Report assignments and notes on the lectures "Group actions and integrations" in Mathematical Omnibus 2021

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1 Report assignments

Report assignment 1 (60/100): Let $\mu : \mathbb{R}^2 \to \mathbb{R}$ be a map satisfying the following five conditions:

Condition 1: $\mu(\lambda v) = \lambda \mu(v)$. for any $\lambda \in \mathbb{R}$ and any $v \in \mathbb{R}^2$.

Condition 2: $\mu(v + w) = \mu(v) + \mu(w)$ for any $v, w \in \mathbb{R}^2$.

Condition 3: $\mu(v) \geq 0$ if $v_1, v_2 \geq 0$ for $v = (v_1, v_2) \in \mathbb{R}^2$.

Condition 4: $\mu(v) = \mu(\sigma v)$ for any $v = (v_1, v_2) \in \mathbb{R}^2$, where we put $\sigma v = (v_2, v_1) \in \mathbb{R}^2$.

Condition 5: $\mu((1, 0)) = 1$.

Show that $\mu(v) = v_1 + v_2$ for any $v = (v_1, v_2) \in \mathbb{R}^2$.

Report assignment 2 (40/100) : For some claims appeared in our lectures, detailed arguments are omitted. Complete two of them (20+20).

2 Introduction

"Integration" is one of the most important concepts in Mathematics. In our lectures, we give a characterization of the integration on $\mathbb{R}$ as an invariant strictly-positive linear functional on the space of all continuous functions on $\mathbb{R}$ with compact supports.

For a continuous function f on $\mathbb{R}$, the support of f is defined by the closure of

$$\{x \in \mathbb{R} \mid f(x) \neq 0\}$$

in $\mathbb{R}$. Note that the support of f is compact if and only if there exists $a, b \in \mathbb{R}$ with $a < b$ such that

$$\{x \in \mathbb{R} \mid f(x) \neq 0\} \subset [a, b].$$

The vector space of continuous functions on $\mathbb{R}$ with compact support is denoted by $C_c(\mathbb{R})$.

Definition 2.1. *Let $\mu : C_c(\mathbb{R}) \to \mathbb{R}$ be a map:*

- *μ is called a linear functional if*

$$\mu(\lambda_1 f + \lambda_2 h) = \lambda_1 \mu(f) + \lambda_2 \mu(h)$$

  for any $f, h \in C_c(\mathbb{R})$ and any $\lambda_1, \lambda_2 \in \mathbb{R}$.

- *μ is called strictly-posiive if*

$$\mu(f) > 0$$

  for any $f \in C_c(\mathbb{R}) \setminus \{0\}$ with $f \geq 0$, where $f \geq 0$ means that $f(x) \geq 0$ for any $x \in \mathbb{R}$.

- *μ is called invariant if*

$$\mu(f) = \mu(\sigma_s f)$$

  for any $f \in C_c(\mathbb{R})$ and any $s \in \mathbb{R}$, where $\sigma_s f \in C_c(\mathbb{R})$ is defined by

$$\sigma_s f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x - s).$$

Main theorem of the lectures is the following:

Theorem 2.2 (Main theorem)**.** *Let $\mu_1, \mu_2 : C_c(\mathbb{R}) \to \mathbb{R}$ be both invariant strictly-positive linear functionals. Then there exists $c \in \mathbb{R}_{>0}$ such that $c \cdot \mu_1 = \mu_2$.*

For a continuous function f with compact support, we have some definitions of the integral "$\int_{\mathbb{R}} f(x)dx$" of f on $\mathbb{R}$:

By differential equations:

$$\int_{\mathbb{R}} f(x)dx := F(b) - F(a)$$

for a function F with $F' = f$ and $a, b \in \mathbb{R}$ with

$$\{x \in \mathbb{R} \mid f(x) \neq 0\} \subset [a, b].$$

As the Riemann integral:

$$\int_{\mathbb{R}} f(x)dx := \lim_{\triangle \to 0} \sum_k f(a_k)|x_{k+1} - x_k|$$

As the Lebesgue integral:

$$\int_{\mathbb{R}} f(x)dx := \int_{\mathbb{R}} f_+(x)dx - \int_{\mathbb{R}} f_-(x)dx$$

where

$$\int_{\mathbb{R}} f_\pm(x)dx = \sup_{s \text{ is a simple function with } 0 \le s \le f_\pm} \int_{\mathbb{R}} s d\mu_{\mathbb{R}}.$$

Then for each definition, one can check that

$$C_c(\mathbb{R}) \to \mathbb{R},\ f \mapsto \int_{\mathbb{R}} f(x)dx$$

is invariant strictly-positive linear functional, and for a function

$$f_0 : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 0 \quad (\text{if } x \le -1) \\ x + 1 \quad (\text{if } -1 < x \le 0) \\ -x + 1 \quad (\text{if } 0 < x \le 1) \\ 0 \quad (\text{if } 1 < x), \end{cases}$$

we have

$$\int_{\mathbb{R}} f_0(x)dx = 1.$$

Then by the main theorem, the definitions of integrals above are equivalent. Furthermore, there uniquely exists an invariant strictly-positive linear functional μ on $C_c(\mathbb{R})$ with $\mu(f_0) = 1$, and

$$\int_{\mathbb{R}} f(x)dx := \mu(f)$$

can be considered as a new definition of the integral of f on $\mathbb{R}$.

The following generalization of the main theorem is well-known and applied for the theory of Fourier analysis on locally-compact Hausdorff groups and their homogeneous spaces (cf. [2]).

Theorem 2.3 (A generalization (see [1] for the details))**.** *Let G be a locally-compact Hausdorff group. Then the following holds:*

1. *There exists a left-invariant strictly-positive linear functional μ on $C_c(G)$.*

2. *Such μ are unique up to positive scalar multiplications.*

3 Terminologies for functions with compact support

3.1 The vector space of continuous functions on $\mathbb{R}$

Definition 3.1. $C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f$ *is continuous on* $\mathbb{R}\}$.

Definition 3.2.

1. *We simply write $0 \in C(\mathbb{R})$ for the zero function on $\mathbb{R}$.*

2. *For each $f_1, f_2 \in C(\mathbb{R})$, we define the summation $f_1 + f_2 \in C(\mathbb{R})$ of f_1 and f_2 by*
$$f_1 + f_2 : \mathbb{R} \to \mathbb{R},\ x \mapsto f_1(x) + f_2(x).$$

3. *For each $f \in C(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we define the scalar multiplication $\lambda f \in C(\mathbb{R})$ of f and λ by*
$$\lambda f : \mathbb{R} \to \mathbb{R},\ x \mapsto \lambda f(x).$$

4. *For each $f_1, f_2 \in C(\mathbb{R})$, we define the product $f_1 \cdot f_2 \in C(\mathbb{R})$ of f_1 and f_2 by*
$$f_1 \cdot f_2 : \mathbb{R} \to \mathbb{R},\ x \mapsto f_1(x) \cdot f_2(x).$$

Theorem 3.3. *$C(\mathbb{R})$ is a vector space with respect to the zero, summations and scalar multiplications defined in Definition 3.2. Furthermore, $C(\mathbb{R})$ is a commutative and associative $\mathbb{R}$-algebra with respect to the product*

$$C(\mathbb{R}) \times C(\mathbb{R}) \to C(\mathbb{R}),\ (f_1, f_2) \mapsto f_1 \cdot f_2.$$

Remark 3.4. *$C(\mathbb{R})$ is not finite-dimensional as a vector space.*

3.2 The vector space of functions with compact support

Definition 3.5. *For each $f \in C(\mathbb{R})$, we denote by $\operatorname{supp} f$, and called the support of f in $\mathbb{R}$, the closure of $\{x \in \mathbb{R} \mid f(x) \neq 0\}$ in $\mathbb{R}$.*

Proposition 3.6. *For each $f \in C(\mathbb{R})$, the following two conditions on f are equivalent:*

(i) *The support of f in $\mathbb{R}$ is compact.*

(ii) *There exists $a, b \in \mathbb{R}$ with $a < b$ such that*

$$\{x \in \mathbb{R} \mid f(x) \neq 0\} \subset [a, b].$$

Proposition 3.7. *For any non-zero polynomial function on $\mathbb{R}$, the support of it in $\mathbb{R}$ is not compact.*

Example 3.8. *Let us define*

$$h : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 0 & (\textit{if } x \leq -1) \\ x + 1 & (\textit{if } -1 < x \leq 0) \\ -x + 1 & (\textit{if } 0 < x \leq 1) \\ 0 & (\textit{if } 1 < x) \end{cases}$$

Then $h \in C(\mathbb{R})$ and $\operatorname{supp} h = [-1, 1]$. In particular $\operatorname{supp} h$ is compact.

Definition 3.9.

$$C_c(\mathbb{R}) := \{f \in C(\mathbb{R}) \mid \operatorname{supp} f \textit{ is compact}\}.$$

Theorem 3.10. *$C_c(\mathbb{R})$ is a linear subspace of $C(\mathbb{R})$. That is, $C_c(\mathbb{R})$ itself is a vector space with respect to the zero, summations and scalar multiplications defined in Definition 3.2. Furthermore, $C_c(\mathbb{R})$ is an ideal of the commutative and associative $\mathbb{R}$-algebra $C(\mathbb{R})$, that is, $f \cdot h \in C_c(\mathbb{R})$ for any $f \in C(\mathbb{R})$ and any $h \in C_c(\mathbb{R})$.*

3.3 Positive functions with compact support

Definition 3.11. *For $f \in C_c(\mathbb{R})$, we say that f is positive if $f(x) \in \mathbb{R}_{\geq 0}$ for any $x \in \mathbb{R}$.*

Definition 3.12.

$$C_c^+(\mathbb{R}) := \{f \in C_c(\mathbb{R}) \mid f \text{ is positive } \}.$$

Proposition 3.13.
1. *For any $f_1, f_2 \in C_c^+(\mathbb{R})$, $f_1 + f_2 \in C_c^+(\mathbb{R})$ and $f_1 \cdot f_2 \in C_c^+(\mathbb{R})$*
2. *For any $f \in C_c^+(\mathbb{R})$ and any $\lambda \in \mathbb{R}_{\geq 0}$, $\lambda f \in C_c^+(\mathbb{R})$.*

In particular, $C_c^+(\mathbb{R})$ is a convex cone in the vector space $C_c(\mathbb{R})$ and closed under the product.

3.4 The $\mathbb{R}$-action on functions with compact support

Definition 3.14. *For each $f \in C_c(\mathbb{R})$ and each $s \in \mathbb{R}$, we define*

$$\sigma_s f : \mathbb{R} \to \mathbb{R}, \ x \mapsto f(x - s).$$

Proposition 3.15. *For any $f \in C_c(\mathbb{R})$ and any $s \in \mathbb{R}$, $\sigma_s f \in C_c(\mathbb{R})$. Furthermore if $f \in C_c^+(\mathbb{R})$, then $\sigma_s f \in C_c^+(\mathbb{R})$.*

Definition 3.16. *For each $s \in \mathbb{R}$, we define the map*

$$\sigma_s : C_c(\mathbb{R}) \to C_c(\mathbb{R}), \ f \mapsto \sigma_s f.$$

Proposition 3.17. *The map*

$$\mathbb{R} \times C_c(\mathbb{R}) \to C_c(\mathbb{R}), \ (s, f) \mapsto \sigma_s f$$

defines a linear representation of the additive group $\mathbb{R}$ on the vector space $C_c(\mathbb{R})$. That is, the following holds:

1. *For any $s \in \mathbb{R}$, the map*

   $$\sigma_s : C_c(\mathbb{R}) \to C_c(\mathbb{R}), \ f \mapsto \sigma_s f.$$

   is linear.

2. *For any $s_1, s_2 \in \mathbb{R}$, the equality*

   $$\sigma_{s_1+s_2} = \sigma_{s_1} \circ \sigma_{s_2}.$$

   as linear endomorphisms on $C_c(\mathbb{R})$ holds.

4 Terminologies for linear functionals on the vector space of functions with compact support

4.1 Positive linear functionals on the vector space of functions with compact support

Definition 4.1. *A map $\mu : C_c(\mathbb{R}) \to \mathbb{R}$ is said to be a linear functional on $C_c(\mathbb{R})$ if the map is linear.*

Definition 4.2. *A linear functional μ on $C_c(\mathbb{R})$ is called positive if $\mu(f) \in \mathbb{R}_{\geq 0}$ for any $f \in C_c^+(\mathbb{R})$. Furthermore, if $\mu(f) > 0$ for any $f \in C_c^+(\mathbb{R}) \setminus \{0\}$, we say that μ is strictly-positive.*

Proposition 4.3. *Let μ be a positive linear functional on $C_c(\mathbb{R})$. Then the following holds:*

1. *Let $f_1, f_2 \in C_c(\mathbb{R})$. Assume that*
$$f_1(x) \leq f_2(x) \quad \text{for any } x \in \mathbb{R}.$$
Then $\mu(f_1) \leq \mu(f_2)$.

2. *Let $f_1, f_2 \in C_c(\mathbb{R})$ and $f_3 \in C_c^+(\mathbb{R})$. Assume that*
$$|f_1(x) - f_2(x)| \leq f_3(x) \quad \text{for any } x \in \mathbb{R}.$$
Then for any positive linear functional μ on $C_c(\mathbb{R})$, the inequality below holds:
$$|\mu(f_1) - \mu(f_2)| \leq \mu(f_3).$$

Theorem 4.4 (Continuity of positive linear functionals)**.** *Let μ be a positive linear functional on $C_c(\mathbb{R})$. Fix any compact set K in $\mathbb{R}$ and take any uniformaly convergent sequence of functions $\{f_i\}_{i=1,2,\dots}$ such that $f_i \in C_c(\mathbb{R})$ with $\operatorname{supp} f_i \subset K$ for each i. Then*
$$\lim_{i\to\infty} f_i \in C_c(\mathbb{R})$$
and its support is included in K. Furthermore, the equality below holds:
$$\lim_{i\to\infty} \mu(f_i) = \mu(\lim_{i\to\infty} f_i).$$

4.2 Invariant linear functionals

Definition 4.5. *A linear functional* μ *on* $C_c(\mathbb{R})$ *is called invariant if* $\mu(\sigma_s f) = \mu(f)$ *for any* $f \in C_c(\mathbb{R})$ *and any* $s \in \mathbb{R}$.

4.3 Main theorem

Proposition 4.6. *Let* $c \in \mathbb{R}_{>0}$ *and* μ *an invariant strictly-positive linear functional on* $C_c(\mathbb{R})$. *Then*

$$c \cdot \mu : C_c(\mathbb{R}) \to \mathbb{R}, \ f \mapsto c \cdot \mu(f)$$

is also an invariant strictly-positive linear functional on $C_c(\mathbb{R})$.

The main theorem of the lectures is the following:

Theorem 4.7 (Main theorem)**.** *Let* $\mu_1, \mu_2 : C_c(\mathbb{R}) \to \mathbb{R}$ *be both invariant strictly-positive linear functionals. Then there exists* $c \in \mathbb{R}_{>0}$ *such that* $c \cdot \mu_1 = \mu_2$.

5 Ratios of pairs of positive functions

Let $f, h \in C_c^+(\mathbb{R})$ with $h \neq 0$. In this section, we define "the ratio" $(f : h) \in \mathbb{R}_{\geq 0}$ of f and h. Furthermore we prove that for any invariant strictly-positive linear functional μ on $C_c(\mathbb{R})$, the inequality

$$\mu(f)/\mu(h) \leq (f : h)$$

holds.

5.1 Definition of ratios of pairs of positive functions

Let $f, h \in C_c^+(\mathbb{R})$ with $h \neq 0$. In this subsection, we define the ratio

$$(f : h) \in \mathbb{R}_{\geq 0}$$

of f and h (see Definition 5.4).

Definition 5.1. *We define*

$$\Omega(f;h) := \bigcup_{N=1}^{\infty} \{((c_1, \ldots, c_N), (s_1, \ldots, s_N)) \mid$$
$$c_1, \ldots c_N \in \mathbb{R}_{\geq 0} \text{ and } s_1, \ldots, s_N \in \mathbb{R}$$
$$\text{with } f(x) \leq \sum_{i=1}^{N} c_i \cdot (\sigma_{s_i} h)(x) \text{ for any } x \in \mathbb{R}\}.$$

Example 5.2. *Let us define $f, h \in C_c^+(\mathbb{R})$ as follows:*

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 0 \quad (\text{if } x \leq -2), \\ \frac{1}{2}x + 1 \quad (\text{if } -2 < x \leq 0), \\ -\frac{1}{2}x + 1 \quad (\text{if } 0 < x \leq 2), \\ 0 \quad (\text{if } 2 < x), \end{cases}$$

$$h : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 0 \quad (\text{if } x \leq -1), \\ x + 1 \quad (\text{if } -1 < x \leq 0), \\ -x + 1 \quad (\text{if } 0 < x \leq 1), \\ 0 \quad (\text{if } 1 < x). \end{cases}$$

Take $N = 3$, $(c_1, c_2, c_3) = (1, 1, 1)$ and $(s_1, s_2, s_3) = (-1, 0, 1)$. Then one can see that, for any $x \in \mathbb{R}$,

$$\sum_{i=1}^{N} c_i \cdot (\sigma_{s_i} h)(x) = \begin{cases} 0 \quad (\text{if } x \leq -2), \\ x + 2 \quad (\text{if } -2 < x \leq -1), \\ 1 \quad (\text{if } -1 < x \leq 1), \\ -x + 2 \quad (\text{if } 1 < x \leq 2), \\ 0 \quad (\text{if } 2 < x). \end{cases}$$

In particular, we have

$$f(x) \leq \sum_{i=1}^{N} c_i \cdot (\sigma_{s_i} h)(x)$$

for any $x \in \mathbb{R}$, and hence

$$((c_1, c_2, c_3), (s_1, s_2, s_3)) = ((1, 1, 1), (-1, 0, 1)) \in \Omega(f; h).$$

Proposition 5.3. *$\Omega(f;h) \neq \emptyset$.*

Proof of Proposition 5.3. The details are omitted. $\square$

Definition 5.4. *We define the ratio*

$$(f : h) \in \mathbb{R}_{\geq 0}$$

by

$$(f : h) := \inf \left\{ \sum_{i=1}^{N} c_i \;\middle|\; ((c_1, \ldots, c_N), (s_1, \ldots, s_N)) \in \Omega(f;h) \right\} \in \mathbb{R}_{\geq 0}.$$

Proposition 5.5. *$(f : h) = 0$ if and only if $f = 0$.*

Proof of Proposition 5.5. The details are omitted. $\square$

5.2 Ratios and invariant positive linear functionals

Theorem 5.6. *Let μ be an invariant positive linear functional on $C_c(\mathbb{R})$. Then for any $f, h \in C_c^+(\mathbb{R})$ with $h \neq 0$, the following inequality holds:*

$$\mu(f) \leq \mu(h) \cdot (f : h).$$

Proof of Theorem 5.6. Take any

$$((c_1, \ldots, c_N), (s_1, \ldots, s_N)) \in \Omega(f;h).$$

We only need to show that

$$\mu(f) \leq \mu(h) \cdot \left(\sum_{i=1}^{N} c_i \right).$$

Since

$$f(x) \leq \sum_{i=1}^{N} c_i \cdot (\sigma_{s_i} h)(x) \quad (\text{ for any } x \in \mathbb{R}),$$

we have

$$\begin{aligned}\mu(f) &\leq \mu(\sum_{i=1}^{N} c_i \cdot (\sigma_{s_i} h)) \quad (\because \text{ Proposition 4.3}) \\ &= \sum_{i=1}^{N} c_i \cdot \mu(\sigma_{s_i} h) \\ &= \sum_{i=1}^{N} c_i \cdot \mu(h) \quad (\because \mu \text{ is invariant}) \\ &= \mu(h) \cdot \left(\sum_{i=1}^{N} c_i \right).\end{aligned}$$

This completes the proof. □

Corollary 5.7. *Let μ be an invariant positive linear functional on $C_c(\mathbb{R})$. Then the following two conditions on μ are equivalent:*

1. *μ is strictly-positive, i.e. $\mu(f) > 0$ for any $f \in C_c^+(\mathbb{R}) \setminus \{0\}$.*
2. *There exists $f \in C_c^+(\mathbb{R}) \setminus \{0\}$ such that $\mu(f) > 0$.*

6 The approximation theorem

In this section, we state the approximation theorem which will plays key roles in our proof of the main theorem 4.7 in Section 7.

Definition 6.1. *For each $h \in C_c^+(\mathbb{R})$, we define* $\mathrm{width}(h) \in \mathbb{R}_{\geq 0}$ *by*

$\mathrm{width}(h) := \min\{r \in \mathbb{R}_{\geq 0} \mid$ *there exists* $a \in \mathbb{R}$ *such that* $\operatorname{supp} h \subset [a, a+r]\}$.

Proposition 6.2. *For any $h \in C_c^+(\mathbb{R})$ and any $s \in \mathbb{R}$,*

$$\mathrm{width}(\sigma_s h) = \mathrm{width}(h).$$

Proposition 6.3. *For any $\delta \in \mathbb{R}_{>0}$, there exists $h \in C_c^+(\mathbb{R})$ with* $\mathrm{width}(h) \leq \delta$.

Definition 6.4. *For each* $f, h \in C_c^+(\mathbb{R})$ *and* $\varepsilon \in \mathbb{R}_{>0}$*, we put*

$$\begin{aligned}\mathcal{A}_\varepsilon(f;h) := \bigcup_{N=1}^{\infty} \{((c_1,\dots,c_N),(s_1,\dots,s_N)) \mid \\ c_1,\dots,c_N \in \mathbb{R}_{\geq 0} \text{ and } s_1,\dots,s_N \in \mathbb{R} \text{ satisfying that} \\ |f(x) - \sum_{i=1}^{N} c_i(\sigma_{s_i}h)(x)| \leq \varepsilon \text{ for any } x \in \mathbb{R} \\ \text{and } \operatorname{supp} f \cap \operatorname{supp}(\sigma_{s_i}h) \neq \emptyset \text{ for any } i = 1,\dots,N\}.\end{aligned}$$

The following theorem will play important roles in our proof of Theorem 4.7 in Section 7.2:

Theorem 6.5 (The approximation theorem)**.** *Fix* $f \in C_c^+(\mathbb{R})$ *and* $\varepsilon \in \mathbb{R}_{>0}$*. Then there exists* $\delta \in \mathbb{R}_{>0}$ *satisfying the following condition:*

Condition: $\mathcal{A}_\varepsilon(f;h) \neq \emptyset$ *for any* $h \in C_c^+(\mathbb{R}) \setminus \{0\}$ *with* $\operatorname{width}(h) \leq \delta$*.*

The proof of Theorem 6.5 is postponed to Section 8.

Corollary 6.6. *Let us fix* $f, f_0 \in C_c^+(\mathbb{R})$ *and* $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$*. Then there exists* $h \in C_c^+(\mathbb{R})$ *with* $\operatorname{width}(h) \leq 1$*,* $\mathcal{A}_{\varepsilon_1}(f;h) \neq \emptyset$ *and* $\mathcal{A}_{\varepsilon_2}(f_0;h) \neq \emptyset$*.*

Proof of Corollary 6.6. Hint: Proposition 6.3 and Theorem 6.5. □

7 Outline of our proof of the main theorem

In this section, we give a proof of the main theorem 4.7 by applying the approximation theorem.

7.1 Restrictions of linear functionals on the convex cone of positive functions

The theorem below claims that each positive linear functional on $C_c(\mathbb{R})$ is characterized by its restriction on $C_c^+(\mathbb{R})$:

Theorem 7.1. *Let* μ_1, μ_2 *be both linear functionals on* $C_c(\mathbb{R})$*. Then the following two conditions on* (μ_1, μ_2) *are equivalent:*

(i): *$\mu_1 = \mu_2$, i.e. $\mu_1(f) = \mu_2(f)$ for any $f \in C_c(\mathbb{R})$.*

(ii): *$\mu_1(f) = \mu_2(f)$ for any $f \in C_c^+(\mathbb{R})$.*

Theorem 7.1 follows from the lemma below:

Lemma 7.2. *For any $f \in C_c(\mathbb{R})$, there exists $f_+, f_- \in C_c^+(\mathbb{R})$ such that $f = f_+ - f_-$.*

Proof of Lemma 7.2. Hint: One can put

$$f_+ : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} f(x) & (\text{ if } f(x) > 0), \\ 0 & (\text{otherwise}), \end{cases}$$

$$f_- : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} -f(x) & (\text{ if } f(x) < 0), \\ 0 & (\text{otherwise}), \end{cases}$$

□

7.2 Proof of the main theorem

Throughout this susbection, we fix $f_0 \in C_c^+(\mathbb{R}) \setminus \{0\}$.

In order to prove the main theorem 4.7, because of Theorem 7.1, we only need to show the following proposition.

Proposition 7.3. *Let μ_1, μ_2 be both invariant strictly-positive linear functionals on $C_c(\mathbb{R})$ with $\mu_1(f_0) = \mu_2(f_0) = 1$. Then $\mu_1(f) = \mu_2(f)$ for any $f \in C_c^+(\mathbb{R})$.*

Proposition 7.3 follows directly from the lemma below:

Lemma 7.4. *Let us fix $f \in C_c^+(\mathbb{R})$ and $\varepsilon \in \mathbb{R}_{>0}$. Then there exists $r = r_{f,\varepsilon} \in \mathbb{R}_{\geq 0}$ such that the inequality below holds*

$$|\mu(f) - r| \leq \varepsilon$$

for any invariant strictly-positive linear functional μ on $C_c(\mathbb{R})$ with $\mu(f_0) = 1$.

We give a proof of Lemma 7.4 by applying the approximation theorem (Corollary 6.6) below:

Remark 7.5. *Idea of the proof of Lemma 7.4: By Corollary 6.6, we can find*

- $h \in C_c^+(\mathbb{R}) \setminus \{0\}$,
- $c_1, \dots, c_N \in \mathbb{R}_{\geq 0}$,
- $s_1, \dots, s_N \in \mathbb{R}$,
- $d_1, \dots, d_{N'} \in \mathbb{R}_{\geq 0}$ *and*
- $t_1, \dots, t_{N'} \in \mathbb{R}$

such that

$$f \risingdotseq \sum_{i=1}^{N} c_i(\sigma_{s_i} h),$$
$$f_0 \risingdotseq \sum_{j=1}^{N'} d_j(\sigma_{t_j} h).$$

We put $r := (\sum_i c_i)/(\sum_j d_j)$. *Then for any invariant strictly-positive linear functional* μ *on* $C_c(\mathbb{R})$ *with* $\mu(f) = 1$, *we have*

$$\mu(f) \risingdotseq \sum_{i=1}^{N} c_i \mu(\sigma_{s_i} h) = \mu(h) \cdot \sum_{i=1}^{N} c_i,$$
$$1 = \mu(f_0) \risingdotseq \sum_{j=1}^{N'} d_j \mu(\sigma_{t_j} h) = \mu(h) \cdot \sum_{i=1}^{N'} d_i,$$

and hence

$$\mu(f) \risingdotseq (\sum_i c_i)/(\sum_j d_j) = r.$$

Proof of Lemma 7.4. Let us fix $f \in C_c^+(\mathbb{R})$ and $\varepsilon \in \mathbb{R}_{>0}$. If $f = 0$, then one can take $r = 0$. Let us consider the cases where $f \neq 0$. We fix $a, b \in \mathbb{R}$ with $a < b$ and

$$\operatorname{supp} f \cup \operatorname{supp} f_0 \subset [a, b].$$

Let us also fix $\psi \in C_c^+(\mathbb{R}) \setminus \{0\}$ satisfying the following conditions:

- $0 \leq \psi(x) \leq 1$ for any $x \in \mathbb{R}$.

- $\psi(x) = 1$ if $x \in [a-1, b+1]$.

Note that such $\psi \in C_c^+(\mathbb{R}) \setminus \{0\}$ exists and $(\psi : f_0) > 0$ by Proposition 5.5.

We define $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ by

$$\varepsilon_1 := \varepsilon/2 > 0$$
$$\varepsilon_2 := \min\left\{\frac{1}{2},\ \frac{\varepsilon}{3(2(f : f_0) + \varepsilon)},\ \frac{(\psi : f_0)}{2}(\max_{x\in\mathbb{R}} f(x))\right\} > 0.$$

Note that one can easily check that the following ineqalities holds:

$$\varepsilon_2 \leq 1/2, \tag{1}$$
$$\varepsilon_1 + \varepsilon_2 \frac{(f : f_0) + \varepsilon_1}{1 - \varepsilon_2} < \varepsilon. \tag{2}$$

Furthermore, let us also put

$$\varepsilon_1' := \varepsilon_1/(\psi : f_0) > 0,$$
$$\varepsilon_2' := \varepsilon_2/(\psi : f_0) > 0.$$

We also note that

$$|f(x)| > \varepsilon_2' \psi(x) \quad \text{for some } x \in \mathbb{R}. \tag{3}$$

By Corollary 6.6, one can find and fix $h \in C_c^+(\mathbb{R})$ with $\mathrm{width}(h) \leq 1$, $\mathcal{A}_{\varepsilon_1'}(f;h) \neq \emptyset$ and $\mathcal{A}_{\varepsilon_2'}(f_0;h) \neq \emptyset$. We also fix $((c_1, \ldots, c_N), (s_1, \ldots, s_N)) \in \mathcal{A}_{\varepsilon_1'}(f;h)$ and $((d_1, \ldots, d_{N'}), (t_1, \ldots, t_{N'})) \in \mathcal{A}_{\varepsilon_2'}(f_0;h)$. By the definitions of ψ, $\mathrm{width}(h)$, $\mathcal{A}_{\varepsilon_1'}(f;h)$ and $\mathcal{A}_{\varepsilon_2'}(f_0;h)$, we see that both inequalities below holds for any $x \in \mathbb{R}$:

$$|f(x) - \sum_{i=1}^{N} c_i(\sigma_{s_i} h)(x)| \leq \varepsilon_1' \psi(x), \tag{4}$$
$$|f_0(x) - \sum_{j=1}^{N'} d_j(\sigma_{t_j} h)(x)| \leq \varepsilon_2' \psi(x). \tag{5}$$

Put

$$c := \sum_{i=1}^{N} c_i,\ d := \sum_{j=1}^{N'} d_j \in \mathbb{R}_{\geq 0}.$$

Then $d = \sum_j d_j > 0$ by (3). We put

$$r := c/d \in \mathbb{R}_{\geq 0}.$$

Take any invariant strictly-positive linear functional μ on $C_c^+(\mathbb{R})$ with $\mu(f_0) = 1$. We shall prove that

$$|\mu(f) - r| \leq \varepsilon.$$

By Proposition 4.3, Theorem 5.6, $\mu(f_0) = 1$, (4) and (5), we have

$$\begin{aligned}
|\mu(f) - c \cdot \mu(h)| &\leq \varepsilon_1' \cdot \mu(\psi) \leq \varepsilon_1' \cdot (\psi : f_0) = \varepsilon_1, \\
|1 - d \cdot \mu(h)| &\leq \varepsilon_2' \cdot \mu(\psi) \leq \varepsilon_2' \cdot (\psi : f_0) = \varepsilon_2.
\end{aligned}$$

In particular, we also have

$$\begin{aligned}
&c \cdot \mu(h) \leq \mu(f) + \varepsilon_1 \leq (f : f_0) + \varepsilon_1, \\
&d \cdot \mu(h) \geq 1 - \varepsilon_2 > 0,
\end{aligned}$$

and hence

$$r \leq \frac{(f : f_0) + \varepsilon_1}{1 - \varepsilon_2}.$$

Therefore we obtain

$$\begin{aligned}
|\mu(f) - r| &= |\mu(f) - c\mu(h) + c\mu(h) - r| \\
&\leq |\mu(f) - c\mu(h)| + r|d\mu(h) - 1| \\
&\leq \varepsilon_1 + r\varepsilon_2 \\
&\leq \varepsilon_1 + \varepsilon_2 \frac{(f : f_0) + \varepsilon_1}{1 - \varepsilon_2} \\
&\leq \varepsilon \quad (\because (2)).
\end{aligned}$$

□

8 Proof of the approximation theorem

In this section, we give a proof of Theorem 6.5.

8.1 Partitions of unity for finite open covers

Proposition 8.1 (Urysohn's lemma on $\mathbb{R}$)**.** *Let C be a compact subset of $\mathbb{R}$ and U an open subset of $\mathbb{R}$ with $C \subset U$. Then there exists $\psi \in C_c^+(\mathbb{R})$ satisfying the following conditions:*

1. $\operatorname{supp} \psi \subset U$.
2. $0 \leq \psi(x) \leq 1$ *for any* $x \in \mathbb{R}$.
3. $\psi(x) = 1$ *for any* $x \in C$.

Theorem 8.2. *Let K be a compact subset of $\mathbb{R}$ and $U_1, \ldots, U_N$ a finite open cover on K in $\mathbb{R}$. Then there exist $\phi_1, \ldots, \phi_N \in C_c^+(\mathbb{R})$ satisfying the following conditions:*

1. $\operatorname{supp} \phi_i \subset U_i$ *for each* $i = 1, \ldots, N$.
2. $0 \leq \phi_i(x) \leq 1$ *for each* $i = 1, \ldots, N$ *and each* $x \in \mathbb{R}$.
3. $\sum_{i=1}^{N} \phi_i(x) = 1$ *for each* $x \in K$.

Remark 8.3. *For Proposition 8.1 and Theorem 8.2, the similar statements hold on any locally-compact Hausdorff topological space. Note that it is not needed to assume that the space is second countable.*

Proof of Theorem 8.2. For each point $x \in K$, we fix $i_x \in \{1, \ldots, N\}$ with $x \in U_{i_x}$ and a compact neighberhood C_x of x included in U_{i_x}. Since K is compact, one can find finite subset $\{x_1, \ldots, x_r\}$ with

$$K \subset \bigcup_{j=1}^{r} C_{x_j}.$$

For each $i = 1, \ldots, N$, we put

$$C_i := \bigcup_{i_{x_j}=i} C_{x_j} \subset U_i.$$

Then each C_i is compact and

$$K \subset \bigcup_{i=1}^{N} C_i.$$

By Proposition 8.1, for each i, we can choose $\psi_i \in C_c^+(\mathbb{R})$ satisfying the following conditions:

1. $\operatorname{supp} \psi_i \subset U_i$.
2. $0 \le \psi_i(x) \le 1$ for any $x \in \mathbb{R}$.
3. $\psi_i(x) = 1$ for any $x \in C_i$.

Note that $(1-\psi_i)(x) \ge 0$ for any i and any $x \in \mathbb{R}$. Let us define $\phi_i \in C_c^+(\mathbb{R})$ $(i = 1, \dots, N)$ as follows:

- $\phi_1 := \psi_1$.
- $\phi_i := \psi_k \prod_{l=1}^{i-1}(1-\psi_l)$ $(i \ge 2)$.

Note that

$$\sum_{i=1}^{N} \psi_i = 1 - \prod_{l=1}^{N}(1-\psi_l).$$

Then one can easily check that the three conditions in the statement of Theorem 8.2. □

8.2 Uniformly continuity of functions with compact support

Definition 8.4. *A function $f : \mathbb{R} \to \mathbb{R}$ is called* uniformly continuous *if for any $\varepsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that the inequality*

$$|f(x) - f(y)| \le \varepsilon$$

holds for any $x, y \in \mathbb{R}$ with $|x - y| \le \delta$.

Proposition 8.5. *Any uniformly continuous function is continuous.*

Theorem 8.6. *Any continuous functions with compact support is uniformly continuous.*

Proof of Theorem 8.6. Hint: "sequentially compactness". □

8.3 Key lemma for Theorem 6.5

In this subsection, we give a proof of the key lemma for Theorem 6.5 below:

Lemma 8.7. *Let us fix a function $h \in C_c^+(\mathbb{R}) \setminus \{0\}$, a compact subset K of $\mathbb{R}$ and $\varepsilon \in \mathbb{R}_{>0}$. Then there exist $N \in \mathbb{Z}_{\geq 0}$, $s_1, \dots, s_N \in K$ and $\phi_1, \dots, \phi_N \in C_c^+(\mathbb{R})$ such that the inequality*

$$\left| (\sigma_s h)(x) - \sum_{i=1}^{N} \phi_i(s)(\sigma_{s_i} h)(x) \right| \leq \varepsilon.$$

holds for any $s \in K$ and any $x \in \mathbb{R}$.

Proof of Lemma 8.7. By Theorem 8.6, the function h is uniformly continuous. Therefore, one can find and fix $\delta \in \mathbb{R}_{>0}$ such that for any $s, t \in \mathbb{R}$ with $|s - t| \leq \delta$, the inequality below holds:

$$|(\sigma_s h)(x) - (\sigma_t h)(x)| \leq \varepsilon \text{ for any } x \in \mathbb{R}. \tag{6}$$

For each $t \in \mathbb{R}$, we define the open neighberhood U_t^δ of t in $\mathbb{R}$ by

$$U_t^\delta := \{s \in \mathbb{R} \mid |s - t| < \delta\} \subset \mathbb{R}.$$

Since K is compact, one can find and fix $s_1, \dots, s_N \in K$ such that

$$K \subset \bigcup_{i=1}^{N} U_{s_i}^\delta.$$

By Theorem 8.2, one can also find and fix $\phi_1, \dots, \phi_N \in C_c^+(\mathbb{R})$ satisfying that

- $\operatorname{supp} \phi_i \subset U_{s_i}^\delta$ for each $i = 1, \dots, N$, and
- $$\sum_{i=1}^{N} \phi_i(s) = 1$$

  for any $s \in K$.

Then for any $i = 1, \dots, N$ and any $s, x \in \mathbb{R}$, the following inequality holds:

$$\phi_i(s) \cdot |(\sigma_s h)(x) - (\sigma_{s_i} h)(x)| \leq \varepsilon \phi_i(s). \tag{7}$$

In fact, $\phi_i(s) = 0$ in the cases where $s \notin U_{s_i}^{\delta}$, and the inequality holds

$$|(\sigma_s h)(x) - (\sigma_{s_i} h)(x)| \leq \varepsilon$$

in the cases where $s \in U_{s_i}^{\delta}$ by (6) above.

Let us consider cases where $s \in K$. Then $\sum_{i=1}^{N} \phi_i(s) = 1$, and hence for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left| (\sigma_s h)(x) - \sum_{i=1}^{N} \phi_i(s)(\sigma_{s_i} h)(x) \right| &= \left| \sum_{i=1}^{N} (\phi_i(s) \cdot (\sigma_s h)(x) - (\sigma_{s_i} h)(x)) \right| \\
&\leq \sum_{i=1}^{N} \phi_i(s) \cdot |(\sigma_s h)(x) - (\sigma_{s_i} h)(x)| \\
&\leq \varepsilon \cdot \sum_{i=1}^{N} \phi_i(s) \quad (\because (7)) \\
&= \varepsilon.
\end{aligned}$$

This completes the proof. □

8.4 Proof of Theorem 6.5

In order to give a proof of Theorem 6.5, we introduce the notation below for functions on $\mathbb{R}$.

Definition 8.8. *For each $h \in C_c^+(\mathbb{R})$ and each $x \in \mathbb{R}$, we define the functions $\tilde{h}$ and $\tilde{h}_x$ by*

$$\tilde{h} : \mathbb{R} \to \mathbb{R},\ s \mapsto h(-s)$$

and

$$\tilde{h}_x : \mathbb{R} \to \mathbb{R},\ s \mapsto h(x - s).$$

Proposition 8.9. *Let us fix any $h \in C_c^+(\mathbb{R})$ and any $x \in \mathbb{R}$. Then the following holds:*

1. *$\tilde{h}, \tilde{h}_x \in C_c^+(\mathbb{R})$.*

2. *If $h \neq 0$, then $\tilde{h}, \tilde{h}_x \in C_c^+(\mathbb{R}) \setminus \{0\}$.*

3. *$\tilde{h}_x(s) = (\sigma_{-x}\tilde{h})(s) = (\sigma_s h)(x)$ for any $s \in \mathbb{R}$.*

In our lectures, for the proof of Theorem 6.5, we also apply the following theorem (see also Remark 8.12 below):

Theorem 8.10. *Let $h \in C_c^+(\mathbb{R})\setminus\{0\}$. Then there exists an invariant strictly-positive linear functional μ on $C_c(\mathbb{R})$ with $\mu(h) = 1$.*

Proof of Theorem 8.10. We know that Riemann integrations and Lebesgue integrations defines invariant strictly-positive linear functionals on $C_c(\mathbb{R})$. By considering positive scalar multiplications of such linear functionals on $C_c(\mathbb{R})$, we have μ. □

Let us give a proof of Theorem 6.5 by applying Lemma 8.7 below:

Remark 8.11. *Idea of proof of Theorem 6.5. Take small $\delta > 0$ and $h \in C_c^+(\mathbb{R}) \setminus \{0\}$ with* width $h \leq \delta$. *Our goal is to show that there exist $N \in \mathbb{Z}_{\geq 0}$, $s_1, \dots, s_N \in \mathbb{R}$ and $c_1, \dots, c_N \in \mathbb{R}_{\geq 0}$ satisfying that*

$$f(x) \fallingdotseq \sum_{i=1}^{N} c_i(\sigma_{s_i}h)(x)$$

for any $x \in \mathbb{R}$. Without loss of the generaliy, we can assume that $h(0) > 0$. Note that $h(t) = 0$ if t is not small since width $h \leq \delta$.

By applying Lemma 8.7, one can find and fix $N \in \mathbb{Z}_{\geq 0}$, $s_1, \dots, s_N \in \mathbb{R}$ and $\phi_1, \dots, \phi_N \in C_c^+(\mathbb{R})$ such that

$$(\sigma_s h)(x) \fallingdotseq \sum_i \phi_i(s)(\sigma_{s_i}h)(x) \tag{8}$$

holds for any $s \in \operatorname{supp} f$ and any $x \in \mathbb{R}$. In particular, we have

$$f(s)(\sigma_s h)(x) \fallingdotseq f(s) \sum_i \phi_i(s)(\sigma_{s_i}h)(x)$$

for any $s \in \mathbb{R}$ and any $x \in \mathbb{R}$.

By Theorem 8.10, we can find and take invariant strictly-positive linear functional μ on $C_c(\mathbb{R})$ with $\mu(\tilde{h}) = 1$. Put

$$c_i := \mu(f \cdot \phi_i) \in \mathbb{R}_{\geq 0}$$

for each $i = 1, \ldots, N$.

Take any $x \in \mathbb{R}$. *Then we have*

$$f(x)\tilde{h}_x(s) = f(x)h(x-s) \fallingdotseq f(s)h(x-s) = f(s)\tilde{h}_x(s)$$

since f *is uniformly continuous and* $h(t) = 0$ *if* t *is not small. Thus*

$$f(x) = f(x)\mu(\tilde{h}_x) \fallingdotseq \mu(f \cdot \tilde{h}_x).$$

Furthermore, we also have

$$f(s)\tilde{h}_x(s) = f(s)(\sigma_s h)(x) \fallingdotseq f(s)\sum_i(\phi_i)(s)(\sigma_{s_i}h)(x) = \sum_i(f\cdot\phi_i)(s)(\sigma_{s_i}h)(x)$$

for any $s \in \mathbb{R}$. *Therefore,*

$$\mu(f \cdot \tilde{h}_x) \fallingdotseq \sum_i \mu(f \cdot \phi_i)(\sigma_{s_i}h)(x) = \sum_i c_i(\sigma_{s_i}h)(x).$$

Hence we have

$$f(x) \fallingdotseq \sum_i c_i(\sigma_{s_i}h)(x).$$

Proof of Theorem 6.5. Put $\varepsilon_1 := \varepsilon/2 \in \mathbb{R}_{>0}$. By Theorem 8.6, one can find and fix $\delta \in \mathbb{R}_{>0}$ such that for any $x, y \in \mathbb{R}$ with $|x - y| \leq \delta$,

$$|f(x) - f(y)| \leq \varepsilon_1. \tag{9}$$

Let us fix any $h \in C_c^+(X) \setminus \{0\}$ with $\operatorname{width} h \leq \delta$.

Our goal is to show that $\mathcal{A}_\varepsilon(f; h) \neq \emptyset$. Note that for any $s \in \mathbb{R}$,

$$\mathcal{A}_\varepsilon(f; h) = \mathcal{A}_\varepsilon(f; \sigma_s h).$$

Therefore, without loss of the generality, we can assume that $h(0) > 0$. Note that we have

$$\operatorname{supp} h \subset [-\delta, \delta]. \tag{10}$$

It is not difficult to see that we only need to prove the existance of $N \in \mathbb{Z}_{\geq 0}$, $s_1, \ldots, s_N \in \mathbb{R}$ and $c_1, \ldots, c_N \in \mathbb{R}_{\geq 0}$ satisfying that

$$|f(x) - \sum_{i=1}^{N} c_i(\sigma_{s_i}h)(x)| \leq \varepsilon \text{ for any } x \in \mathbb{R}.$$

First, we choose $N \in \mathbb{Z}_{\geq 0}$, $s_1, \dots, s_N \in \mathbb{R}$ and $c_1, \dots, c_N \in \mathbb{R}_{\geq 0}$ as follows: Recall that $\tilde{h} \in C_c^+(\mathbb{R}) \setminus \{0\}$ by Proposition 8.9. In particular, the ratio

$$(f : \tilde{h}) \in \mathbb{R}_{>0}$$

is defined as in Section 5.1. Put

$$\varepsilon_2 := \frac{\varepsilon}{2(f : \tilde{h})} \in \mathbb{R}_{>0}.$$

Let us denote by

$$K := \operatorname{supp} f \subset \mathbb{R}.$$

Then K is compact subset of $\mathbb{R}$. By applying Lemma 8.7, one can find and fix $N \in \mathbb{Z}_{\geq 0}$, $s_1, \dots, s_N \in \mathbb{R}$ and $\phi_1, \dots, \phi_N \in C_c^+(\mathbb{R})$ such that the inequality

$$|(\sigma_s h)(x) - \sum_i \phi_i(s)(\sigma_{s_i} h)(x)| \leq \varepsilon_2 \tag{11}$$

holds for any $s \in K$ and any $x \in \mathbb{R}$. Let us take any invariant strictly-positive linear functional μ on $C_c(\mathbb{R})$ with $\mu(\tilde{h}) = 1$ (see Theorem 8.10). Put

$$c_i := \mu(f \cdot \phi_i) \in \mathbb{R}_{\geq 0}$$

for each $i = 1, \dots, N$.

Next, take any $x \in \mathbb{R}$. We only need to show that

$$|f(x) - \sum_{i=1}^{N} c_i(\sigma_{s_i} h)(x)| \leq \varepsilon. \tag{12}$$

To prove (12), let us show that the inequality below holds for any $s \in \mathbb{R}$:

$$|f(x) \cdot \tilde{h}_x(s) - \sum_{i=1}^{N} (\sigma_{s_i} h)(x) \cdot \phi_i(s) \cdot f(s)| \leq \varepsilon_1 \tilde{h}_x(s) + \varepsilon_2 f(s). \tag{13}$$

Fix any $s \in \mathbb{R}$. By the definitions of $\tilde{h}_x(s)$, the left hand side of (13) can be evaluated as

$$\begin{aligned} |f(x) \cdot \tilde{h}_x(s) - &\sum_{i=1}^{N} (\sigma_{s_i} h)(x) \cdot \phi_i(s) \cdot f(s)| \\ &= |f(x) \cdot (\sigma_s h)(x) - f(s) \cdot \sum_{i=1}^{N} \phi_i(s) \cdot (\sigma_{s_i} h)(x)| \\ &\leq (\sigma_s h)(x) \cdot |f(x) - f(s)| + f(s) \cdot |(\sigma_s h)(x) - \sum_{i=1}^{N} \phi_i(s)(\sigma_{s_i} h)(x)|. \end{aligned}$$

Therefore, for (13), it is enough to show both inequalities below:

$$(\sigma_s h)(x) \cdot |f(x) - f(s)| \leq \varepsilon_1 (\sigma_s h)(x) \ (= \varepsilon_1 \tilde{h}_x(s)), \tag{14}$$

$$f(s) \cdot |(\sigma_s h)(x) - \sum_{i=1}^{N} \phi_i(s)(\sigma_{s_i} h)(x)| \leq \varepsilon_2 f(s) \ (= \varepsilon_2 f(s)). \tag{15}$$

The inequality (14) follows from the observation that if $|x - s| \leq \delta$, then

$$|f(x) - f(s)| \leq \varepsilon_1$$

by (9), otherwise $(\sigma_s h)(x) = h(x-s) = 0$ by (10). The inequality (15) comes from the fact that if $s \in K$ then the inequality (11) holds, and otherwise $f(s) = 0$ sinse $K := \operatorname{supp} f$. Thus the inequality (13) holds.

Finally, let us give a proof of the inequality (12). Note that by Proposition 8.9, we have

$$\mu(\tilde{h}_x) = \mu(\sigma_{-x}\tilde{h}) = \mu(\tilde{h}) = 1.$$

Then we have

$$\begin{aligned}
|f(x) - &\sum_i c_i (\sigma_{s_i} h)(x)| \\
&= |\mu(\tilde{h}_x) f(x) - \sum_i \mu(\phi_i \cdot f)(\sigma_{s_i} h)(x)| \\
&= |\mu(f(x) \cdot \tilde{h}_x - \sum_i (\sigma_{s_i} h)(x) \cdot \phi_i \cdot f)| \\
&\leq \mu(\varepsilon_1 \tilde{h}_x + \varepsilon_2 f) \quad (\because (13)) \\
&= \varepsilon_1 \cdot \mu(\tilde{h}_x) + \varepsilon_2 \cdot \mu(f) \\
&\leq \varepsilon_1 + \varepsilon_2 \cdot (f : \tilde{h}) \quad (\because \text{Theorem 5.6}) \\
&= \varepsilon \quad (\because \text{definitions of } \varepsilon_1, \varepsilon_2).
\end{aligned}$$

This completes the proof. □

Remark 8.12. *For the proof of Theorem 6.5 above, we apply Theorem 8.10. By giving careful arguments for invariant strictly-positive "subadditive" operators on $C_c(\mathbb{R})$, one can prove Theorem 6.5 without applying Theorem 8.10. Furthermore, Theorem 8.10 can be obtained as a corollary to Theorem 6.5 without any arguments for Riemann integrations nor Lebesgue integrations. See Nachbin [1] for more details.*

References

[1] Leopoldo Nachbin, *The Haar integral*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London (1965).

[2] Joseph A. Wolf, *Harmonic Analysis on Commutative Spaces*, Mathematical Surveys and Monographs, **142**. American Mathematical Society, Providence, RI, (2007).