block-diag}\left(\, S_{11} \ \cdots \ S_{NN} \, \right), \\
&= \mathrm{block-diag} \\
&\quad \left(\, B_{11}R_1^{-1}B_{11}^T \ \cdots \ B_{NN}R^{-1}B_{NN}^T \, \right), \\
B_e &:= \begin{bmatrix} B_0 \\ \Pi_e^{-1}B_f \end{bmatrix}, \ B_0 := \left[\, B_{01} \ \cdots \ B_{0N} \, \right], \\
B_f &:= \mathrm{block-diag}\left(\, B_{11} \ \cdots \ B_{NN} \, \right), \\
R &:= \mathrm{block-diag}\left(\, R_1 \ \cdots \ R_N \, \right), \\
Q &:= \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix}, \ Q_{00} \ := \ \sum_{j=0}^{N} C_{j0}^T C_{j0}, \\
Q_{0f} &:= \left[\, Q_{01} \ \cdots \ Q_{0N} \, \right] \\
&= \left[\, C_{10}^T C_{11} \ \cdots \ C_{N0}^T C_{NN} \, \right], \\
Q_f &:= \mathrm{block-diag}\left(\, Q_{11} \ \cdots \ Q_{NN} \, \right) \\
&= \mathrm{block-diag}\left(\, C_{11}^T C_{11} \ \cdots \ C_{NN}^T C_{NN} \, \right).
\end{aligned}$$

In the following analysis, we need some assumptions.

Assumption 1: The triples $(A_{jj}, \ B_{jj}, \ C_{jj}), \ j = 1, \ 2, \ \cdots, N$ are stabilizable and detectable.

Assumption 2:

$$\mathrm{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & B_0 \\ -A_{f0} & -A_f & B_f \end{bmatrix} = \bar{n}, \tag{6a}$$

$$\mathrm{rank} \begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} = \bar{n}, \tag{6b}$$

where $\bar{n} := \sum_{j=0}^{N} n_j$,

$$C_0 := \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix},\ C_f := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_{11} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{NN} \end{bmatrix},$$

with $\mathrm{Re}[s] \geq 0, \quad s \in \mathrm{C}$.

Assumption 3: The Hamiltonian matrices $T_{jj},\ j = 1,\ 2,\ \cdots, N$ are nonsingular, where

$$T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix}.$$

Before investigating the optimal control problem, we investigate the asymptotic structure of the MARE (5). Let us introduce the scaling matrices

$$\Phi_e := \mathrm{block-diag}\left(I_{n_0}\ \varepsilon_1 I_{n_1}\ \cdots\ \varepsilon_N I_{n_N} \right) = \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix}.$$

In order to avoid the ill–conditioned caused by the large parameter ε_j^{-1} which is included in the MARE (5), we introduce the following useful lemma.

Lemma 1: The MARE (5) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (7)

$$\mathcal{G}(P) = A^T P + P^T A - P^T S P + Q = 0, \tag{7}$$

where

$$A := \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix},\ S := \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^T & S_f \end{bmatrix},$$

$$P := \begin{bmatrix} P_{00} & P_{0f} \\ P_{f0} & P_f \end{bmatrix},\ P_{f0} := \begin{bmatrix} P_{10} \\ \vdots \\ P_{N0} \end{bmatrix},$$

$$P_{0f} = P_{f0}^T \Pi_e := \left[\varepsilon_1 P_{10}^T \cdots \varepsilon_N P_{N0}^T \right],$$

$$P_f := \begin{bmatrix} P_{11} & \alpha_{12} P_{21}^T & \cdots & \alpha_{1N} P_{N1}^T \\ P_{21} & P_{22}^T & \cdots & \alpha_{2N} P_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ P_{N-11} & P_{N-12} & \cdots & \alpha_{N-1N} P_{NN-1}^T \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix}.$$

Proof: Firstly, by direct calculation we verify that $P_e = \Phi_e P$. Secondly, it is easy to verify that $A = \Phi_e A_e$, $S = \Phi_e S_e \Phi_e$. Hence,

$$A^T P = A_e^T \Phi_e \Phi_e^{-1} P_e = A_e^T P_e.$$

By using the similar calculation, we can immediately rewrite (5) as (7). □

The GMARE (7) can be partitioned into

$$\begin{aligned} f_1 &= P_{00}^T A_{00} + A_{00}^T P_{00} + P_{f0}^T A_{f0} + A_{f0}^T P_{f0} \\ &\quad - P_{00}^T S_{00} P_{00} - P_{f0}^T S_f P_{f0} - P_{00}^T S_{0f} P_{f0} \\ &\quad - P_{f0}^T S_{0f}^T P_{00} + Q_{00} = 0, \end{aligned} \tag{8a}$$

$$\begin{aligned} f_2 &= A_{00}^T P_{f0}^T \Pi_e + A_{f0}^T P_f + P_{00}^T A_{0f} + P_{f0}^T A_f \\ &\quad - P_{00}^T S_{00} P_{f0}^T \Pi_e - P_{f0}^T S_{0f}^T P_{f0}^T \Pi_e \\ &\quad - P_{00}^T S_{0f} P_f - P_{f0}^T S_f P_f + Q_{0f} = 0, \end{aligned} \tag{8b}$$

$$\begin{aligned} f_3 &= P_f^T A_f + A_f^T P_f + \Pi_e P_{f0} A_{0f} + A_{0f}^T P_{f0}^T \Pi_e \\ &\quad - P_f^T S_f P_f - P_f^T S_{0f}^T P_{f0}^T \Pi_e - \Pi_e P_{f0} S_{0f} P_f \\ &\quad - \Pi_e P_{f0} S_{00} P_{f0}^T \Pi_e + Q_f = 0. \end{aligned} \tag{8c}$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., Khalil and Kokotović, 1978, 1979), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij}. \tag{9}$$

Let $\bar{P}_{00}$, $\bar{P}_{f0}$ and $\bar{P}_f$ be the limiting solutions of the above equation (8) as $\varepsilon_j \to +0,\ j = 1,\ \cdots,\ N$, then we obtain the following equations

$$\begin{aligned} &\bar{P}_{00}^T A_{00} + A_{00}^T \bar{P}_{00} + \bar{P}_{f0}^T A_{f0} + A_{f0}^T \bar{P}_{f0} \\ &\quad - \bar{P}_{00}^T S_{00} \bar{P}_{00} - \bar{P}_{f0}^T S_f \bar{P}_{f0} - \bar{P}_{00}^T S_{0f} \bar{P}_{f0} \\ &\quad - \bar{P}_{f0}^T S_{0f}^T \bar{P}_{00} + Q_{00} = 0, \end{aligned} \tag{10a}$$

$$\begin{aligned} &A_{f0}^T \bar{P}_f + \bar{P}_{00}^T A_{0f} + \bar{P}_{f0}^T A_f - \bar{P}_{00}^T S_{0f} \bar{P}_f \\ &\quad - \bar{P}_{f0}^T S_f \bar{P}_f + Q_{0f} = 0, \end{aligned} \tag{10b}$$

$$\bar{P}_f^T A_f + A_f^T \bar{P}_f - \bar{P}_f^T S_f \bar{P}_f + Q_f = 0, \tag{10c}$$

where

$$\bar{P}_f := \begin{bmatrix} \bar{P}_{11} & \bar{\alpha}_{12} \bar{P}_{21}^T & \cdots & \bar{\alpha}_{1N} \bar{P}_{N1}^T \\ \bar{P}_{21} & \bar{P}_{22} & \cdots & \bar{\alpha}_{2N} \bar{P}_{N2}^T \\ \bar{P}_{31} & \bar{P}_{32} & \cdots & \bar{\alpha}_{3N} \bar{P}_{N3}^T \\ \vdots & \vdots & \vdots & \vdots \\ \bar{P}_{N-11} & \bar{P}_{N-12} & \cdots & \bar{\alpha}_{N-1N} \bar{P}_{NN-1}^T \\ \bar{P}_{N1} & \bar{P}_{N2} & \cdots & \bar{P}_{NN} \end{bmatrix},$$

$$\bar{P}_{jj} := \bar{P}_{jj}^T,\ j = 0,\ 1,\ 2,\ \cdots, N.$$

The ARE (10c) obtained is nonsymmetric. However, it is easy to verify that the ARE (10c) is an ARE which admits at least a unique symmetric positive semidefinite stabilizing solution.

Theorem 1: Under the assumption 1, the ARE (10c) admits a unique symmetric positive semidefinite stabilizing solution $\bar{P}_f$ which can be written as

$$\bar{P}_f^* := \mathrm{block-diag}\left(\bar{P}_{11}^* \cdots \bar{P}_{NN}^* \right), \tag{11}$$

where $\bar{P}_{jj}^*$ is a unique symmetric positive semidefinite stabilizing solution for the following AREs, respectively

$$
\begin{aligned}
&A_{jj}^T\bar{P}_{jj}^* + \bar{P}_{jj}^*A_{jj} - \bar{P}_{jj}^*S_{jj}\bar{P}_{jj}^* + Q_{jj} = 0,\\
&j = 1,\ 2, \cdots, N.
\end{aligned}
$$

Proof: Substituting (11) into the ARE (10c) as $\bar{P}_f^* \to \bar{P}_f$, it is easy to verify that $\bar{P}_f^*A_f + A_f^T\bar{P}_f^* - \bar{P}_f^*S_f\bar{P}_f^* + Q_f = 0$. Furthermore, it can be seen that $\bar{P}_f^* = \bar{P}_f^{*T} \geq 0$ and $A_f - S_f\bar{P}_f^*$ is stable from (12) under the assumption 1.

$$
\begin{aligned}
&A_f - S_f\bar{P}_f^* = \mathrm{block-diag}\\
&\quad\left(A_{11} - S_{11}\bar{P}_{11}^* \ \cdots \ A_{NN} - S_{NN}\bar{P}_{NN}^* \right) \qquad (12)
\end{aligned}
$$

Consequently, there exists a unique solution of the ARE (10c) and its solution is (11) itself. □

The assumption 1 ensures that $A_{jj} - S_{jj}\bar{P}_{jj}^*,\ j = 1,\ 2,\ \cdots,\ N$ are nonsingular. Substituting the solution of (10c) into (10b) and substituting $\bar{P}_{f0}^*$ into (10a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following 0–order equations (13)

$$
\begin{aligned}
&\bar{P}_{00}^*\mathcal{A} + \mathcal{A}^T\bar{P}_{00}^* - \bar{P}_{00}^*\mathcal{S}\bar{P}_{00}^* + \mathcal{Q} = 0, && (13a)\\
&\bar{P}_{f0}^* = -N_2^T + N_1^T\bar{P}_{00}^*, && (13b)\\
&\Leftrightarrow\ P_{j0}^T = -[P_{00}^*D_{0j} + (A_{j0}^TP_{jj}^* + Q_{0j})]D_{jj}^{-1} &&\\
&\bar{P}_f^*A_f + A_f^T\bar{P}_f^* - \bar{P}_f^*S_f\bar{P}_f^* + Q_f = 0, && (13c)\\
&\Leftrightarrow\ \bar{P}_{jj}^*A_{jj} + A_{jj}^T\bar{P}_{jj}^* - \bar{P}_{jj}^*S_{jj}\bar{P}_{jj}^* + Q_{jj} = 0, &&
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A} &:= A_{00} + N_1A_{f0} + S_{0f}N_2^T + N_1S_fN_2^T,\\
\mathcal{S} &:= S_{00} + N_1S_{0f}^T + S_{0f}N_1^T + N_1S_fN_1^T,\\
\mathcal{Q} &:= Q_{11} - N_2A_{f0} - A_{f0}^TN_2^T - N_2S_fN_2^T,\\
N_2^T &:= \bar{A}_f^{-T}\bar{Q}_{0f}^T,\ N_1^T := -\bar{A}_f^{-T}\bar{A}_{0f}^T,\\
\bar{A}_{0f} &:= A_{0f} - S_{0f}\bar{P}_f^*,\ \bar{A}_f := A_f - S_f\bar{P}_f^*,\\
\bar{Q}_{0f} &:= Q_{0f} + A_{f0}^T\bar{P}_f^*,\ D_{0j} := A_{0j} - S_{0j}P_{jj}^*,\\
D_{jj} &:= A_{jj} - S_{0j}P_{jj}^*,\ j = 1,\ 2,\ \cdots, N.
\end{aligned}
$$

In the following we established the relation between the GMARE (7) and the 0–order equations (13). Before doing that, we give the results for the AREs (13).

Lemma 2: Under the assumptions 1–3, the following results hold.

(i) The matrices $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{Q}$ do not depend on $\bar{P}_{jj}^*,\ j = 1,\ 2,\ \cdots,\ N$.

(ii) There exist a matrix $\mathcal{B} \in \mathbf{R}^{\bar{n}\times\bar{m}},\ \bar{m} := \sum_{j=0}^{N} m_j$ and a matrix $\mathcal{C}$ with the same dimension as C_0 such that $\mathcal{S} = \mathcal{B}R^{-1}\mathcal{B}^T$, $\mathcal{Q} = \mathcal{C}^T\mathcal{C}$. Moreover, the triple $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ is stabilizable and detectable.

Proof: Firstly, we introduce the following coordinate matrix

$$
\Omega = \begin{bmatrix}
I_{n_1} & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & I_{n_2} & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & I_{n_N} & 0\\
0 & I_{n_1} & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & I_{n_2} & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & I_{n_N}
\end{bmatrix}. \qquad (14)
$$

Furthermore, let us define four partitioned matrices (Wang et al., 1994)

$$
\begin{aligned}
&H_1 = T_{00} := \begin{bmatrix} A_{00} & -S_{00}\\ -Q_{00} & -A_{00}^T \end{bmatrix},\\
&H_2 := \begin{bmatrix} A_{0f} & -S_{0f}\\ -Q_{0f} & -A_{f0}^T \end{bmatrix},\\
&H_3 := \begin{bmatrix} A_{f0} & -S_{0f}^T\\ -Q_{0f}^T & -A_{0f}^T \end{bmatrix},\ H_4 := \begin{bmatrix} A_f & -S_f\\ -Q_f & -A_f^T \end{bmatrix}.
\end{aligned}
$$

It is well known from Xu et al. (1997) and Wang et al. (1994) that

$$
H_0 = \begin{bmatrix} \mathcal{A} & -\mathcal{S}\\ -\mathcal{Q} & -\mathcal{A}^T \end{bmatrix} = H_1 - H_2H_4^{-1}H_3.
$$

Using the above relation under the assumption 3, we get

$$
\begin{aligned}
H_0 &= H_1 - H_2\Omega(\Omega^TH_4\Omega)^{-1}\Omega^TH_3\\
&= T_{00} - \sum_{j=1}^{N} T_{0j}T_{jj}^{-1}T_{j0}, \qquad (15)
\end{aligned}
$$

where

$$
\begin{aligned}
&T_{0j} = \begin{bmatrix} A_{0j} & -S_{0j}\\ -Q_{0j} & -A_{j0}^T \end{bmatrix},\ T_{j0} = \begin{bmatrix} A_{j0} & -S_{0j}^T\\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix},\\
&j = 1,\ 2,\ \cdots, N.
\end{aligned}
$$

Therefore, it suffices the proof of (i) to show that the Hamilton matrix H_0 can be computed by using $T_{pq},\ pq = 00, \cdots, 0N,\ 11, \cdots, NN,\ 10, \cdots, N0$ which are independent of $\bar{P}_{jj}^*$.

The rest of the proof of Lemma 2 is omitted since it is similar to the proof of Xu et al. (1997). □

Since the triple $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ is stabilizable and detectable, the ARE (13a) admits a unique positive semidefinite symmetric stabilizing solution, denoted by $\bar{P}_{00}^*$, and $\mathcal{A} - \mathcal{S}\bar{P}_{00}^*$ is stable.

The limiting behavior of P_e as the parameter $\|\mu\| := \sqrt[N]{\varepsilon_1\varepsilon_2\cdots\varepsilon_N} \to +0$ is described by the following theorem.

Theorem 2: Under the assumptions 1–3, there exists a small σ^* such that for all $\|\mu\| \in (0,\ \sigma^*)$ the MARE (5) admits a symmetric positive semidefinite stabilizing solution P_e which can be written as

$$
P_e = \Phi_e \begin{bmatrix} \bar{P}_{00}^* + O(\|\mu\|) & [\bar{P}_{f0}^* + O(\|\mu\|)]^T \Pi_e \\ \bar{P}_{f0}^* + O(\|\mu\|) & \bar{P}_f^* + O(\|\mu\|) \end{bmatrix}
= \begin{bmatrix} \bar{P}_{00}^* + O(\|\mu\|) & [\bar{P}_{f0}^* + O(\|\mu\|)]^T \Pi_e \\ \Pi_e[\bar{P}_{f0}^* + O(\|\mu\|)] & \Pi_e[\bar{P}_f^* + O(\|\mu\|)] \end{bmatrix}. \quad (16)
$$

Proof: We apply the implicit function theorem (Gajic, 1988) to (8). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (8) in the limit as $\|\mu\| \to 0$ is given by

$$
\mathbf{J} = \nabla \mathbf{F} = \left. \frac{\partial \mathrm{vec}(f_1,\ f_2,\ f_3)}{\partial \mathrm{vec}(P_{00},\ P_{f0},\ P_f)^T} \right|_{\|\mu\|=0}
= \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix}, \quad (17)
$$

where vec denotes an ordered stack of the columns of its matrix and

$$
\begin{aligned}
&\mathbf{J}_{00} = (I_{n_0} \otimes \bar{A}_{00}^T) U_{n_0 n_0} + \bar{A}_{00}^T \otimes I_{n_0}, \\
&\mathbf{J}_{01} = (I_{n_0} \otimes \bar{A}_{f0}^T) U_{n_0 \hat{n}} + \bar{A}_{f0}^T \otimes I_{n_0}, \\
&\mathbf{J}_{10} = \bar{A}_{0f}^T \otimes I_{n_0},\ \mathbf{J}_{11} = \bar{A}_f^T \otimes I_{n_0}, \\
&\mathbf{J}_{22} = (I_{\hat{n}} \otimes \bar{A}_f^T) U_{\hat{n}\hat{n}} + \bar{A}_f^T \otimes I_{\hat{n}}, \\
&\bar{A}_{00} = A_{00} - S_{00} \bar{P}_{00}^* - S_{0f} \bar{P}_{f0}^*, \\
&\bar{A}_{f0} = A_{f0} - S_{0f}^T \bar{P}_{00}^* - S_f \bar{P}_{f0}^*, \\
&\bar{A}_0 = \bar{A}_{00} - \bar{A}_{0f} \bar{A}_f^{-1} \bar{A}_{f0},\ \hat{n} = \sum_{j=1}^{N} n_j,
\end{aligned}
$$

where $\otimes$ denotes Kronecker products and $U_{n_0 n_0}$ is the permutation matrix in the Kronecker matrix sense.

The Jacobian (17) can be expressed as

$$
\det \mathbf{J} = \det \mathbf{J}_{22} \cdot \det \mathbf{J}_{11} \cdot \det[I_{n_0} \otimes \bar{A}_0^T U_{n_0 n_0} + \bar{A}_0^T \otimes I_{n_0}], \quad (18)
$$

where $\bar{A}_0 \equiv \bar{A}_{00} - \bar{A}_{0f} \bar{A}_f^{-1} \bar{A}_{f0}$. Obviously, $\mathbf{J}_{jj},\ j = 1,\ 2$ are nonsingular because the matrices $\bar{A}_f = A_f - S_f \bar{P}_f^*$ is stable under the assumption 1. After some straightforward but tedious algebra, we see that $\mathcal{A} - \mathcal{S} \bar{P}_{00}^* = \bar{A}_{00} - \bar{A}_{0f} \bar{A}_f^{-1} \bar{A}_{f0} = \bar{A}_0$. Therefore, the matrix $\bar{A}_0$ is also stable if the assumption 2 holds. Thus, $\det \mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Theorem 2 is obtained directly by using the implicit function theorem.

The remainder of the proof is to show that P_e is the positive semidefinite stabilizing solution. Firstly, from (16), we get

$$
P_e = \begin{bmatrix} \bar{P}_{00}^* & 0 \\ 0 & 0 \end{bmatrix} + O(\|\mu\|),
$$

Taking into consideration the fact that the solution $\bar{P}_{00}^*$ is positive semidefinite, we have $P_e \geq 0$. Secondly, using (16), we obtain

$$
A_e - S_e P_e = \Phi_e^{-1} \left(\begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \bar{A}_{f0} & \bar{A}_f \end{bmatrix} + O(\|\mu\|) \right).
$$

The matrix $\bar{A}_f$ and $\bar{A}_0$ are stable since the assumptions 1 and 2 holds. Therefore, if parameter $\|\mu\|$ is very small, $A_e - S_e P_e$ is stable by applying the Theorem 1 in Khalil and Kokotović (1979). □

3. NEAR–OPTIMAL CONTROL FOR THE MSPS

The required solution of the MARE (5) exists under the assumptions 1–3. Our attention is focused on the specific linear state feedback controller which does not depend on the values of the small parameters. Such the linear state feedback controller is obtained by eliminating $O(\|\mu\|)$ item of the linear state feedback controller (4). If $\|\mu\|$ is very small, it is obvious that the linear state feedback controller (4) can be approximated as

$$
\begin{aligned}
u_{\mathrm{app}} &= \begin{bmatrix} u_{1\mathrm{app}}^T & \cdots & u_{N\mathrm{app}}^T \end{bmatrix}^T \\
&= -R^{-1} B^T P_{\mathrm{app}} x = -R^{-1} B^T \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix} x \\
&= -R^{-1} B^T \begin{bmatrix} \bar{P}_{00}^* & 0 & \cdots & 0 \\ \bar{P}_{10}^* & \bar{P}_{11}^* & \cdots & 0 \\ \bar{P}_{20}^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{N0}^* & 0 & \cdots & \bar{P}_{NN}^* \end{bmatrix} x, \quad (19)
\end{aligned}
$$

where $B = \Phi_e B_e$.

Remark 1: Even though our control design is quite different from the composite controller design (Khalil and Kokotović, 1979; Wang et al., 1994; Xu et al., 1997), we can be shown that the resulting controller (19) is similar to the existing one.

When $\|\mu\|$ is sufficiently small, we know from Theorem 2 that the resulting controller (19) will be close to the optimal controller (4). In an optimization problem it is of interest to check whether the resulting value of the cost function will be near its optimal value. The optimal value J_{opt} is obtained with the controller (4) which optimizes the cost for the actual system (1).

Theorem 3: Under the assumptions 1–3, the use of the reduced–order controller (19) results in J_{app} satisfying

$$
J_{\mathrm{app}} = J_{\mathrm{opt}} + O(\|\mu\|^2), \quad (20)
$$

where

$$
J_{\mathrm{opt}} = \frac{1}{2} x(0)^T P_e x(0).
$$

Before proving this theorem, we introduce the following useful lemma (Mukaidani et al., 2001).

Lemma 3: Consider the iterative algorithm which is based on the Kleinman algorithm

$$\begin{aligned}&(A-SP^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A-SP^{(i)})\\&\quad +P^{(i)T}SP^{(i)} + Q = 0, \quad i=0,\ 1,\ \cdots, \qquad (21a)\\&P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & P_{f0}^{(i)T}\Pi_e \\ P_{f0}^{(i)} & P_f^{(i)} \end{bmatrix}, \qquad (21b)\end{aligned}$$

with the initial condition obtained from

$$P^{(0)} = P_{\mathrm{app}} = \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix}. \qquad (22)$$

Under the assumptions 1–3, there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0,\ \bar{\sigma}),\ \bar{\sigma} \le \sigma^*$ the iterative algorithm (21) converges to the exact solution of $P_e = \Phi_e P = P^T \Phi_e$ with the rate of quadratic convergence, where $P_e^{(i)} = \Phi_e P^{(i)} = P^{(i)T}\Phi_e$ is positive semidefinite.

$$\|P^{(i)} - P\| = O(\|\mu\|^{2^i}),\ i=0,\ 1,\ 2,\ \cdots, \qquad (23)$$

where

$$\begin{aligned}&\gamma = 2\|S\| < \infty,\ \beta = \|[\nabla\mathcal{G}(P^{(0)})]^{-1}\|,\\&\eta = \beta\cdot\|\mathcal{G}(P^{(0)})\|,\ \theta = \beta\eta\gamma,\ \nabla\mathcal{G}(P) = \frac{\partial \mathrm{vec}\mathcal{G}(P)}{\partial(\mathrm{vec}P)^T}.\end{aligned}$$

Proof: When u_{app} is used, the value of the performance index is

$$J_{\mathrm{app}} = \frac{1}{2}x(0)^T W_e x(0), \qquad (24)$$

where W_e is a positive semidefinite solution of the multiparameter algebraic Lyapunov equation (MALE)

$$\begin{aligned}&(A_e - S_e P_{\mathrm{appe}})^T W_e + W_e(A_e - S_e P_{\mathrm{appe}})\\&\quad + P_{\mathrm{appe}} S_e P_{\mathrm{appe}} + Q = 0, \qquad (25)\end{aligned}$$

where $P_{\mathrm{appe}} = \Phi_e P_{\mathrm{app}}$. Subtracting (5) from (25) we find that $V_e = W_e - P_e$ satisfies the following MALE

$$\begin{aligned}&(A_e - S_e P_{\mathrm{appe}})^T V_e + V_e(A_e - S_e P_{\mathrm{appe}})\\&\quad + (P_e - P_{\mathrm{appe}})S_e(P_e - P_{\mathrm{appe}}) = 0. \qquad (26)\end{aligned}$$

Similarly, subtracting (5) from (21a) we also get the MALE

$$\begin{aligned}&(A_e - S_e P_e^{(i)})^T (P_e^{(i+1)} - P_e)\\&\quad + (P_e^{(i+1)} - P_e)(A_e - S_e P_e^{(i)})\\&\quad + (P_e - P_e^{(i)})S_e(P_e - P_e^{(i)}) = 0, \qquad (27)\end{aligned}$$

where $P_e^{(i)} = \Phi_e P^{(i)}$. When $i = 0$, we have

$$\begin{aligned}&(A_e - S_e P_e^{(0)})^T (P_e^{(1)} - P_e)\\&\quad + (P_e^{(1)} - P_e)(A_e - S_e P_e^{(0)})\\&\quad + (P_e - P_e^{(0)})S_e(P_e - P_e^{(0)})\\&= (A_e - S_e P_{\mathrm{appe}})^T (P_e^{(1)} - P_e)\\&\quad + (P_e^{(1)} - P_e)(A_e - S_e P_{\mathrm{appe}})\\&\quad + (P_e - P_{\mathrm{appe}})S_e(P_e - P_{\mathrm{appe}}) = 0.\end{aligned}$$

Therefore, it is easy to verify that $V_e = P_e^{(1)} - P_e$ because $A_e - S_e P_{\mathrm{appe}}$ is stable from Theorem 1 in Khalil and Kokotović (1979). Using Lemma 3 we obtain that

$$\begin{aligned}&\|V_e\| = \|W_e - P_e\| = \|P_e^{(1)} - P_e\|\\&\le \|\Phi_e\|\cdot\|P^{(1)} - P\| \le \|P^{(1)} - P\| = O(\|\mu\|^2). \qquad (28)\end{aligned}$$

Hence, we have $V_e = W_e - P_e = O(\|\mu\|^2)$, which implies (20). □

4. CONCLUSION

In this paper, we have studied the optimal control problem associated with the MSPS. The main contribution of this paper is to propose the new design method of the ε_j–independent reduced–order controller. Note that our design method is quite different from the existing method such as the two–time–scale design method and the descriptor approach. Furthermore, we have shown that the resulting controller achieves $O(\|\mu\|^2)$ approximation of the optimal cost compared with the previously proposed controller.

References

Coumarbatch, C. and Z. Gajić (2000). Parallel optimal Kalman filtering for stochastic systems in multimodeling form. *Trans. ASME, J. Dynamic Systems, Measurement, and Control*, **122**, 542–550.

Gajić, Z. (1988). The existence of a unique and bounded solution of the algebraic Riccati equation of multimodel estimation and control problems. *Sys. & Con. Lett.*, **10**, 185–190.

Khalil, H. K. and P. V. Kokotović (1978). Control strategies for decision makers using different models of the same system. *IEEE Trans. A.C.*, **23**, 289–298.

Khalil, H. K. and P. V. Kokotović (1979). Control of linear systems with multiparameter singular perturbations. *Automatica*, **15**, 197–207.

Laub, A. J. (1979). A Schur method for solving algebraic Riccati equations. *IEEE Trans. A.C.*, **24**, 913–921.

Mukaidani, H., H. Xu and K. Mizukami (2001). New iterative algorithm for algebraic Riccati equation related to H_∞ control problem of singularly perturbed systems. *IEEE Trans. A.C.*, **46**, 1659-1666.

Wang, Y-Y., M. Paul and N. E. Wu (1994). Near–optimal control of nonstandard singularly perturbed systems. *Automatica*, **30**, 277–292.

Xu, H., H. Mukaidani and K. Mizukami (1997). : New Method for Composite Optimal Control of Singularly Perturbed Systems, International. *Int. J. Systems Sciences*, **28**, 161–172.