合成関数偏微分

[基本事項] $z=f(x,y)$, $x=x(u,v)$, $y=y(u,v)$ とする.

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial u^2}&=\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial u}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial u}\right]\frac{\partial x}{\partial u}+\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u^2}+\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial u}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u}\right]\frac{\partial y}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial u^2}\\&=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2+2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2+\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u^2}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial u^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial v^2}&=\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial v}\right]\frac{\partial x}{\partial v}+\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial v^2}+\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial v}\right]\frac{\partial y}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial v^2}\\&=\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2+2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}+\frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2+\frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial v^2}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial v^2}\end{aligned}$$

[1] $z=f(x,y)$, $x=e^{-r}\cos\theta$, $y=e^{-r}\sin\theta$ とする.

$$(x^2+y^2)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)=\frac{\partial^2 z}{\partial r^2}+\frac{\partial^2 z}{\partial \theta^2}$$

[2] $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ とする.

(1) $f=f(x,y,z)$ であるなら,

$$f_r=f_x\frac{\partial x}{\partial r}+f_y\frac{\partial y}{\partial r}+f_z\frac{\partial z}{\partial r}=\sin\theta\cos\phi f_x+\sin\theta\sin\phi f_y+\cos\theta f_z$$

$$f_\theta=f_x\frac{\partial x}{\partial \theta}+f_y\frac{\partial y}{\partial \theta}+f_z\frac{\partial z}{\partial \theta}=r\cos\theta\cos\phi f_x+r\cos\theta\sin\phi f_y-r\sin\theta f_z$$

$$f_\phi=f_x\frac{\partial x}{\partial \phi}+f_y\frac{\partial y}{\partial \phi}+f_z\frac{\partial z}{\partial \phi}=-r\sin\theta\sin\phi f_x+r\sin\theta\cos\phi f_y$$

$$\begin{aligned}f_{rr}=&\ \sin\theta\cos\phi(\sin\theta\cos\phi f_{xx}+\sin\theta\sin\phi f_{xy}+\cos\theta f_{xz})\\&+\sin\theta\sin\phi(\sin\theta\cos\phi f_{yx}+\sin\theta\sin\phi f_{yy}+\cos\theta f_{yz})\\&+\cos\theta(\sin\theta\cos\phi f_{zx}+\sin\theta\sin\phi f_{zy}+\cos\theta f_{zz})\\=&\ \sin^2\theta\cos^2\phi f_{xx}+\sin^2\theta\sin^2\phi f_{yy}+\cos^2\theta f_{zz}\\&+2\sin^2\theta\sin\phi\cos\phi f_{xy}+2\sin\theta\cos\theta\sin\phi f_{yz}+2\sin\theta\cos\theta\cos\phi f_{zx}\end{aligned}$$

$$\begin{aligned}f_{\theta\theta}=&\ r\cos\theta\cos\phi(r\cos\theta\cos\phi f_{xx}+r\cos\theta\sin\phi f_{xy}-r\sin\theta f_{xz})\\&+r\cos\theta\sin\phi(r\cos\theta\cos\phi f_{yx}+r\cos\theta\sin\phi f_{yy}-r\sin\theta f_{yz})\\&-r\sin\theta(r\cos\theta\cos\phi f_{zx}+r\cos\theta\sin\phi f_{zy}-r\sin\theta f_{zz})\\&-r\sin\theta\cos\phi f_x-r\sin\theta\sin\phi f_y-r\cos\theta f_z\\=&\ r^2\cos^2\theta\cos^2\phi f_{xx}+r^2\cos^2\theta\sin^2\phi f_{yy}+r^2\sin^2\theta f_{zz}\\&+2r^2\cos^2\theta\sin\phi\cos\phi f_{xy}-2r^2\sin\theta\cos\theta\sin\phi f_{yz}-2r^2\sin\theta\cos\theta\cos\phi f_{yz}\\&-r\sin\theta\cos\phi f_x-r\sin\theta\sin\phi f_y-r\cos\theta f_z\end{aligned}$$

$$\begin{aligned}f_{\phi\phi}=&\ -r\sin\theta\sin\phi(-r\sin\theta\sin\phi f_{xx}+r\sin\theta\cos\phi f_{xy})\\&+r\sin\theta\cos\phi(-r\sin\theta\sin\phi f_{yx}+r\sin\theta\cos\phi f_{yy})\\&-r\sin\theta\cos\phi f_x-r\sin\theta\sin\phi f_y\\=&\ r^2\sin^2\theta\sin^2\phi f_{xx}-2r^2\sin^2\theta\sin\phi\cos\phi f_{xy}+r^2\sin^2\theta\cos^2\phi f_{yy}\\&-r\sin\theta\cos\phi f_x-r\sin\theta\sin\phi f_y\end{aligned}$$

$$f_{xx}+f_{yy}+f_{zz}=f_{rr}+\frac{1}{r^2}f_{\theta\theta}+\frac{1}{r^2\sin^2\theta}f_{\phi\phi}+\frac{2}{r}f_r+\frac{\cos\theta}{r^2\sin\theta}f_\theta$$

このとき,

$$\nabla^2=\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

をラプラシアンという.

(2) 特に, $f(x,y,z)$ が, $r=\sqrt{x^2+y^2+z^2}$ のみの関数であるとき, すなわち, $f=f(r)$ であるなら,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f(r)=\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)f(r)$$

$$\Leftrightarrow f_{xx}+f_{yy}+f_{zz}=f_{rr}+\frac{2}{r}f_r$$

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$z = f(x, y)$, $x(r, \theta) = r\cos\theta$, $y(r, \theta) = r\sin\theta$ に対し, 次の問いに答えよ.

(1) z_r, z_θ を計算せよ.

(2) $z_x^2 + z_y^2 = z_r^2 + \frac{1}{r^2} z_\theta^2$ を示せ.

(3) $z_{xx} + z_{yy} = z_{rr} + \frac{1}{r^2} z_{\theta\theta} + \frac{1}{r} z_r$ を示せ.

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(1)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

(2)

$$z_r^2 + \frac{1}{r^2}z_\theta^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}\right)^2 = z_x^2 + z_y^2$$

(3)

$$\begin{aligned}
z_{rr} = \frac{\partial^2 z}{\partial r^2} &= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2} \\
&= \frac{\partial^2 z}{\partial x^2}(\cos\theta)^2 + 2\frac{\partial^2 z}{\partial x \partial y}(\cos\theta)(\sin\theta) + \frac{\partial^2 z}{\partial y^2}(\sin\theta)^2 \\
&= \cos^2\theta\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta)\frac{\partial^2 z}{\partial y^2} \\
z_{\theta\theta} = \frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial \theta}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial \theta}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial \theta^2} \\
&= \frac{\partial^2 z}{\partial x^2}(-r\sin\theta)^2 + 2\frac{\partial^2 z}{\partial x \partial y}(-r\sin\theta)(r\cos\theta) + \frac{\partial^2 z}{\partial y^2}(r\cos\theta)^2 + \frac{\partial z}{\partial x}(-r\cos\theta) + \frac{\partial z}{\partial y}(-r\sin\theta) \\
&= r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x \partial y} + r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2} - r\cos\theta\frac{\partial z}{\partial x} - r\sin\theta\frac{\partial z}{\partial y} \\
&= r^2\left(\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\left\{\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right\}\right) \\
&= r^2\left(\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2} - \frac{1}{r}z_r\right)
\end{aligned}$$

以上より,

$$z_{rr} + \frac{1}{r^2}z_{\theta\theta} = z_{xx} + z_{yy} - \frac{1}{r}z_r$$

より成立.