§4: Equivariant vector bundles.

§4.1 : Def of eq. vect. b'dles

Setting : $\mathcal{V} := (X, \mathbb{R}^n, E, \pi, S)$: a vect. b'dle of rank n.

G : a group.

$\rho : G \to \mathrm{Homeo}(E)$
$\tau : G \to \mathrm{Homeo}(X)$: group homomorphisms

(※ $\mathrm{Homeo}(M) := \{ \varphi : M \to M \mid \text{homeo.} \}$)

Def 4.1.1: The vector b'dle $\mathcal{V}$ is

G - equivariant (w.r.t. ρ, τ)

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall g \in G$,

① $\pi \circ \rho(g) = \tau(g) \circ \pi$ $\left(\begin{tikzcd} E \arrow[r, "\rho(g)"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\pi"] \\ X \arrow[r, "\tau(g)"] & X \end{tikzcd} \right)$

For each $x \in X$,

② we write $\rho_x(g) : E_x \to E_{\tau(x)}$.

$$v \mapsto \rho(g) v$$

Then $\rho_x(g)$ is linear.

Ex 4.1.2 Let us consider

$$\mathcal{V} = (S^2, \mathbb{R}^2, TS^2, \pi, S) \quad (\text{Ex } 3.1.4).$$

$$G = O(3) = \{ g \in M(3;\mathbb{R}) \mid {}^t g = g^{-1} \}$$

$$\rho : O(3) \to \mathrm{Homeo}(TS^2)$$

$$g \mapsto \rho(g) : TS^2 \to TS^2$$

$$(x, v) \mapsto (g \cdot x, g \cdot v)$$

matrix products.

$$\tau : O(3) \to \mathrm{Homeo}(S^2)$$

$$g \mapsto \tau(g) : S^2 \to S^2, \; x \mapsto g \cdot x$$

$\rightsquigarrow$ $\mathcal{V}$ is a G-eq. vect. b'dle.

§4.2: The regular representation on an equivariant vector bundles.

Def 4.2.1.

For each $g \in G$,

we define

$$\rho^*(g) : P(\pi) \to P(\pi)$$

$$S \mapsto \rho^*(g)S := \rho(g) \circ S \circ \tau(g^{-1})$$

$$\left(\begin{array}{l} X \to E \\ x \mapsto \rho(g)\big(S(\tau(g^{-1})x)\big) \end{array} \right)$$

Prop 4.2.2

(1) For each $g \in G$,

$$\rho^*(g) : P(\pi) \to P(\pi) \text{ is a linear isom.}$$

(2) $$\rho^* : G \to GL(P(\pi)) \text{ is a group hom.}$$
$$g \mapsto \rho^*(g)$$

(the regular representation)

$$\left(GL(P(\pi)) := \{ \xi : P(\pi) \to P(\pi) \mid \text{linear isom} \} \right)$$

Ex 4.2.3 : In the setting of Ex 4.1.2.

We have a group hom

$$\rho^* : O(3) \to GL(\Gamma(\pi))$$

$$\xi = (S^2, \mathbb{R}^2, TS^2, \pi, \delta)$$

Study of $\rho^* : O(3) \to GL(\Gamma(\pi))$

$\doteqdot$ Harmonic analysis on ξ