

Section 10:

Spherical Fourier transforms

on finite homogeneous
spaces.

設定 : G : a finite group

M : a homogeneous
 G -space
s.t.

the regular representation

$$G \cong \mathbb{C}^M$$

is
multiplicity-free.

Section 10.1 : Q_V

記号:

$$I := (\text{diag } G) \setminus M \times M$$

$$J := \left\{ V \subset \mathbb{C}^M \mid \begin{array}{l} \neq \{0\} \\ G\text{-stable} \end{array} \right\}$$

設定 $V \in J$ と可也.

Prop (0.1.1) : $\forall x \in M, g \in G$

(= 247)

$$\lfloor g \delta_v^x = \delta_v^{g^x}$$

Pf of Prop (0.1.1) :

$h \in V \subset \mathbb{C}^M$ is fix

$$\textcircled{1} (h, g \delta_v^x)_M = h(g^x)$$

$$\textcircled{2} = (h, g \delta_v^x)_M = (g^{-1} h, \delta_v^x)$$

$$= (g^{-1} h)(x)$$

$$= h(g^x) = \textcircled{1} \quad \square$$

Prop 10.1.2:

$K_V : M \times M \rightarrow \mathbb{C}$
再生核 if diag G -inv

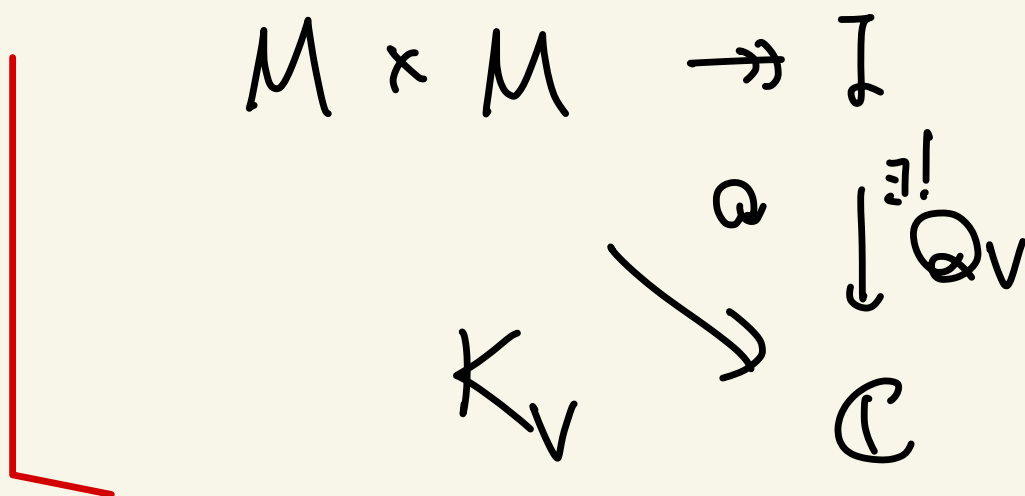
Pf of Prop 10.1.2:

$x, y \in M, g \in G \text{ \textit{z} fix}$

$$\textcircled{\text{示}} K_V(gx, gy) = K_V(x, y)$$

$$\begin{aligned} \textcircled{\text{左}} &= (\delta_V^{gx}, \delta_V^{gy})_M \\ &= (g\delta_V^x, g\delta_V^y)_M \\ &= (\delta_V^x, \delta_V^y)_M = \textcircled{\text{右}} \end{aligned} \quad \textcircled{\text{证}}$$

Cor 10.1.3 :



(cf. Prop 8.3.5)

$Q_V \in \mathbb{C}^I$ 即定义 I 中! !

$$\alpha \in M \text{ fix } \vec{0} \text{ etc}$$

$$\begin{array}{ccc} \mathbb{C}^I & \rightarrow & \mathbb{C}^M \\ f & \mapsto & \tilde{f} \end{array} \quad \text{is linear isometric isomorphism}$$

$$\tau \text{ is } \tau^{-1}$$

(Thm 8.5.2.)

$$\tau = \tau^{-1}$$

$$\tilde{f} : M \rightarrow \mathbb{C}, \gamma \mapsto f(R(\alpha, \gamma))$$

Prop 10.1.4:

$$\tau \text{ is } \tau^{-1}$$

$$\tilde{Q}_V = \sum_V^x$$

Thm 10.1.5

$$\left\{ \sqrt{\frac{\#M}{m_v}} Q_v \mid v \in J \right\}$$

\mathbb{C}^I a o.n.b

Proof of Thm 10.1.5

準備中

Section 10.2: Spherical Fourier transforms

Def 10.2.1:

$$\text{Let } f \in \mathbb{C}^I \text{ where}$$

$$\hat{f} \in \mathbb{C}^J \text{ is}$$

$$\hat{f}(v) = (f, Q_v)_I \frac{\#M}{m_v}$$

$$= \frac{\#M}{m_v} \sum_{i \in I} f(i) \overline{Q_v(i)} k_i$$

is $\delta_i < \dots$

Def 10.2.2:

$$\mathbb{C}^2 \rightarrow \mathbb{C}^J, f \mapsto \hat{f}$$

$\approx G \approx M$ a spherical

Fourier transform

& "big" \mathbb{J}_k

Thm 10.2.3:

$$\mathbb{C}^I \rightarrow \mathbb{C}^J, f \mapsto \hat{f}$$

is linear isometric

isomorphism

pf:

for $f \in \mathbb{C}^I$ is ok?

$$f = \sum_{v \in J} \hat{f}(v) Q_v$$

Pf of Thm (0.2.3) :

準備中

Ex 10.2.4

$$G := \mathbb{Z}/n\mathbb{Z}$$

$M :=$ 正 n 角形, a 頂点集合

$G \curvearrowright M$: 回転

a 場合

$x_0 \in M$ ε fix \exists .

$G \curvearrowright M$ ε isotropy-free τ \exists

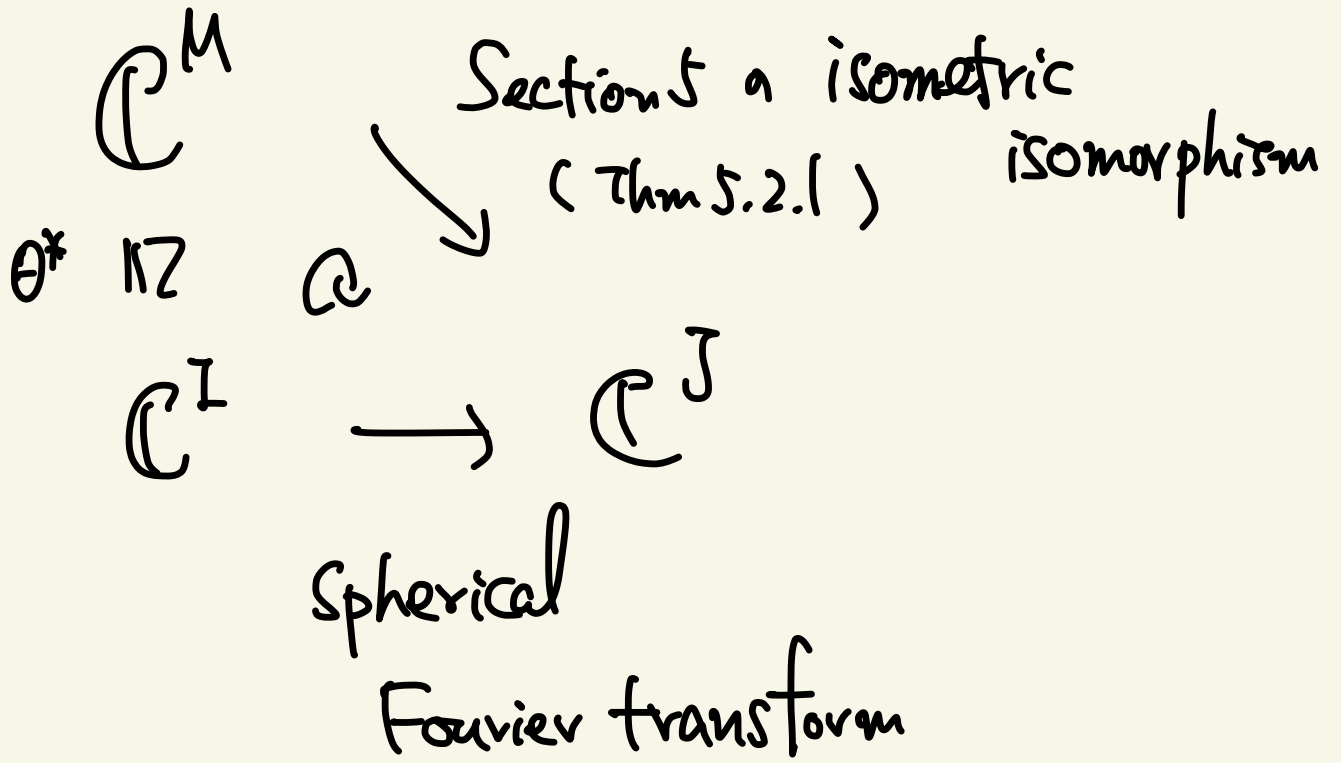
(i.e. $\forall x \in M, G^x = \{e\}$)

ε ρ 's

$\theta: M \rightarrow \mathbb{I}$ ε 全単射 ε \exists $\rho = \psi$

$\gamma \mapsto R(x_0, \gamma)$ ρ 's ε \exists ρ .

$$\bar{\mathbb{R}}^1 = \mathbb{C}_{x_0}^M$$



$$\in \frac{2}{\pi} \lambda d .$$

Ex 10.2.5

$H(n, 2)$ について

$$I \cong \{0, 1, \dots, n\}$$

$$J \cong \{0, 1, \dots, n\}$$

と同視
可也。

∴ かつ 各 $i \in I, l \in J$ について

$$Q_l(i) = \frac{1}{2^n} \sum_{s=0}^i (-1)^s \binom{i}{s} \binom{n-i}{l-s}$$

(cf. Thm 9.4.6)

$n = 8$ の場合を考へる。

$$f: I \rightarrow \mathbb{C}$$

と仮定。

$$i \mapsto \frac{1}{2}(i-4)(i-8)$$

Claim: $\hat{f}: J \rightarrow \mathbb{C}$

L

$$l \mapsto \begin{cases} 2^8 & (l=0,1) \\ 2^6 & (l=2) \\ 0 & (l=3, \dots, 8) \end{cases}$$

(後で使う)

$$\textcircled{\vdots} \quad Q_0(i) = \frac{1}{2^8}$$

$$Q_1(i) = \frac{1}{2^8}(-2i+8)$$

$$Q_2(i) = \frac{1}{2^8}(2i^2 - 16i + 32)$$

7)

$$f = 2^8 Q_0 + 2^6 Q_1 + 2^6 Q_2$$

□