§3 : Vector bundles

We introduce the concept of vector b'dles,

and study the vector space str on the set of continuous sections of vector b'dles.

§3.1 : Definition of vector bundles

Setting : $(X, \mathbb{R}^n, E, \pi)$: a fiber bundle
$(n \in \mathbb{Z}_{\geq 0})$

For each $x \in X$,

we fix a vector space str S_x on $E_x \subset E$.

$$S_x : \begin{cases} E_x \times E_x \to E_x & (\text{the summation}) \\ \mathbb{R} \times E_x \to E_x & (\text{the scalar product}) \end{cases}$$

Put $S := \{S_x\}_{x \in X}$.

Def 3.1.1 :

$\mathcal{L}T_S(\pi : \mathbb{R}^n) := \{ (U, \psi) \in \mathcal{L}T(\pi : \mathbb{R}^n) \mid$

$\forall x \in U,$

$\psi_x : (E_x, S_x) \to \mathbb{R}^n$ is linear $\}$

$\Downarrow$

(linear isom)

Def 3.1.2 $(X, \mathbb{R}^n, E, \pi, S = \{S_x\}_{x\in X})$ is

a vector bundle of rank n

$\overset{\text{def}}{\iff}$ $\forall x \in X$,

$\exists (U, \varphi) \in \mathcal{LT}_S(\pi ; \mathbb{R}^n)$ with $x \in U$.

Ex 3.1.3 : X : a topological space

$$E := X \times \mathbb{R}^n$$

$$\pi : X \times \mathbb{R}^n \to X, \quad (x, v) \mapsto x$$

$$\left(\rightsquigarrow E_x := \{x\} \times \mathbb{R}^n \quad (x \in X) \right)$$

S_x : the "usual" vector space str on $E_x = \{x\} \times \mathbb{R}^n$.

$$\rightsquigarrow \begin{cases} U := X \\ \varphi : \pi^{-1}(U) = X \times \mathbb{R}^n \xrightarrow{id} X \times \mathbb{R}^n \end{cases}$$

Then $(U, \varphi) \in \mathcal{LT}_S(\pi : \mathbb{R}^n)$ $\left(\begin{array}{l} \varphi_x : E_x \xrightarrow{\sim} \mathbb{R}^n : \text{lin. isom.} \\ \quad\;\; \| \\ \quad \{x\} \times \mathbb{R}^n \\ \quad\;\; (x, v) \mapsto v \end{array} \right)$

Ex 3.1.4. Let us consider the vector b'dle

$$(S^2, \mathbb{R}^2, TS^2, \pi) \text{ in Ex 2.1.5.}$$

For each $x \in S^2$,

$$E_x = \{x\} \times T_x S^2 = \{(x, v) \mid \underset{\langle x, v \rangle_{\mathbb{R}^3} = 0}{v \in \mathbb{R}^3}\} \subset \mathbb{R}^2$$

can be considered as a linear subspace of $\mathbb{R}^3$.

We write S_x for such the vector space structure on E_x.

Then $(S^2, \mathbb{R}^2, TS^2, \pi, \{S_x\}_{x \in S^2})$

is a vector b'dle of rank 2.

For example,

for $(U, \psi) \in \mathcal{LT}(\pi ; \mathbb{R}^2)$ in Ex 2.1.5,

$$\left(\begin{array}{l} U := \{ x \in S^2 \mid x_3 > 0 \}, \ \psi : \\ \pi^{-1}(U) = \{ (x, v) \in S^2 \times \mathbb{R}^3 \mid x_3 > 0, \ \langle x, v \rangle_{\mathbb{R}^3} = 0 \} \\ \psi : \pi^{-1}(U) \to U \times \mathbb{R}^2, \ (x, v) \mapsto (x, (v_1, v_2)) \end{array} \right)$$

we have $(U, \psi) \in \mathcal{LT}_S(\pi ; \mathbb{R}^2)$.

$$\left(\begin{array}{ll} \psi_x : \{x\} \times T_x S^2 \to \mathbb{R}^2 & \text{is linear} \\ \quad (x, v) \mapsto (v_1, v_2) & \\ \quad (\langle x, v \rangle_{\mathbb{R}^3} = 0) & \end{array} \right)$$

§3.2: Vector space of continuous sections of vector bundles.

Setting: $(X, \mathbb{R}^n, E, \pi, S = \{S_x\}_{x \in X})$: a vect. b'dle of rank n.

Recall: $\mathrm{Sect}(\pi) := \{ \text{sections of } \pi \}$

$\cup$

$\Gamma(\pi) := \{ \text{continuous sections of } \pi \}$

$\mathrm{Sect}(\pi)$ can be identified with a direct product set $\prod_{x \in X} E_x$

by

$$\begin{aligned} \mathrm{Sect}(\pi) &\overset{1:1}{\longleftrightarrow} \prod_{x \in X} E_x \\ s &\longmapsto (s(x))_{x \in X} \\ \begin{array}{l} s : X \to E \\ \quad x \mapsto v_x \end{array} &\longleftarrow\!\shortmid (v_x)_{x \in X} \end{aligned}$$

(Note : one of definitions of "$\prod_{x \in X} E_x$" is $\mathrm{Sect}(\pi)$.)

Let us consider $\mathrm{Sect}(\pi)$

as a vector space $\prod_{x \in X} E_x = \prod_{x \in X} (E_x, S_x)$

- Zero vector : $0 : X \to E,\ x \mapsto 0_x$ ← zero in E_x.
- Sum : $S_1 + S_2 : X \to E,\ x \mapsto S_1(x) + S_2(x)$ ← sum in E_x
- Scalar product : $\lambda s : X \to E,\ x \mapsto \lambda \cdot s(x)$ ← scalar prod in E_x.

$(s, s_1, s_2 \in \mathrm{Sect}(\pi),\ \lambda \in \mathbb{R})$

Thm 3.2.1: $\Gamma(\pi)$ is a linear subspace of $\mathrm{Sect}(\pi)$

Proof: Check list:

① $0 \in \Gamma(\pi)$

② $s_1 + s_2 \in \Gamma(\pi)$ $\quad (s_1, s_2 \in \Gamma(\pi))$

③ $\lambda s \in \Gamma(\pi)$ $\quad (s \in \Gamma(\pi),\ \lambda \in \mathbb{R})$

We shall prove ② (① and ③ : report problems)

② Fix $s_1, s_2 \in P(\pi)$.

Goal : $s_1 + s_2 : X \to E$: conti.

Take any $(U, \psi) \in LT_S(\pi : \mathbb{R}^n)$.

Then we only need to show that

Goal : $(s_1 + s_2)|_U : U \to \pi^{-1}(U)$: conti.

Since $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is homeo,

our goal is reduced to the following

Goal : $\psi \circ (s_1 + s_2)|_U : U \to U \times \mathbb{R}^n$: conti.

$$x \mapsto (x, p_{\mathbb{R}^n}(\psi((s_1+s_2)(x))))$$

where $p_{\mathbb{R}^n} : U \times \mathbb{R}^n \to \mathbb{R}^n$, $(x, v) \mapsto v$.

Obviously $\begin{aligned} U &\xrightarrow{\mathrm{id}_U} U \\ x &\longmapsto x \end{aligned}$ is conti.

Thus it suffices to show that:

<u>Goal</u>: $\quad P_{\mathbb{R}^n} \circ \psi \circ (s_1 + s_2)|_U : U \to \mathbb{R}^n$

$$x \longmapsto P_{\mathbb{R}^n}\left(\psi\left((s_1+s_2)(x)\right)\right)$$

Note that $\quad P_{\mathbb{R}^n} \circ \psi \circ (s_1 + s_2)|_U = P_{\mathbb{R}^n} \circ \psi \circ (s_1|_U) + P_{\mathbb{R}^n} \circ \psi \circ (s_2|_U)$

(Report : Hint: $\underbrace{\psi_x}_{\text{linear } E_x \to \mathbb{R}^n}\left(s_1(x) \underbrace{+}_{\text{sum in } E_x} s_2(x)\right) = \psi_x(s_1(x)) + \psi_x(s_2(x))$)

Since s_1, s_2 are both continuous,

$$P_{\mathbb{R}^n} \circ \psi \circ (s_1 + s_2)|_U = P_{\mathbb{R}^n} \circ \psi \circ (s_1|_U) + P_{\mathbb{R}^n} \circ \psi \circ (s_2|_U)$$

is also conti.

$\boxed{\text{Q}}$