§1: Sections of surjective maps

In this section,

we study definitions and examples

of "sections" of surjective maps.

§ 1.1 : Definition of sections.

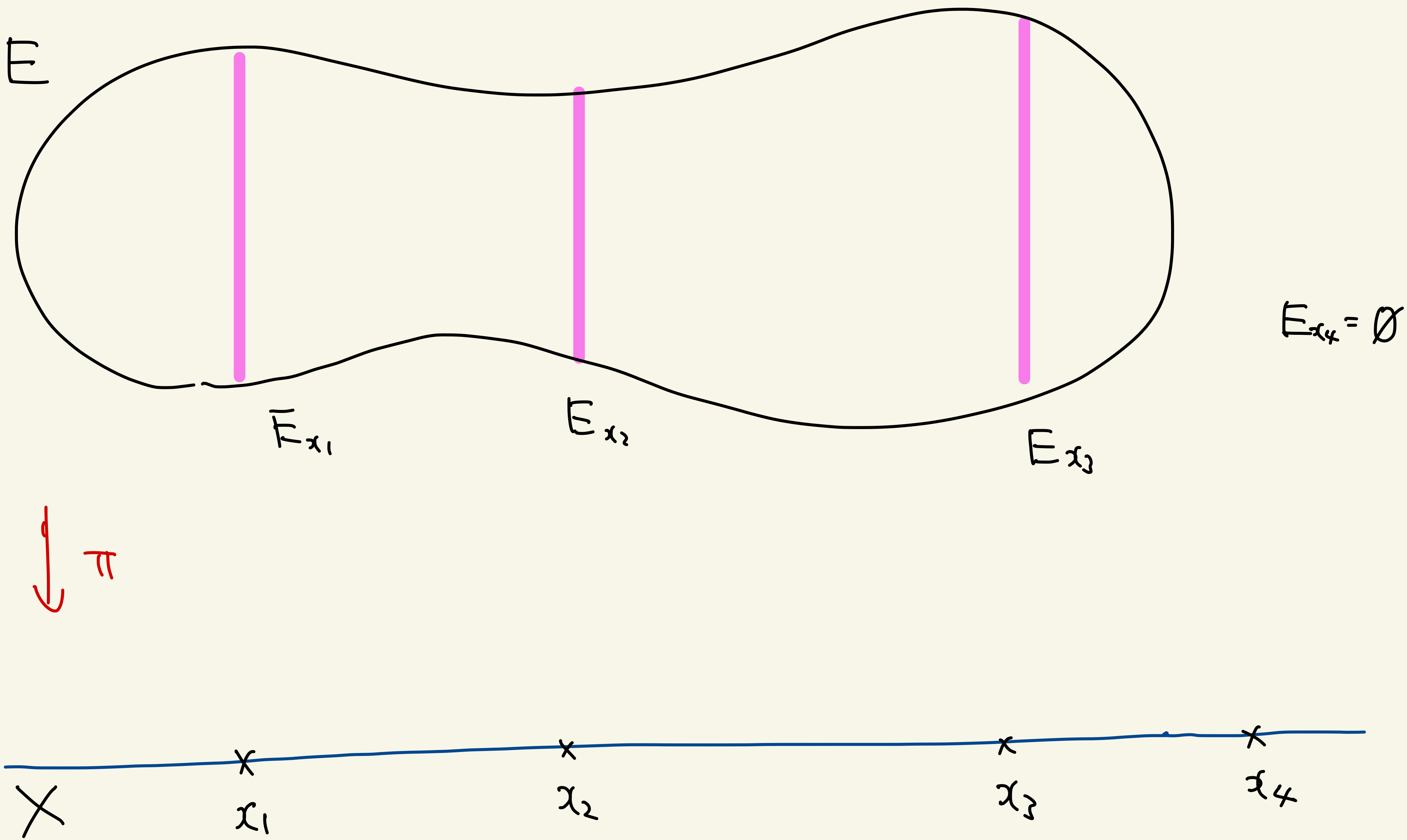
Setting : X, E : sets

$\pi : E \to X$: a map

Def 1.1.1 : For each $x \in X$,

$$E_x := E_x^{\pi} := \pi^{-1}(\{x\}) := \{ y \in E \mid \pi(y) = x \} \subset E$$

is called the fiber of x by π.



E
E_{x_1}
E_{x_2}
E_{x_3}
E_{x_4} = ∅
π
X
x_1
x_2
x_3
x_4

Def 1.1.2 :

A map $s : X \to E$ is called a section of π

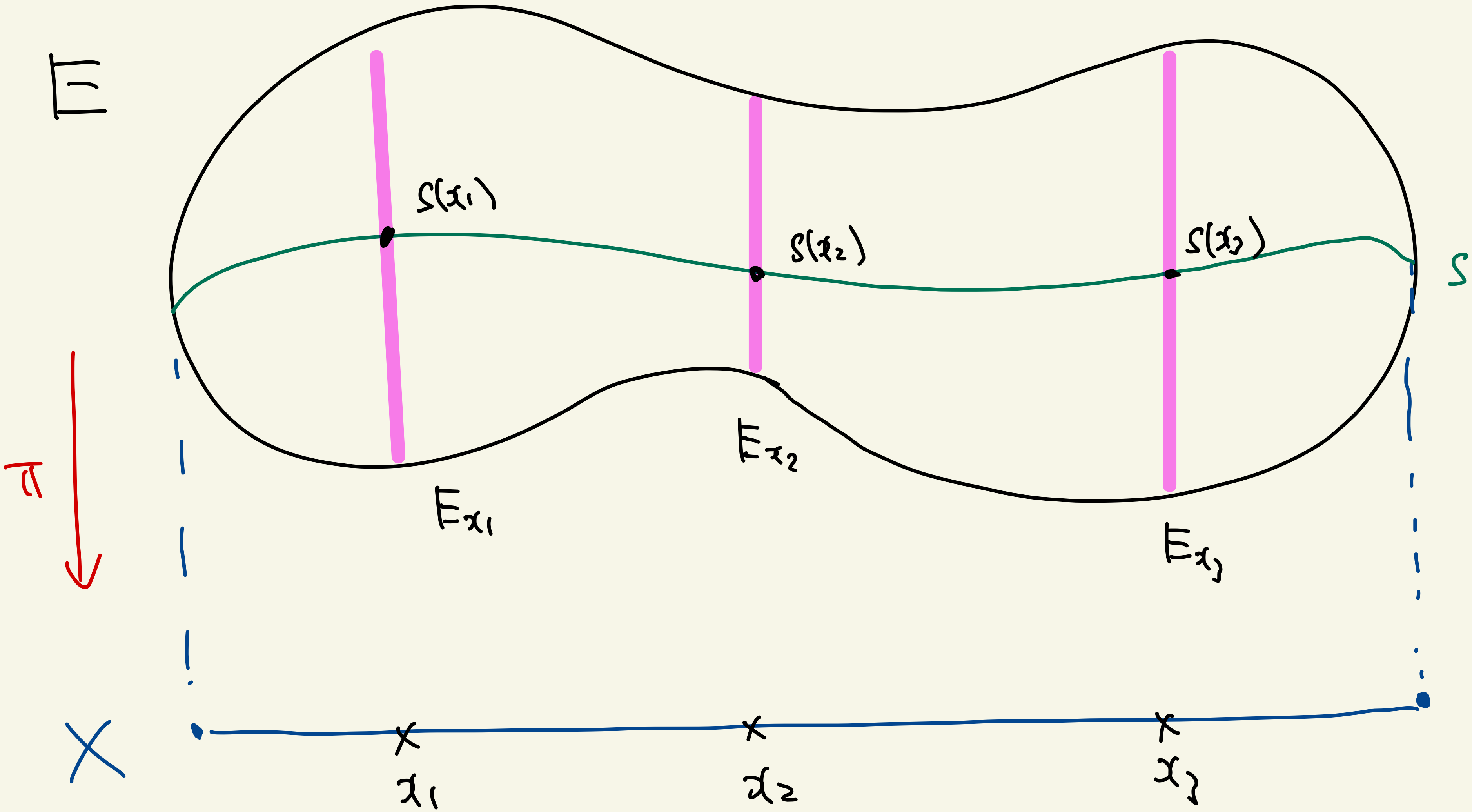
$$\underset{\text{def}}{\Longleftrightarrow} \quad \pi \circ s = \mathrm{id}_X$$

We put $\mathrm{Sect}(\pi) := \{ \text{sections of } \pi \}$ $\left(\subset \underbrace{\mathrm{Map}(X, E)}_{:= \{\text{maps from } X \text{ to } E\}} \right)$

Observation 1.1.3 : Let $s \in \mathrm{Sect}(\pi)$.

$$\forall x \in X, \quad s(x) \in E_x$$

Note : a section $s \in \mathrm{Sect}(\pi)$ choices $s(x) \in E_x$ for each $x \in X$.



E
$s(x_1)$
$s(x_2)$
$s(x_3)$
s
π
E_{x_1}
E_{x_2}
E_{x_3}
X
x_1
x_2
x_3

Observation 1.1.4 : For π, the following two conditions are equivalent:

(i) $\pi : E \to X$ is surjective

$\Updownarrow$

(ii) $\mathrm{Sect}(\pi) \neq \emptyset$.

(memo : "(i) $\Rightarrow$ (ii)" is called "the axiom of choice")

Thus we focus on the case where $\pi : E \to X$ is surjective.

Ex 1.1.5 :

Let X, Y : sets

$$E := X \times Y$$

$$\pi : E = X \times Y \to X, \quad (x, y) \mapsto x.$$

Then for each $x \in X$,

$$E_x = \{x\} \times Y.$$

Furthermore, $\mathrm{Sect}(\pi) \overset{1:1}{\leftrightarrow} \mathrm{Map}(X, Y)$ where

$$s \mapsto \mathrm{Pr}_Y \circ s \qquad \mathrm{Pr}_Y : X \times Y \to Y,\ (x, y) \mapsto y$$

$$s_f \leftarrow f \qquad s_f : X \to X \times Y,\ x \mapsto (x, f(x))$$

"maps" $\rightsquigarrow$ "sections"

a kind of generalization

Ex 1.1.6 :

$$X := S^2 := \{ x \in \mathbb{R}^3 \mid \sum_{i=1}^{3} x_i^2 = 1 \} \subset \mathbb{R}^3$$

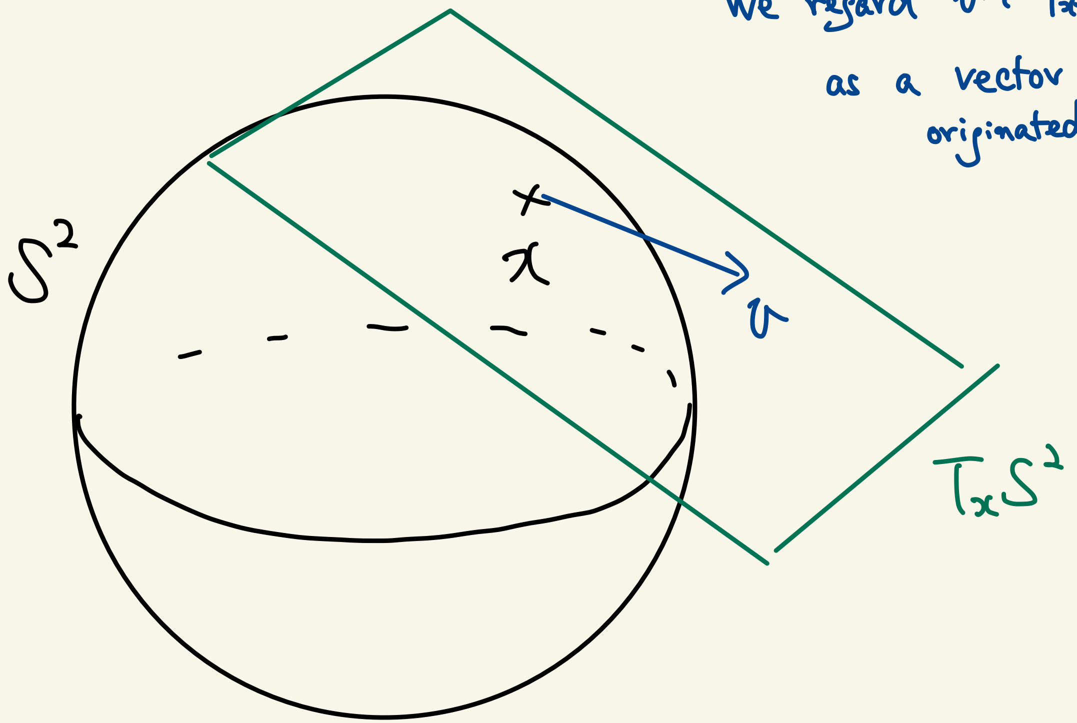
$$E := TS^2 := \{ (x, v) \mid x \in S^2,\ v \in T_x S^2 \}$$

where $T_x S^2 := \{ v \in \mathbb{R}^3 \mid \langle x, v \rangle_{\mathbb{R}^3} = 0 \}$

the inner product on $\mathbb{R}^3$

(the space of tangent vectors of S^2 at x).

We regard $v \in T_x S^2$ as a vector originated from the point x.



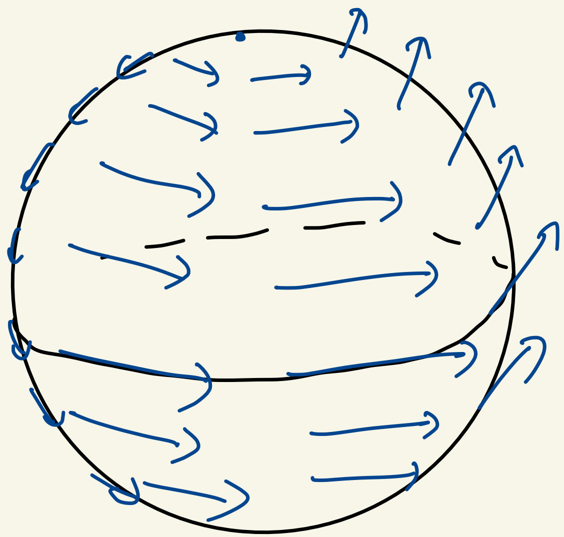
Put $\pi : TS^2 \to S^2,\ (x, v) \mapsto x$.

Then for $x \in S^2$,

$$(TS^2)_x = \{ (x, v) \mid v \in T_x S^2 \} \cong T_x S^2.$$

For example,

$$s : S^2 \to TS^2,\ x \mapsto (x, \underset{\in T_x S^2}{(-x_2, x_1, 0)}) \text{ defines a section.}$$



§ 1.2 : Continuous sections

Setting : X, E : topological spaces.

$\pi : E \twoheadrightarrow X$: a surjective continuous map.

Def 1.2.1 : $\Gamma(\pi) := \{ s \in \mathrm{Sect}(\pi) \mid s : X \to E : \text{continuous} \}$

$\subset \mathrm{Sect}(\pi)$

Ex 1.2.2: Let X, Y : topological spaces,

$E := X \times Y$: the direct product space

(equipped with the product topology)

$\pi : E = X \times Y \to X, \ (x, y) \mapsto x$

(surjective, continuous)

In this situation, we have

$$\Gamma(\pi) \overset{1:1}{\longleftrightarrow} C(X, Y) := \{ f : X \to Y \mid \text{continuous} \}$$

$$s \mapsto \mathrm{Pr}_Y \circ s$$

$$s_f \leftarrow f$$

(see Ex 1.1.5)

Ex 1.2.3: As in Ex 1.1.6, we take

$$X := S^2 := \{ x \in \mathbb{R}^3 \mid \sum_{i=1}^{3} x_i^2 = 1 \} \subset \mathbb{R}^3,$$

$$E := TS^2 := \{ (x, v) \mid x \in S^2,\ v \in T_x S^2 \} \subset S^2 \times \mathbb{R}^3.$$

We regard S^2 and TS^2 as topological spaces equipped with the relative topology.

Then $\pi : TS^2 \twoheadrightarrow S^2,\ (x, v) \mapsto x$ is surjective, continuous.

The section $s : S^2 \to TS^2,\ x \mapsto (x, (-x_2, x_1, 0))$ appeared in Ex 1.1.6 is continuous.

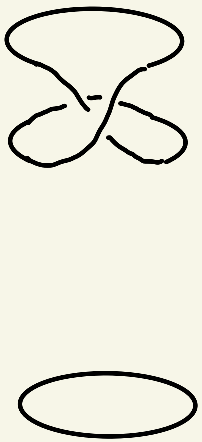
Ex 1.2.4:

Let $E = X = S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$, and

$$\pi : E = S^1 \twoheadrightarrow X = S^1, \quad z \mapsto z^2$$

(surjective, continuous)

$$\begin{array}{c} E = S^1 \\ \downarrow \pi \\ X = S^1 \end{array}$$



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In this situation,

$$\Gamma(\pi) = \emptyset$$ (i.e. π has no continuous sections!)

(although $\mathrm{Sect}(\pi) \neq \emptyset$).