§2 : Topological fiber bundles.

In this section,

we introduce the concept of

fiber bundles

§2.1 : Definition of (topological) fiber bundles.

Setting : X, F, E : topological spaces.

$\pi : E \twoheadrightarrow X$: a surjective continuous map.

Def 2.1.1 (Local trivializations)

(U, ψ) is a local trivialization of π by F

$\overset{\text{def}}{\iff}$ $U \underset{\text{open}}{\subset} X$, $\psi : \pi^{-1}(U) \to U \times F$: a homeomorphism

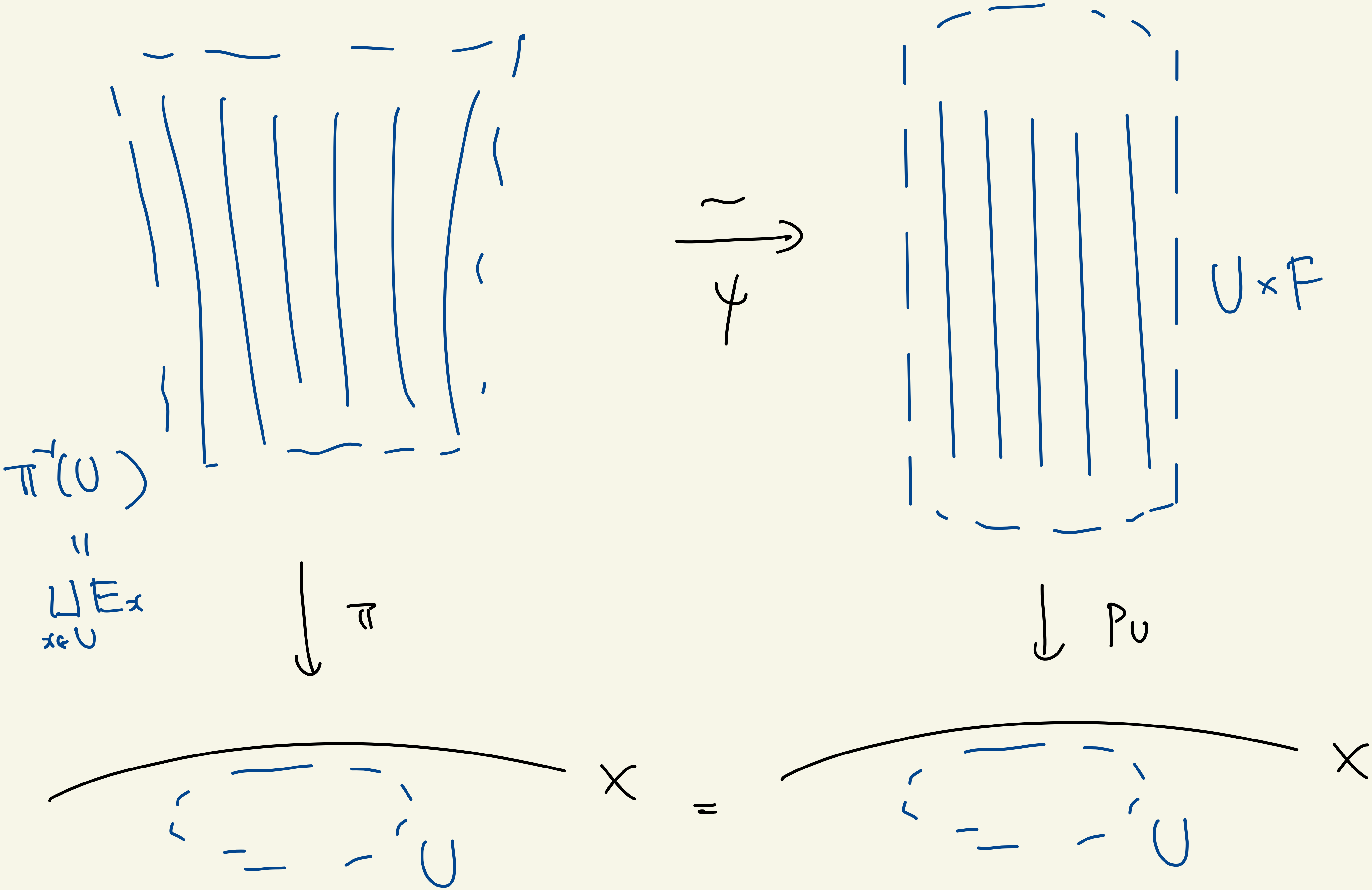
s.t. $\pi(y) = \underset{\sim}{P_U}(\psi(y))$ $({}^{\forall} y \in \pi^{-1}(U))$

$P_U : U \times F \to U,\ (u, a) \mapsto u.$

$$\left(\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, phantom, "\circlearrowleft"] \arrow[dl, "P_U"] \\ & U \end{tikzcd}\right)$$

We put

$$LT(\pi : F) := \{(U, \psi) \mid \text{a local trivialization of } \pi \text{ by } F\}$$


$\pi^{-1}(U)$
$= \bigsqcup_{x \in U} E_x$
$\xrightarrow[\psi]{\sim}$
$U \times F$
π
p_U
X
$=$
X
U
U

Observation 2.1.2:

Let $(U, \varphi) \in LT(\pi ; F)$.

Then for each $x \in U \subset X$,

we have an homeomorphism

$$\varphi_x : \underset{\pi^{-1}(U)}{\overset{\cup}{E_x}} \xrightarrow{\sim} F$$

$$y \longmapsto P_F(\varphi(y))$$

$\left(\begin{array}{l} P_F : U \times F \to F \\ \quad (u, a) \mapsto a \end{array} \right)$

$$\varphi^{-1}(x, a) \longleftarrow a$$

Def 2.1.3 (Topological fiber bundle)

(X, F, E, π) is called a (topological) fiber bundle

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall x \in X$, $\exists (U, \varphi) \in LT(\pi ; F)$ s.t. $x \in U$.

We call

X : the base space

F : the (abstract) fiber

E : the total space

Ex 2.1.4 : Let X, Y : topological spaces.

$$E := X \times Y$$

$$\pi : E = X \times Y \to X, \ (x, y) \mapsto x.$$

Then

$$\left(X, \ \psi : \pi^{-1}(X) = E = X \times Y \to Y, \ (x, y) \mapsto y \right) \in \mathcal{L}T(\pi ; Y)$$

and thus

$(X, Y, X \times Y, \pi)$ is a fiber bundle.

the trivial Y-bundle on X.

Ex 2.1.5 Let us consider

(S^2, TS^2, π) as in Ex 1.2.3.

Then $(S^2, \mathbb{R}^2, TS^2, \pi)$

is a fiber bundle

For example, for $x_0 = (0,0,1) \in S^2$,

one can take $(U, \psi) \in LT(\pi : \mathbb{R}^2)$ with $x_0 \in U$

defined by $U := \{x \in S^2 \mid x_3 > 0\}$, ψ:

$$\pi^{-1}(U) = \{(x, v) \in S^2 \times \mathbb{R}^3 \mid x_3 > 0, \langle x, v \rangle_{\mathbb{R}^3} = 0\}$$

$$\psi : \pi^{-1}(U) \to U \times \mathbb{R}^2, \quad (x, v) \mapsto (x, (v_1, v_2))$$

(Note: $\nexists (U, \psi) \in LT(\pi : \mathbb{R}^2)$ s.t. $U = S^2$!)

not easy to prove

Ex 2.1.6

Let $X = S^1 = \mathbb{R}/\mathbb{Z}$, $F = [-1,1] \subset \mathbb{R}$

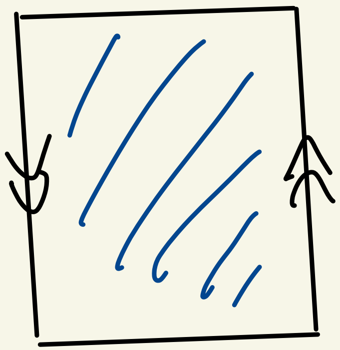
$$E = (\mathbb{R} \times [-1,1])/\sim \quad \text{where}$$

$$(r,s) \sim (r',s') \overset{\text{def}}{\iff} r - r' \in \mathbb{Z} \text{ and } s' = (-1)^{r-r'} s$$

$$\pi : E \to X = \mathbb{R}/\mathbb{Z}$$

$$[(r,s)] \mapsto [r]$$

(E =



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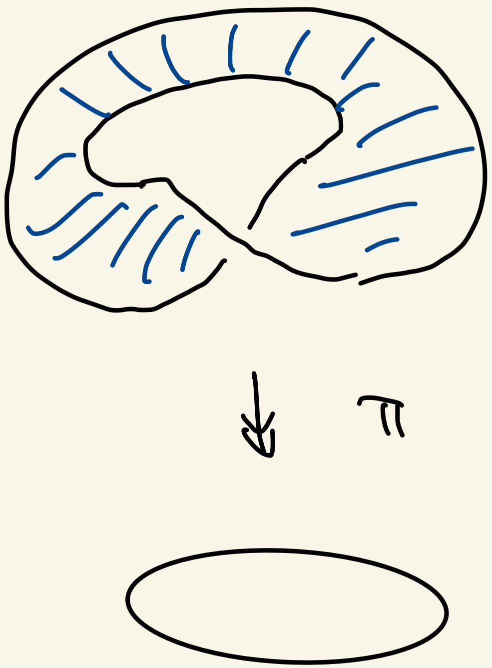

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(X, F, E, π) is a fiber bundle

(the Möbius band)

Remark:

Let (X, F, E, π) be a fiber bundle.

If F is a discrete,

then $\pi : E \twoheadrightarrow X$ is a covering map