微分積分学 I　期末試験: 担当 向谷　博明

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座席番号	学部	学籍番号	氏名	得点

【 1 】理系は (1a), (2a), (3a), (4a), (5a), (6a), (7)〜(10). 文系は (1b), (2b), (3b), (4b), (5b), (6b), (7)〜(10) を計算せよ.

(1a) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{x^2+1} = \frac{\pi}{\boxed{3}}$

(1b) $\displaystyle\int_1^{e} x\log x dx = \frac{e^2+\boxed{1}}{\boxed{4}}$

(2a) $\displaystyle\int_0^{1} \sin^{-1} x dx = \frac{\pi}{\boxed{2}} - \boxed{1}$

(2b) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{\boxed{6}}$

(3a) $\displaystyle\int_0^{1} \frac{x}{x^2-x+1}dx = \frac{\sqrt{\boxed{3}}}{\boxed{9}}\pi$

(3b) $\displaystyle\int_0^{1} \frac{dx}{x^2+x+1} = \frac{\sqrt{\boxed{3}}}{\boxed{9}}\pi$

(4a) $\displaystyle\int_{\log\frac{1}{2}}^{0} \frac{dx}{e^x+1} = \log\frac{\boxed{3}}{\boxed{2}}$

(4b) $\displaystyle\int_2^{3} \frac{dx}{x^2-1} = \frac{1}{\boxed{2}}\log\frac{\boxed{3}}{\boxed{2}}$

(5a) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{1+\sin x} = \sqrt{\boxed{3}} - \boxed{1}$

(5b) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{\sin x} = \frac{\log\boxed{3}}{\boxed{2}}$

(6a) $\displaystyle\int_0^{2} x\sqrt{2x-x^2}dx = \frac{\pi}{\boxed{2}}$

(6b) $\displaystyle\int_0^{\frac{3}{4}} \frac{dx}{\sqrt{x^2+1}} = \log\boxed{2}$

(7) $\displaystyle\int_0^{2\pi} \cos 2018x\cos 8x dx = \boxed{0}$

(8) $\displaystyle\int_{-1}^{1} (x+1)\sqrt{x^2+1}dx = \sqrt{\boxed{2}} + \log\left(\boxed{1} + \sqrt{\boxed{2}}\right)$

(9) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x\cos^5 x dx = \frac{1}{\boxed{6\ 0}}$

(10) $\displaystyle\int_0^{1} x^4\sqrt{1-x^2}dx = \frac{\pi}{\boxed{3\ 2}}$

【２】広義積分 $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)^5}dx$ が収束することを示し, その値を計算せよ.

【加点問題 (+15 点)】広義積分 $\displaystyle\int_0^1 (\log x)^6 dx$ が収束することを示し, その値を計算せよ.

まず, 広義積分が存在することを示す.

$$\int_0^\infty \frac{x^2}{(x^2+1)^5}dx < \int_0^1 \frac{x^2}{(x^2+1)^5}dx + \int_1^\infty \frac{dx}{x^8} = \int_0^1 \frac{x^2}{(x^2+1)^5}dx + \left[-\frac{1}{7x^7}\right]_1^\infty = \int_0^1 \frac{x^2}{(x^2+1)^5}dx + \frac{1}{7}$$

ここで, $\displaystyle\int_0^1 \frac{x^2}{(x^2+1)^5}dx$ は有限確定値より広義積分は存在する. また, その値は, $x = \tan\theta$ と置換して,

$$\begin{aligned}\int_0^\infty \frac{x^2}{(x^2+1)^5}dx &= \int_0^{\frac{\pi}{2}} \frac{\tan^2\theta}{\frac{1}{\cos^{10}\theta}} \cdot \frac{d\theta}{\cos^2\theta} = \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^6\theta d\theta = \int_0^{\frac{\pi}{2}} (\cos^6\theta - \cos^8\theta)d\theta \\ &= \left(1-\frac{7}{8}\right)\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{256}\pi\end{aligned}$$

$0 < x \leq 1$ で $0 \leq (\log x)^6 \leq \dfrac{1}{\sqrt{x}}$ を示す. $f(x) = x(\log x)^{12}$ とおく. このとき,

$$f'(x) = (\log x)^{12} + 12(\log x)^{11} = (\log x)^{11}(\log x + 12)$$

したがって, 増減表から $x = e^{-12} < 1$ で最大値をとることが分かり, $f(x) \leq f(e^{-12}) = e^{-12}(-12)^{12} = \left(\dfrac{12}{e}\right)^{12}$.
以上より,

$$f(x) = x(\log x)^{12} \leq \left(\frac{12}{e}\right)^{12} \ \Rightarrow\ 0 \leq (\log x)^{12} \leq \left(\frac{12}{e}\right)^{12}\frac{1}{x} \ \Rightarrow\ 0 \leq (\log x)^6 \leq \left(\frac{12}{e}\right)^6 \frac{1}{\sqrt{x}}$$

この不等式の両辺を積分して,

$$0 \leq \int_0^1 (\log x)^6 dx \leq \left(\frac{12}{e}\right)^6 \int_0^1 \frac{dx}{\sqrt{x}} = 2\left(\frac{12}{e}\right)^6$$

したがって, 有限である. すなわち, 広義積分は収束する.
一方, $\log x = -t$ と置換すれば,

$$\int_0^1 (\log x)^6 dx = \int_0^\infty t^6 e^{-t} dt = \Gamma(7) = 6! = 720$$

ただし, $\Gamma(s) = \displaystyle\int_0^\infty t^{s-1}e^{-t}dt$ で表される Γ 関数である. なお, 広義積分の存在の証明は, Γ 関数の存在と同様に示される.

【３】定積分 $I = \displaystyle\int_0^1 \frac{dx}{x\sqrt{x}+1}$ を求めたい.

(1) $\sqrt{x} = t$ と置換することによって, $I = 2\displaystyle\int_0^1 \frac{t}{t^3+1}dt$ を示せ.

(2) t についての恒等式 $\dfrac{t}{t^3+1} = \dfrac{a}{t+1} + \dfrac{bt+c}{t^2-t+1}$ が成立するとき, 定数 a, b, c を求めよ.

(3) I を求めよ. 【加点問題 (+10 点)】 $J = \displaystyle\int_0^\infty \frac{dx}{x\sqrt{x}+1}$ を求めよ.

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(1) $\sqrt{x} = t$ より, $x = t^2$, $dx = 2tdt$ より成立.
(2)

$$\frac{A}{t+1} + \frac{Bt+C}{t^2-t+1} = \frac{(A+B)t^2 + (-A+B+C)t + A + C}{t^3+1} = \frac{t}{t^3+1}$$

が恒等的に成立するので, $A+B=0$, $-A+B+C=1$, $A+C=0$ が成立する.
これを解いて, $A = -\dfrac{1}{3}$, $B = C = \dfrac{1}{3}$.

(3) まず, 不定積分 $\mathcal{I}$ を求める.

$$
\begin{aligned}
\mathcal{I} &= \int \frac{t}{t^3+1}dt = \frac{1}{3}\int\left\{-\frac{1}{t+1}+\frac{t+1}{t^2-t+1}\right\}dt \\
&= -\frac{1}{3}\log|t+1| + \left\{\frac{1}{6}\int\frac{2t-1}{t^2-t+1}dt + \frac{1}{2}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} \\
&= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log(t^2-t+1) + \frac{1}{\sqrt{3}}\arctan\frac{2t-1}{\sqrt{3}}
\end{aligned}
$$

以上から,

$$
\begin{aligned}
I &= 2\int_0^1 \frac{t}{t^3+1} = 2\left[-\frac{1}{3}\log|t+1| + \frac{1}{6}\log(t^2-t+1) + \frac{1}{\sqrt{3}}\arctan\frac{2t-1}{\sqrt{3}}\right]_0^1 = -\frac{2}{3}\log 2 + \frac{2}{3\sqrt{3}}\pi \\
J &= 2\int_0^\infty \frac{t}{t^3+1} = 2\left[-\frac{1}{3}\log|t+1| + \frac{1}{6}\log(t^2-t+1) + \frac{1}{\sqrt{3}}\arctan\frac{2t-1}{\sqrt{3}}\right]_0^\infty = \frac{4}{3\sqrt{3}}\pi
\end{aligned}
$$

【 4 】定積分 $I=\displaystyle\int_0^1\sqrt{\frac{1-x}{1+x}}dx$, $J=\displaystyle\int_0^1\frac{1}{1+x}\sqrt{\frac{1-x}{1+x}}dx$, $K=\displaystyle\int_0^1 x\sqrt{\frac{1-x}{1+x}}dx$ を求めたい.

(1) $t=\sqrt{\dfrac{1-x}{1+x}}$ とおくとき, x を t を用いて表せ. さらに, dx を計算せよ.

(2) 理系は I. 文系は J を計算せよ. ただし, 理系は $t=\tan\theta$ と置換して計算せよ.

【加点問題 (+10 点)】K を計算せよ.

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(1) $t=\sqrt{\dfrac{1-x}{1+x}}$ とおく. このとき, $t^2=\dfrac{1-x}{1+x}$ かつ, $x=\dfrac{1-t^2}{t^2+1}$ である. さらに,

$$
dx = \frac{-2t(t^2+1)-(1-t^2)2t}{(t^2+1)^2}dt = \frac{-4t}{(t^2+1)^2}dt
$$

であるので, 以下となる.

$$
\begin{aligned}
I &= \int_0^1\sqrt{\frac{1-x}{1+x}}dx = \int_1^0\frac{-4t^2}{(t^2+1)^2}dt = 4\int_0^1\frac{t^2}{(t^2+1)^2}dt = 4\int_0^{\frac{\pi}{4}}\tan^2\theta\cos^4\theta\frac{d\theta}{\cos^2\theta} = 4\int_0^{\frac{\pi}{4}}\sin^2\theta d\theta \\
&= 2\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)d\theta = 2\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{2}-1
\end{aligned}
$$

$$
\begin{aligned}
J &= \int_0^1\frac{1}{1+x}\sqrt{\frac{1-x}{1+x}}dx = \int_1^0\frac{t^2+1}{2}\cdot\frac{-4t^2}{(t^2+1)^2}dt = 2\int_0^1\frac{t^2}{t^2+1}dt \\
&= 2\int_0^1\left(1-\frac{1}{t^2+1}\right)dt = 2\Big[t-\arctan t\Big]_0^1 = 2-\frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
K &= \int_0^1 x\sqrt{\frac{1-x}{1+x}}dx = \int_1^0\frac{1-t^2}{t^2+1}\cdot\frac{-4t^2}{(t^2+1)^2}dt = 4\int_0^1\frac{t^2(1-t^2)}{(t^2+1)^3}dt \\
&= 4\int_0^{\frac{\pi}{4}}\tan^2\theta(1-\tan^2\theta)\cos^6\theta\frac{d\theta}{\cos^2\theta} = 4\int_0^{\frac{\pi}{4}}\sin^2\theta(\cos^2\theta-\sin^2\theta)d\theta = 4\int_0^{\frac{\pi}{4}}\sin^2\theta(1-2\sin^2\theta)d\theta \\
&= 2\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\cos 2\theta d\theta = \int_0^{\frac{\pi}{4}}(2\cos 2\theta-1-\cos 4\theta)d\theta = \left[\sin 2\theta-\theta-\frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}} = 1-\frac{\pi}{4}
\end{aligned}
$$

【 3 】定積分 $I = \displaystyle\int_0^1 \frac{dx}{x^4+x^2+1}$ を求めたい.

(1) $x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$ を示せ.

(2) 恒等式

$$\frac{1}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$

が成立するとき, 定数 A, B, C, D を求めよ.

(3) I を求めよ. また, 余力があれば, $J = \displaystyle\int_0^\infty \frac{dx}{x^4+x^2+1}$ を求めよ. ボーナスとして加点する.

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(2)

$$\begin{aligned}\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} &= \frac{(A+C)x^3+(-A+B+C+D)x^2+(A-B+C+D)x+B+D}{(x^2+x+1)(x^2-x+1)} \\ &= \frac{1}{(x^2+x+1)(x^2-x+1)}\end{aligned}$$

が恒等的に成立するので, $A+C=0$, $-A+B+C+D=0$, $A-B+C+D=0$, $B+D=1$ が成立する. これを解いて, $A=B=D=\frac{1}{2}$, $C=-\frac{1}{2}$.

(3) まず, 不定積分 $\mathcal{I}$ を求める.

$$\begin{aligned}\mathcal{I} &= \int \frac{dx}{x^4+x^2+1} = \frac{1}{2}\int \left\{\frac{x+1}{x^2+x+1} - \frac{x-1}{x^2-x+1}\right\} dx \\ &= \frac{1}{4}\int \left\{\frac{2x+1}{x^2+x+1} - \frac{2x-1}{x^2-x+1}\right\} dx + \frac{1}{4}\left\{\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\} \\ &= \frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + \frac{1}{2\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} = \frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}x}{1-x^2}\end{aligned}$$

以上から,

$$\begin{aligned}I &= \int_0^1 \frac{dx}{x^4+x^2+1} = \left[\frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}x}{1-x^2}\right]_0^1 = \frac{1}{4}\log 3 + \frac{1}{2\sqrt{3}}\cdot\frac{\pi}{2} = \frac{1}{4}\log 3 + \frac{\pi}{4\sqrt{3}} \\ J &= \int_0^\infty \frac{dx}{x^4+x^2+1} = \left[\frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}x}{1-x^2}\right]_0^\infty = \frac{\pi}{2\sqrt{3}}\end{aligned}$$

【 4 】定積分 $I=\displaystyle\int_0^1 x\sqrt{\frac{1-x}{1+x}}dx,\ J=\int_0^1\sqrt{\frac{1-x}{1+x}}dx,\ K=\int_0^1\frac{1}{1+x}\sqrt{\frac{1-x}{1+x}}dx$ を求めたい.

(1) $t=\sqrt{\dfrac{1-x}{1+x}}$ とおくとき, x を t を用いて表せ. さらに, dx を計算せよ.

(2) $a_n=\displaystyle\int_0^{\frac{\pi}{4}}\sin^n\theta d\theta$ とおくとき, $a_n=\dfrac{n-1}{n}a_{n-2}-\dfrac{1}{n}\left(\dfrac{1}{\sqrt{2}}\right)^n$, $a_0=\dfrac{\pi}{4}$ を示せ.

(3) 理系は I. 文系は J を計算せよ. ただし, $t=\tan\theta$ と置換せよ.

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(1) $t=\sqrt{\dfrac{1-x}{1+x}}$ とおく. このとき, $t^2=\dfrac{1-x}{1+x}$ かつ, $x=\dfrac{1-t^2}{t^2+1}$ である. さらに,

$$dx=\frac{-2t(t^2+1)-(1-t^2)2t}{(t^2+1)^2}dt=\frac{-4t}{(t^2+1)^2}dt$$

であるので, 以下となる.

$$\begin{aligned}
I &= \int_0^1\sqrt{\frac{1-x}{1+x}}dx=\int_1^0\frac{-4t^2}{(t^2+1)^2}dt=4\int_0^1\frac{t^2}{(t^2+1)^2}dt=4\int_0^{\frac{\pi}{4}}\tan^2\theta\cos^4\theta\frac{d\theta}{\cos^2\theta}=4\int_0^{\frac{\pi}{4}}\sin^2\theta d\theta\\
&= 2\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)d\theta=2\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}=\frac{\pi}{2}-1
\end{aligned}$$

$$\begin{aligned}
J &= \int_0^1\sqrt{\frac{1-x}{(1+x)^3}}dx=\int_0^1\frac{1}{1+x}\sqrt{\frac{1-x}{1+x}}dx=\int_1^0\frac{t^2+1}{2}\cdot\frac{-4t^2}{(t^2+1)^2}dt=2\int_0^1\frac{t^2}{t^2+1}dt\\
&= 2\int_0^{\frac{\pi}{4}}\left(1-\frac{1}{t^2+1}\right)dt=2\Big[t-\arctan t\Big]_0^{\frac{\pi}{4}}=2-\frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
K &= \int_0^1 x\sqrt{\frac{1-x}{1+x}}dx=\int_1^0\frac{1-t^2}{t^2+1}\cdot\frac{-4t^2}{(t^2+1)^2}dt=4\int_0^1\frac{t^2(1-t^2)}{(t^2+1)^3}dt\\
&= 4\int_0^{\frac{\pi}{4}}\tan^2\theta(1-\tan^2\theta)\cos^6\theta\frac{d\theta}{\cos^2\theta}=4\int_0^{\frac{\pi}{4}}\sin^2\theta(\cos^2\theta-\sin^2\theta)d\theta=4\int_0^{\frac{\pi}{4}}\sin^2\theta(1-2\sin^2\theta)d\theta\\
&= 2\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\cos 2\theta d\theta=\int_0^{\frac{\pi}{4}}(2\cos 2\theta-1-\cos 4\theta)d\theta=\left[\sin 2\theta-\theta-\frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}}=1-\frac{\pi}{4}
\end{aligned}$$

(2) 部分積分を利用する.

$$\begin{aligned}
&a_n=\int_0^{\frac{\pi}{4}}\sin^n\theta d\theta=\int_0^{\frac{\pi}{4}}(-\cos\theta)'\sin^{n-1}\theta d\theta\\
&=\Big[-\cos\theta\sin^{n-1}\theta\Big]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\cos\theta(n-1)\sin^{n-2}\theta\cos\theta d\theta=-\left(\frac{1}{\sqrt{2}}\right)^n+(n-1)\int_0^{\frac{\pi}{4}}\sin^{n-2}\theta\cos^2\theta d\theta\\
&=-\left(\frac{1}{\sqrt{2}}\right)^n+(n-1)\int_0^{\frac{\pi}{4}}\sin^{n-2}\theta(1-\sin^2\theta)d\theta=-\left(\frac{1}{\sqrt{2}}\right)^n+(n-1)(a_{n-2}-a_n)
\end{aligned}$$

したがって, I_n について解き,

$$a_n=\frac{n-1}{n}a_{n-2}-\frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^n$$

が得られる. ただし, $a_1=\displaystyle\int_0^{\frac{\pi}{4}}\sin\theta d\theta=1-\frac{1}{\sqrt{2}},\ a_0=\int_0^{\frac{\pi}{4}}d\theta=\frac{\pi}{4}$.

(3) (2) を利用する.

$$a_2=\frac{1}{2}a_0-\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2=\frac{\pi}{8}-\frac{1}{4},\quad a_4=\frac{3}{4}a_2-\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2=\frac{3}{4}\left(\frac{\pi}{8}-\frac{1}{4}\right)-\frac{1}{16}=\frac{3}{32}\pi-\frac{1}{4}$$

したがって,

$$\begin{aligned}
&I=4\int_0^{\frac{\pi}{4}}\sin^2\theta(1-2\sin^2\theta)d\theta=4a_2-8a_4=4\left(\frac{\pi}{8}-\frac{1}{4}\right)-8\left(\frac{3}{32}\pi-\frac{1}{4}\right)=1-\frac{\pi}{4}\\
&J=4\int_0^{\frac{\pi}{4}}\sin^2\theta d\theta=4a_2=4\left(\frac{\pi}{8}-\frac{1}{4}\right)=\frac{\pi}{2}-1
\end{aligned}$$