# 幾何学 A (Geometry A): 2023

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# Contents

# Chapter 1

# Introduction

Main problem of the lectures: How to define "differentials" on topological spaces?

One of the answers in the lectures is the following: On "smooth manifolds" (which is a suitable topological space with a certain structure), we can define

1. Smooth functions
2. Directional differentiations, and
3. Total differentiations.

Plan of lectures:

**Part I: Chapter 2–5:** We recall and study some basic facts for smooth functions and their differentiations on open sets in Euclidean space. In particular, "algebraic characterizations" of directional differentiations and total differentiations will be given.

**Part II: Chapter 6–12:** We give a definition of "smooth structure" on topological spaces. A (suitable) topological space equipped with a smooth structure is called a smooth manifold.

**Part III: Chapter 13–17:** The definitions of directional differentiations and total differentiations are given algebraically.

# Chapter 2

# $\mathbb{R}$-algebras

## 2.1 Definitions of $\mathbb{R}$-algebras

**Definition 2.1.1** (Bi-linear maps)**.** Let $V_1, V_2, W$ be real vector spaces. A map $F : V_1 \times V_2 \to W$ is said to be bi-linear if $F$ is linear on each variable, i.e. both of the following two conditions hold:

1. For any $b \in V_2$, the map
$$F^b : V_1 \to W,\ a \mapsto F(a, b)$$
is linear.

2. For any $a \in V_1$, the map
$$F_a : V_2 \to W,\ b \mapsto F(a, b)$$
is linear.

**Definition 2.1.2.** Let $V$ be a real vector space. We fix a binary operation (which will be called a product)

$$V \times V \to V,\ (a, b) \mapsto a \cdot b.$$

The pair $(V, \cdot)$ is called an $\mathbb{R}$-algebra if the product fixed above is bi-linear.

**Example 2.1.3.** "$\mathbb{R}$" itself is an $\mathbb{R}$-algebra with respect to the usual product.

For a set $S$ and a real vector space $W$, we use the symbol $W^S$ as the set of all maps from $S$ to $W$.

**Proposition 2.1.4.** The set $W^S$ defined above is a real vector space with respect to the following sum and the scalar product:

**The pointwise sum:** For each $f, g \in W^S$, the sum $f + g \in W^S$ is defined by
$$f + g : S \to W,\ s \mapsto f(s) + g(s).$$

**The pointwise scalar product:** For each $f \in W^S$ and each $\lambda \in \mathbb{R}$, the scalar product $\lambda f \in W^S$ is defined by
$$\lambda f : S \to W,\ s \mapsto \lambda \cdot (f(s)).$$

We give two types of important examples of $\mathbb{R}$-algebras.

**Example 2.1.5.** Let $S$ be a set. Then the set $\mathbb{R}^S$ of all real valued functions on $S$ is an $\mathbb{R}$-algebra with respect to the following "pointwise product":

**The pointwise product:** For each $f, g \in \mathbb{R}^S$, the product $f \cdot g \in \mathbb{R}^S$ is defined by
$$f \cdot g : S \to \mathbb{R},\ s \mapsto f(s) \cdot g(s).$$

**Proposition 2.1.6.** Let $V, W$ be vector spaces. Then
$$\mathcal{L}(V, W) := \{\xi : V \to W \mid \xi \text{ is linear }\}$$
gives a linear subspace of $W^V$. In particular, $\mathcal{L}(V, W)$ itself is a real vector space.

**Example 2.1.7.** Let $V$ be a real vector space. We put
$$\mathrm{End}(V) := \mathcal{L}(V, V).$$
Then the vector space $\mathrm{End}(V)$ is an $\mathbb{R}$-algebra with respect to the composition.

## 2.2 Sub $\mathbb{R}$-algebras and $\mathbb{R}$-algebra homomorphisms

**Definition 2.2.1.** Let $(V, \cdot)$ be an $\mathbb{R}$-algebra and $W$ a linear subspace. We say that $W$ is a sub $\mathbb{R}$-algebra of $(V, \cdot)$ if $W$ is closed under the product "$\cdot$", i.e. $a \cdot b \in W$ for any $a, b \in W$.

**Proposition 2.2.2.** Each sub $\mathbb{R}$-algebra itself is an $\mathbb{R}$-algebra.

**Example 2.2.3.** Let $S$ be a topological space. We define

$$C(S) := \{f \in \mathbb{R}^S \mid f \text{ is continuous}\}.$$

Then $C(S)$ is a sub $\mathbb{R}$-algebra of $\mathbb{R}^S$. In particular, $C(S)$ itself is an $\mathbb{R}$-algebra.

**Proposition 2.2.4.** Let $(V, \cdot)$ be an $\mathbb{R}$-algebra and $\{W_\lambda\}_{\lambda \in \Lambda}$ a family of sub $\mathbb{R}$-algebras of $V$. Then the intersection $\bigcap_{\lambda \in \Lambda} W_\lambda$ is a sub $\mathbb{R}$-algebra of $V$.

**Proposition 2.2.5.** Let $S$ be a topological space. For each $f \in \mathbb{R}^S$, we define the support $\operatorname{supp} f$ of $f$ by the closure in $S$ of

$$\{s \in S \mid f(s) \neq 0\}.$$

Then

$$\{f \in \mathbb{R}^S \mid \operatorname{supp} f \text{ is compact}\}$$

gives a sub $\mathbb{R}$-algebra of $\mathbb{R}^S$. Furthermore,

$$C_c(S) := C(S) \cap \{f \in \mathbb{R}^S \mid \operatorname{supp} f \text{ is compact}\}$$

is also a sub $\mathbb{R}$-algebra of $\mathbb{R}^S$.

Let us define homomorphisms between $\mathbb{R}$-algebras:

**Definition 2.2.6.** Let $(V, \cdot_V)$, $(W, \cdot_W)$ be both $\mathbb{R}$-algebras. A linear map $\psi : V \to W$ is called an $\mathbb{R}$-algebra homomorphism if

$$\psi(a \cdot_V b) = \psi(a) \cdot_W \psi(b)$$

for any $a, b \in V$.

**Example 2.2.7.** Let $S$ be a set and fix a point $p \in S$. Then the map

$$\operatorname{ev}_p : \mathbb{R}^S \to \mathbb{R}, \ f \mapsto f(p)$$

is an $\mathbb{R}$-algebra homomorphism.

## 2.3 Some properties of $\mathbb{R}$-algebras

Throughout this section, we fix an $\mathbb{R}$-algebra $V = (V, \cdot)$.

**Definition 2.3.1.**

- $V = (V, \cdot)$ is called associative if
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
for any $a, b, c \in V$.

- $V = (V, \cdot)$ is called unital if $V$ admits a unit, where we say that an element $1_V$ of $V$ is an unit if $a \cdot 1_V = 1_V \cdot a = a$ for any $a \in V$.

- $V = (V, \cdot)$ is commutative if $a \cdot b = b \cdot a$ for any $a, b \in V$.

**Proposition 2.3.2.** For each sub $\mathbb{R}$-algebra of an associative [resp. commutative] $\mathbb{R}$-algebra is associative [resp. commutative] as $\mathbb{R}$-algebra.

We note that a sub $\mathbb{R}$-algebra of a unital $\mathbb{R}$-algebra is not needed to be unital.

**Example 2.3.3.** Let $S$ be a topological space. Then

- $\mathbb{R}^S$ is associative, commutative and unital,

- $C(S)$ is associative, commutative and unital,

- $C_c(S)$ is associative, commutative (but not unital if $S$ is non-compact).

**Example 2.3.4.** Let $V$ be a real vector space. Then the $\mathbb{R}$-algebra $\mathrm{End}(V)$ is associative and unital. $\mathrm{End}(V)$ is commutative if and only if $\dim V \leq 1$.

**Example 2.3.5.** Let $(V, \cdot)$ be an associative $\mathbb{R}$-algebra. We shall define the binary operator $[\cdot, \cdot]$ (which will be called the bracket product),

$$[a, b] := a \cdot b - b \cdot a$$

for $a, b \in V$. Then $(V, [\cdot, \cdot])$ is an $\mathbb{R}$-algebra. It should be noted that $(V, [\cdot, \cdot])$ is not needed to be associative.

## 2.4 Exercise

1. Let us consider the vector space $\mathbb{R}^2$. For each map defined below, check the map is bi-linear or not:

   (a) $\nu : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2,\ ((x_1, x_2), (y_1, y_2)) \mapsto (x_1 y_1, x_2 y_2)$.

   (b) $\mu : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2,\ ((x_1, x_2), (y_1, y_2)) \mapsto (x_1 x_2, y_1 y_2)$.

   (c) $\xi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2,\ ((x_1, x_2), (y_1, y_2)) \mapsto (y_2, x_1)$.

2. Show that $\mathbb{R}$ is an $\mathbb{R}$-algebra with respect to the usual product (Example 2.1.3).

3. Let $S$ be a set and $W$ a real vector space. Show that the space of all $W$-valued functions $W^S := \{f : S \to W\}$ is a real vector space with respect to the pointwise sum and the pointwise scalar product (Proposition 2.1.4).

4. Let $S$ be a set. Show that the real vector space $\mathbb{R}^S$ is an $\mathbb{R}$-algebra with respect to the pointwise product (Example 2.1.5).

5. Let $V, W$ be both real vector spaces. Show that $\mathcal{L}(V, W) := \{f : V \to W \mid f \text{ is linear }\}$ is a linear subspace of $W^V$ (Proposition 2.1.6).

6. Let $V$ be a real vector space. We put $\mathrm{End}(V) := \mathcal{L}(V, V)$. Show that the real vector space $\mathrm{End}(V)$ is an $\mathbb{R}$-algebra with respect to the composition of maps (Example 2.1.7).

7. Show that "each sub $\mathbb{R}$-algebra itself is an $\mathbb{R}$-algebra" (Proposition 2.2.2).

8. Show that each of the following maps is continuous:

   (a) $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x, y) \mapsto x + y$.

   (b) $\nu : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x, y) \mapsto xy$.

   (c) $\xi : \mathbb{R} \to \mathbb{R},\ x \mapsto -x$.

9. Let $S$ be a topological space. Show that $C(S) := \{f : S \to \mathbb{R} \mid f \text{ is continuous }\}$ is a sub $\mathbb{R}$-algebra of the $\mathbb{R}$-algebra $\mathbb{R}^S$ (Example 2.2.3; Hint: the exercise 8 above).

10. Let $V = (V, \cdot)$ be an $\mathbb{R}$-algebra, $\{W_\lambda\}_{\lambda\in\Lambda}$ a family of sub $\mathbb{R}$-algebras of $V$. Show that $\bigcap_{\lambda\in\Lambda} W_\lambda$ is a sub $\mathbb{R}$-algebra of $V$ (Proposition 2.2.4).

11. Let $S$ be a topological space. Show that $C_c(S) := \{f \in C(S) \mid \operatorname{supp} f \text{ is compact}\}$ is a sub $\mathbb{R}$-algebra of the $\mathbb{R}$-algebra $C(S)$ (Example 2.2.5).

12. Let $S$ be a set and fix a point $p \in S$. Show that the map

$$\mathrm{ev}_p : \mathbb{R}^S \to \mathbb{R},\ f \mapsto f(p)$$

is an $\mathbb{R}$-algebra homomorphism (Example 2.2.7).

13. Let $S_1$ and $S_2$ be both topological spaces and $\varphi : S_1 \to S_2$ a continuous map. Show that the pullback

$$\varphi^* : C(S_2) \to C(S_1),\ f \mapsto f \circ \varphi$$

gives an $\mathbb{R}$-algebra homomorphism (Example 2.2.8).

14. Let $(V, \cdot)$ be an $\mathbb{R}$-algebra. We define the binary operator $[\ ,\ ]$ on $V$ by

$$[a, b] := a \cdot b - b \cdot a.$$

Show that the following holds (Example 2.3.5):

(a) $(V, [\ ,\ ])$ is an $\mathbb{R}$-algebra.

(b) Assume that $V$ is associative. Then for any $a, b, c \in V$, the following equality (the Jacobi identity) holds:

$$[a, [b.c]] = [[a, b], c] + [b, [a, c]].$$

# Chapter 3

# Multivariable smooth functions

In this chapter, we define the $\mathbb{R}$-algebras of multivariable smooth functions.

## 3.1 Definition of multivariable smooth functions

Throughout this section, we fix $n \in \mathbb{Z}_{\geq 0}$ and an open set $U$ of $\mathbb{R}^n$.

For a real vector space $\mathbb{R}^n$, we denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$.

Let us recall the definition of smooth functions:

**Definition 3.1.1.** A function $f : U \to \mathbb{R}$ is said to be of class $C^0$ on $U$ if $f$ is continuous on $U$.

**Definition 3.1.2.** For each $k \in \mathbb{Z}_{\geq 1}$, a function $f : U \to \mathbb{R}$ is said to be $C^k$ on $U$ if for each $i = 1, \ldots, n$,

$$\frac{\partial f}{\partial x_i} : U \to \mathbb{R},\ p \mapsto \lim_{h \to 0} \frac{f(p + he_i) - f(p)}{h}$$

is well-defined and $C^{k-1}$ on $U$ (inductive definition).

**Proposition 3.1.3.** For any $k \in \mathbb{Z}_{\geq 1}$, any $C^k$-function $f$ is $C^{k-1}$.

We note that in the proposition above, the case $k = 1$ (i.e. any $C^1$-function is continuous) is essential and non-trivial. The key is that any $C^1$-function is "totally differentiable".

**Definition 3.1.4.** We say that a function $f : U \to \mathbb{R}$ is $C^\infty$ on $U$ if $f$ is $C^k$ for any $k \in \mathbb{Z}_{\geq 0}$.

**Proposition 3.1.5.** Let us fix open sets $U$ and $V$ of $\mathbb{R}^n$. Suppose that $U \subset V$. Then for any $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and any $C^k$-function $f$ on $V$, the restriction

$$f|_U : U \to \mathbb{R},\ u \mapsto f(u)$$

gives a $C^k$-function on $U$.

## 3.2 Constructions of multivariable smooth functions

**Proposition 3.2.1.** Any $n$-variable polynomial functions on $\mathbb{R}^n$ is $C^\infty$.

**Proposition 3.2.2.** Let us fix $\alpha \in \mathbb{R}$. Then

$$\mathbb{R}_{>0} \to \mathbb{R},\ x \mapsto x^\alpha$$

is $C^\infty$ on $\mathbb{R}_{>0}$.

**Proposition 3.2.3.** Let $U$ and $V$ be open sets of $\mathbb{R}^n$ and $\mathbb{R}$, respectively. We fix $C^\infty$-functions $g : U \to \mathbb{R}$ and $h : V \to \mathbb{R}$. Then the composition function

$$U \cap g^{-1}(V) \to \mathbb{R},\ x \mapsto h(g(x))$$

of $g|_{g^{-1}(V)}$ and $h$ is $C^\infty$ on $U \cap g^{-1}(V)$.

**Example 3.2.4.** Let

$$U := \{x \in \mathbb{R}^n \mid \sum_i x_i^2 < 1\} \subset \mathbb{R}^n.$$

Then the function

$$f : U \to \mathbb{R},\ x \mapsto \sqrt{1 - \sum_i x_i^2}$$

is $C^\infty$ on $U$.

*Proof.* Define $g : U \to \mathbb{R}$ and $h : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$g : U \to \mathbb{R},\ x \mapsto 1 - \sum_i x_i^2,$$
$$h : \mathbb{R}_{>0} \to \mathbb{R},\ t \mapsto t^{1/2}.$$

Then $g$ and $h$ are both $C^\infty$ (by Propositions 3.2.1 and 3.2.2). Note that $g^{-1}(\mathbb{R}_{>0}) = U$ and our function $f$ is a composition of $g$ and $h$. Therefore $f$ is $C^\infty$ by Proposition 3.2.3. □

**Example 3.2.5.** Let us define

$$\rho : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} e^{-\frac{1}{x}} & (\text{ if } x > 0), \\ 0 & (\text{ if } x \leq 0). \end{cases}$$

Then $\rho$ is $C^\infty$ on $\mathbb{R}$. Note that $\rho$ does not admit the Taylor series expansion at $x = 0$.

The following theorem will play important roles in our lectures:

**Theorem 3.2.6.** Let us fix $n \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{R}^n$ and $0 < r_1 < r_2$. Then there exists a $C^\infty$-function $b$ on $\mathbb{R}^n$ satisfying the following two conditions:

1. $b(x) = 1$ for any $x \in \mathbb{R}^n$ with $\|x - p\| \leq r_1$, where we put $\|y\| := \sqrt{\sum_i y_i^2}$.
2. $b(x) = 0$ for any $x \in \mathbb{R}^n$ with $\|x - p\| \geq r_2$.

## 3.3 The ℝ-algebras of multivariable smooth functions

Throughout this section, we fix $n \in \mathbb{Z}_{\geq 0}$ and an open set $U$ in $\mathbb{R}^n$. Recall that we use the symbol $C(U)$ for the $\mathbb{R}$-algebra of all continuous real-valued functions on $U$.

**Definition 3.3.1.** For each $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the set of all $C^k$-functions on $U$ is denoted by $C^k(U)$.

Note that $C^\infty(U) = \bigcap_{k \in \mathbb{Z}_{\geq 0}} C^k(U)$.

The theorem below is one of the most important theorems in our lectures.

**Theorem 3.3.2.** For each $k \in \mathbb{Z}_{\geq 0}$, $C^k(U)$ is a sub $\mathbb{R}$-algebra of $C(U)$.

**Corollary 3.3.3.** $C^\infty(U)$ is a sub $\mathbb{R}$-algebra of $C(U)$. In particular, $C^\infty(U)$ itself is an $\mathbb{R}$-algebra.

To prove Theorem 3.3.2, we give the following proposition:

**Proposition 3.3.4.** Let us fix $k \geq 1$, $i = 1, \ldots, n$, $f, g \in C^k(U)$ and $\lambda \in \mathbb{R}$. Then

$$\frac{\partial(f+g)}{\partial x_i}, \ \frac{\partial(\lambda f)}{\partial x_i}, \ \frac{\partial(f \cdot g)}{\partial x_i}$$

are all well-defined as functions on $U$, and the following holds:

- $\frac{\partial(f+g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}$.
- $\frac{\partial(\lambda f)}{\partial x_i} = \lambda \cdot \frac{\partial f}{\partial x_i}$.
- $\frac{\partial(f \cdot g)}{\partial x_i} = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}$.

*Hint of the proof of Theorem 3.3.2.* Theorem 3.3.2 can be proved by induction of $k$ by applying Proposition 3.3.4. □

We note that as the following proposition follows from Proposition 3.3.4.

**Proposition 3.3.5.** Let $k = \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Then for each $i = 1, \ldots, n$,

$$\frac{\partial}{\partial x_i} : C^k(U) \to C^{k-1}(U), \ f \mapsto \frac{\partial f}{\partial x_i}$$

is linear (if $k = \infty$, we put $k - 1 := \infty$). Furthermore, the following equality (which will be called the Leibniz rule):

$$\frac{\partial}{\partial x_i}(f \cdot g) = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}.$$

It should be remarked that one cannot expect that $C^k(U)$ is finite-dimensional as below.

**Proposition 3.3.6.** $C^\infty(U)$ is not finite-dimensional as real vector space if $n \geq 1$ and $U \neq \emptyset$.

## 3.4 Exercise

Let us fix $n \in \mathbb{Z}_{\geq 0}$ and an non-empty open set $U$ of $\mathbb{R}^n$.

15. (Analysis) Show that any $C^1$-function on $U$ is totally differentiable.

16. (Analysis) Show that any totally differentiable function on $U$ is continuous.

17. (Analysis) Let $k \in \mathbb{Z}_{\geq 1}$. Show that any $C^k$-function on $U$ is also $C^{k-1}$ on $U$ (Proposition 3.1.3).

18. (Analysis) Let $k \geq 1$ and $i = 1, \ldots, n$. Show that the following holds for each $f, g \in C^k(U)$, $\lambda \in \mathbb{R}$ (Proposition 3.3.4).

$$\frac{\partial(f+g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}, \tag{3.1}$$

$$\frac{\partial(\lambda f)}{\partial x_i} = \lambda\frac{\partial f}{\partial x_i}, \tag{3.2}$$

$$\frac{\partial(f \cdot g)}{\partial x_i} = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}. \tag{3.3}$$

19. (Analysis) Show that any $n$ variable polynomial function is $C^\infty$ on $\mathbb{R}^n$ (Proposition 3.2.1).

20. (Analysis) Fix $\alpha$ as a real number.

    (a) For each positive real number $t$, describe the definition of the real number $t^\alpha$, and discuss the wellness of the definition.

    (b) Show that the function

    $$\phi_\alpha : \mathbb{R}_{>0} \to \mathbb{R},\ t \mapsto t^\alpha$$

    is $C^\infty$ on $\mathbb{R}_{>0}$ (Proposition 3.2.2).

21. (Analysis) Let $V$ be a non-empty open set in $\mathbb{R}$. For each pair of functions $g : U \to \mathbb{R}$ and $h : V \to \mathbb{R}$, we define

$$h \circ g : U \cap g^{-1}(V) \to \mathbb{R},\ x \mapsto h(g(x)).$$

For each $k \in \mathbb{Z}_{\geq 0}$, we shall consider the following claim $P_k$:

**Claim $P_k$:** Let $g : U \to \mathbb{R}$ and $h : V \to \mathbb{R}$ be both $C^k$-functions. Then the function $h \circ g$ is also $C^k$ on $U \cap g^{-1}(V)$.

(a) Fix $p \in g^{-1}(V)$. Suppose that the function $g : U \to \mathbb{R}$ is partially differentiable at $p$, and the function $h : V \to \mathbb{R}$ is differentiable at $g(p) \in V$. Show that $h \circ g$ is partially differentiable at $p$. Furthermore, write the partial derivative of $h \circ g$ at $p$ in terms of the partial derivative of $g$ at $p$ and the derivative of $h$ at $g(p)$ (the chain rule).

(b) Prove that the claim $P_k$ is true for each $k \in \mathbb{Z}_{\geq 0}$.

(c) Suppose that the functions $g : U \to \mathbb{R}$ and $h : V \to \mathbb{R}$ are both $C^\infty$. Show that the function $h \circ g$ is $C^\infty$ on $U \cap g^{-1}(V)$ (Proposition 3.2.3).

22. For each one variable real polynomial $P(t)$ with a variable $t$, let us define the function

$$f_P : \mathbb{R} \to \mathbb{R}, \ x \mapsto \begin{cases} P(\frac{1}{x})e^{-1/x} & (x > 0) \\ 0 \quad (x \leq 0). \end{cases}$$

(a) Show that $f_P$ is continuous on $\mathbb{R}$ for any $P(t)$.

(b) Show that $f_P$ is $C^\infty$ on $\mathbb{R}$ for any $P(t)$ (a generalization of Example 3.2.5).

(c) Show that the following function does not admit Taylor series expansions at $x = 0$:

$$\rho : \mathbb{R} \to \mathbb{R}, \ x \mapsto \begin{cases} e^{-1/x} & (x > 0) \\ 0 \quad (x \leq 0). \end{cases}$$

23. Let $p \in \mathbb{R}^n$ and $r_1, r_2 \in \mathbb{R}$ with $0 < r_1 < r_2$. Show that there exists a $C^\infty$-function on $\mathbb{R}^n$

$$b = b_{p,r_1,r_2} : \mathbb{R}^n \to \mathbb{R}$$

satisfying the following two conditions (Theorem 3.2.6):

**Condition (i):** $b(x) = 1$ for any $x \in \mathbb{R}^n$ with $\|x - p\| \leq r_1$.

**Condition (ii):** $b(x) = 0$ for any $x \in \mathbb{R}^n$ with $\|x - p\| \geq r_2$.

24. Let $p \in U$, $r_1, r_2, r_3 \in \mathbb{R}$ with $0 < r_1 < r_2 < r_3$ and $\mathcal{U}_{r_3}(p) \subset U$, where $\mathcal{U}_{r_3}(p)$ denotes the open ball of radius $r_3$ centered at $p$ (in $\mathbb{R}^n$). Furthermore, we fix a $C^\infty$-function $b_{p,r_1,r_2}$ on $\mathbb{R}^n$ satisfying the two conditions stated in the previous problem. For each $f \in C^\infty(U)$, we also define the function

$$\tilde{f} : \mathbb{R}^n \to \mathbb{R},\ x \mapsto \begin{cases} f(x) \cdot b_{p,r_1,r_2}(x) & (\text{if } x \in U), \\ 0 \quad (\text{if } x \notin U). \end{cases}$$

Show that the following holds for any $f \in C^\infty(U)$:

(a) $\tilde{f}$ is $C^\infty$ on $\mathbb{R}^n$.

(b) For each $i = 1, \dots, n$, $\frac{\partial f}{\partial x_i}(p) = \frac{\partial \tilde{f}}{\partial x_i}(p)$.

25. Show that $C^k(U)$ is a sub $\mathbb{R}$-algebra of $C(U)$ (Theorem 3.3.2).

26. Show that $C^\infty(U)$ is a sub $\mathbb{R}$-algebra of $C(U)$ (Corollary 3.3.3).

27. Fix $n \geq 1$. Show that $C^\infty(U)$ is not finite-dimensional as a real vector space (Proposition 3.3.6).

# Chapter 4

# Directional differentiations (on Euclidean spaces)

## 4.1 Definition and some properties of directional differentiations

Throughout this section, we fix $n \in \mathbb{Z}_{\geq 0}$, a non-empty open set $U$ of $\mathbb{R}^n$, a point $p \in U$ and a vector $v \in \mathbb{R}^n$. We write $\{e_1, \ldots, e_n\}$ for the standard basis of the vector space $\mathbb{R}^n$ and $C^\infty(U)$ the $\mathbb{R}$-algebra of smooth functions on $U$.

**Definition 4.1.1.** For each smooth function $f \in C^\infty(U)$, we define

$$v_p(f) := \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h} \in \mathbb{R}.$$

Such $v_p(f)$ is called the $v$-directional differentiation of $f$ at $p$.

**Proposition 4.1.2.** In the setting in Definition 4.1.1, $v_p(f) \in \mathbb{R}$ is well-defined. Furthermore, by putting

$$v = \sum_{i=1}^{n} a_i e_i \quad (\text{ for } a_i \in \mathbb{R}),$$

the equality below holds:

$$v_p(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(p)$$

It should be emphasized that the proposition above is non-trivial. The key is that any $C^1$-function is "totally differentiable".

**Proposition 4.1.3.** The map $v_p : C^\infty(U) \to \mathbb{R}$ is a linear map.

**Proposition 4.1.4.** Fix any $f, g \in C^\infty(U)$. Then the equality below (which will be called the Leibniz rule at the point $p$) holds:

$$v_p(f \cdot g) = v_p(f) \cdot g(p) + f(p) \cdot v_p(g).$$

Let us introduce the following notion:

**Definition 4.1.5.** For each $i = 1, \ldots, n$, we put

$$\left(\frac{\partial}{\partial x_i}\right)_p : C^\infty(U) \to \mathbb{R},\ f \mapsto \frac{\partial f}{\partial x_i}(p).$$

We note the following holds:

**Proposition 4.1.6.** For each $i = 1, \ldots, n$,

$$(e_i)_p = \left(\frac{\partial}{\partial x_i}\right)_p$$

as linear maps from $C^\infty(U)$ to $\mathbb{R}$.

## 4.2 Tangent vectors and tangent spaces

Throughout this section, we fix $n \in \mathbb{Z}_{\geq 0}$, a non-empty open set $U$ of $\mathbb{R}^n$ and a point $p \in U$. We write $\{e_1, \ldots, e_n\}$ for the standard basis of the vector space $\mathbb{R}^n$ and $C^\infty(U)$ the $\mathbb{R}$-algebra of smooth functions on $U$. The set of all linear maps from $C^\infty(U)$ to $\mathbb{R}$ is denoted by

$$\mathcal{L}(C^\infty(U), \mathbb{R}) := \{\eta : C^\infty(U) \to \mathbb{R} \mid \eta \text{ is linear}\}.$$

Recall that $\mathcal{L}(C^\infty(U), \mathbb{R})$ can be considered as a vector space in the sense of Proposition 2.1.6. Usually, an element of $\mathcal{L}(C^\infty(U), \mathbb{R})$ is called *a linear functional* on $C^\infty(U)$.

**Definition 4.2.1.** We say that a linear functional $\eta : C^\infty(U) \to \mathbb{R}$ satisfies "the Leibniz rule at $p \in U$" if the following equality holds for each $f, g \in C^\infty(U)$:

$$\eta(f \cdot g) = \eta(f) \cdot g(p) + f(p) \cdot \eta(g).$$

Let us define the tangent space of $U$ at $p$ as below:

**Definition 4.2.2.** The tangent space of $U$ at the point $p$ is defined by

$$T_pU := \{\eta \in \mathcal{L}(C^\infty(U), \mathbb{R}) \mid \eta \text{satisfies the Leibniz rule at } p\}$$

Each element of $T_pU$ is called a tangent vector of $U$ at $p$.

We note that the following holds:

**Proposition 4.2.3** (cf. Propositions 4.1.3 and 4.1.4)**.** For each $v \in \mathbb{R}^n$, the directional differentiation

$$v_p : C^\infty(U) \to \mathbb{R}$$

is a tangent vector of $U$ at $p$, i.e. a member of $T_pU$.

**Question:** Does there exist a tangent vector which is not a directional differentiation?

**Answer:** No! Surprisingly, the following holds:

$$T_pU = \{v_p \mid v \in \mathbb{R}^n\}.$$

This claims that directional differentiations can be characterized by the linearity and the Leibniz rule, algebraically (see the next section).

## 4.3 Tangent vectors vs directional differentiations

In this section, we study a map

$$\mathbb{R}^n \to T_pU,\ v \mapsto v_p$$

defined in the previous subsection.

First, we note the following:

**Proposition 4.3.1.** $T_pU$ is a linear subspace of $\mathcal{L}(C^\infty(U), \mathbb{R})$. In particular, $T_pU$ itself is also a real vector space.

**Definition 4.3.2.** We use the symbol $\Psi_{p,U}$ for the map

$$\mathbb{R}^n \to T_pU, \ v \mapsto v_p.$$

**Proposition 4.3.3.** The map $\Psi_{p,U} : \mathbb{R}^n \to T_pU$ is a linear map.

Hint: Proposition 4.1.2.

**Proposition 4.3.4.** The map $\Psi_{p,U} : \mathbb{R}^n \to T_pU$ is injective.

*Proof.* We only need to show that $\operatorname{Ker}\Psi_{p,U} = 0$. Take any $v \in \operatorname{Ker}\Psi_{p,U}$, that is, $v_p = 0$ as a linear functional on $C^\infty(U)$. Our goal is to show that $v = 0$ as in $\mathbb{R}^n$. Put $v = \sum_i a_i e_i$. Then we only need to show that $a_i = 0$ for each $i$. Let us fix $i$. We shall define the $C^\infty$-function

$$\boldsymbol{x}_i : U \to \mathbb{R}, \ x \mapsto x_i,$$

then we have

$$a_i = v_p(\boldsymbol{x}_i).$$

By our assumption, we have $a_i = 0$. □

Surprisingly, the following theorem holds:

**Theorem 4.3.5.** The map $\Psi_{p,U} : \mathbb{R}^n \to T_pU$ is surjective, and in particular $\Psi_{p,U}$ is a linear isomorphism.

A proof of the theorem above is omitted in this note (we give it in the Japanese version (see Sections 4.4 and 4.5 of the Japanese version)).

**Corollary 4.3.6.** The family of tangent vectors

$$\left\{ \left( \frac{\partial}{\partial x_i} \right)_p \right\}_{i=1,\dots,n}$$

gives a basis of $T_pU$.

## 4.4 Exercise

Let us fix $n \in \mathbb{Z}_{\geq 0}$, an open subset $U$ of $\mathbb{R}^n$ and a point $p \in U$.

28. (Analysis) Show the followings (Proposition 4.1.2; Hint: Exercise 15):

    (a) For each $v \in \mathbb{R}^n$ and each $f \in C^\infty(U)$ the map

    $$v_p(f) := \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h} \in \mathbb{R}$$

    is well-defined.

    (b) Let us write for $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$. Fix $a_1, \ldots, a_n \in \mathbb{R}$ and put $v := \sum_{i=1} a_i e_i \in \mathbb{R}^n$. Then for any $f \in C^\infty(U)$, the equality below holds:

    $$v_p(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(p).$$

29. Determine $v_p(f)$ for each setting below:

    (a) $U = \mathbb{R}^2$, $p = (1, 0)$, $f : U \to \mathbb{R},\ (x, y) \mapsto x^2 + y^2, \quad v = (1, 0)$.

    (b) $U = \mathbb{R}^2$, $p = (1, 0)$, $f : U \to \mathbb{R},\ (x, y) \mapsto x^2 + y^2, \quad v = (0, 1)$.

    (c) $U = \mathbb{R}^2$, $p = (1, 0)$, $f : U \to \mathbb{R},\ (x, y) \mapsto x^2 + y^2, \quad v = (1, 1)$.

    (d) $U = \mathbb{R}^2 \setminus \{(0, 0)\}$, $p = (1, 0)$, $f : U \to \mathbb{R},\ (x, y) \mapsto \sqrt{x^2 + y^2}$, $v = (1, 0)$.

30. Let $v \in \mathbb{R}^n$. Show that the map

    $$v_p : C^\infty(U) \to \mathbb{R},\ f \mapsto v_p(f)$$

    is linear (Proposition 4.1.3).

31. (Analysis) Let $v \in \mathbb{R}^n$, $f, g \in C^\infty(U)$. Show that the equality below holds:

    $$v_p(f \cdot g) = v_p(f) \cdot g(p) + f(p) \cdot v_p(g)$$

    (Proposition 4.1.4).

32. (Analysis) Show that $v_p \in T_p U$ for each $v \in \mathbb{R}^n$ (Proposition 4.2.3: easy).

33. Show that $T_pU$ is a linear subspace of $\mathcal{L}(C^\infty(U), \mathbb{R})$ (Proposition 4.3.1).

34. Show that the map

$$\Psi_{p,U} : \mathbb{R}^n \to T_pU,\ v \mapsto v_p$$

is linear (Proposition 4.3.3).

35. Show that the map

$$\Psi_{p,U} : \mathbb{R}^n \to T_pU,\ v \mapsto v_p$$

is injective (Proposition 4.3.4).

# Chapter 5

# Total differentiations of smooth maps (on Euclidean spaces)

## 5.1 Smooth maps

In this section, we give an "algebraic" definition of smooth maps.

Throughout this section, we fix $n_i \in \mathbb{Z}_{\geq 0}$ and open set $U_i \subset \mathbb{R}^{n_i}$ for $i = 1, 2$.

For a continuous map $\varphi : U_1 \to U_2$, we write

$$\varphi^* : C(U_2) \to C(U_1),\ f \mapsto f \circ \varphi$$

for the pullback by $\varphi$. Recall that $\varphi^*$ is an $\mathbb{R}$-algebra homomorphism from $C(U_2)$ to $C(U_1)$ (see Example 2.2.8).

Let us give an algebraic definition of smooth maps from $U_1$ to $U_2$:

**Definition 5.1.1.** A continuous map $\varphi : U_1 \to U_2$ is said to be $C^\infty$ or smooth if

$$\varphi^*(C^\infty(U_2)) \subset C^\infty(U_1).$$

**Proposition 5.1.2.** Assume that $\varphi : U_1 \to U_2$ is smooth. Then

$$\varphi^* : C^\infty(U_2) \to C^\infty(U_1),\ f \mapsto f \circ \varphi$$

is an $\mathbb{R}$-algebra homomorphism.

The following proposition gives an equivalent definition of smoothness of maps:

**Proposition 5.1.3.** Let $\varphi : U_1 \to U_2$ be a map. Then the following two conditions on $\varphi$ are equivalent:

**(1)** The map $\varphi : U_1 \to U_2$ is smooth in the sense of Definition 5.1.1 (i.e. $\varphi$ is continuous and $\varphi^*(C^\infty(U_2)) \subset C^\infty(U_1)$).

**(2)** We put
$$\varphi : U_1 \to U_2 \subset \mathbb{R}^{n_2},\ x \mapsto (\varphi_1(x), \dots, \varphi_{n_2}(x)).$$
Then $\varphi_1, \dots, \varphi_{n_1} \in C^\infty(U_1)$.

A sketch of proof of the proposition above will be given later.

**Example 5.1.4.** Let $n_1 = n_2 = 2$ and put

$$U_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0\},$$
$$U_2 := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 < 1,\ y_2 > 0\}.$$

Then
$$\varphi : U_1 \to U_2, (x_1, x_2) \mapsto (x_2, \sqrt{1 - (x_1^2 + x_2^2)})$$
is well-defined and smooth (cf. Proposition 5.1.3). Note that the smooth map above will be applied to obtain “the smooth structure” on the 2-sphere.

To give a proof of Proposition 5.1.3, we first give the following lemma and two propositions:

**Lemma 5.1.5.** For each $j = 1, \dots, n_2$, we put

$$\boldsymbol{y}_j : U_2 \to \mathbb{R},\ y \mapsto y_j.$$

Then $\varphi_j = \varphi^*(\boldsymbol{y}_j)$ as functions on $U_1$.

**Proposition 5.1.6** (Topology)**.** Let $X$ be a topological space and $\{Y_\lambda\}_{\lambda\in\Lambda}$ a family of topological spaces. We write $\prod_{\lambda\in\Lambda} Y_\lambda$ for the direct product space of $\{Y_\lambda\}_{\lambda\in\Lambda}$ (equipped with the direct product topology). Then for each map

$$\varphi : X \to \prod_{\lambda\in\Lambda} Y_\lambda,\ x \mapsto (\varphi_\lambda(x))_{\lambda\in\Lambda},$$

the following two conditions are equivalent:

1. The map $\varphi$ is continuous.

2. For each $\lambda \in \Lambda$, the map $\varphi_\lambda : X \to Y_\lambda$ is continuous.

**Proposition 5.1.7** (Analysis: the chain rule)**.** Let

$$\varphi : U_1 \to U_2 \subset \mathbb{R}^{n_2},\ x \mapsto (\varphi_1(x), \dots, \varphi_{n_2}(x))$$

be a map. Assume that $\varphi_1, \dots, \varphi_{n_2} \in C^1(U_2)$. Then for any $f \in C^1(U_2)$, $\varphi^*(f) \in C^1(U_1)$ and for each $j = 1, \dots, n_2$, the equality below holds:

$$\frac{\partial(\varphi^*(f))}{\partial x_j} = \sum_{i=1}^{n_2} (\varphi^*(\frac{\partial f}{\partial y_i})) \cdot \frac{\partial \varphi_i}{\partial x_j}$$

as functions on $U_1$.

Let us give a sketch of proof of Proposition 5.1.3:

*Proof.* First, let us give a proof of the implication (1) $\Rightarrow$ (2) in Proposition 5.1.3. Assume (1). Then by Lemma 5.1.5, we have $\varphi_j = \varphi^*(\boldsymbol{y}_j)$ for each $j$. Since $\boldsymbol{y}_j \in C^\infty(U_2)$, we have $\varphi_j = \varphi^*(\boldsymbol{y}_j) \in C^\infty(U_1)$ by the assumption. This proves (2).

Conversely, suppose (2). The continuity of $\varphi$ comes from Proposition 5.1.6. To prove (1), we only need to show that $\varphi^*(C^k(U_2)) \subset C^k(U_1)$ for any $k \in \mathbb{Z}_{\geq 0}$. By applying Proposition 5.1.7, one can prove the claim above by induction of $k$. □

Let us study relationship between smooth functions and smooth maps. We consider the cases where $n_2 = 1$ and $U_2 = \mathbb{R}(= \mathbb{R}^{n_2})$, and put $U := U_1$. Then for each map $f : U \to \mathbb{R}$, the following two conditions are equivalent:

1. $f \in C^\infty(U)$ (i.e. $f$ is a smooth function on $U$).

2. $f$ is a smooth map from $U$ to $\mathbb{R}$.

It should be remarked that for a pair of smooth functions $f, g \in C^\infty(U)$, the pointwise product $f \cdot g \in C^\infty(U)$ is defined. However, by considering general open sets $U_i \subset \mathbb{R}^{n_i}$, for a pair of smooth maps $\varphi, \psi : U_1 \to U_2$, the "pointwise product" $\varphi \cdot \psi$ cannot be defined (because $U_2$ does not admit a "product").

## 5.2 Total differentiations of Smooth maps

In this section, we give an algebraic definition of the total differentiations of smooth maps.

Throughout this section, we fix $n_i \in \mathbb{Z}_{\geq 0}$ and open set $U_i \subset \mathbb{R}^{n_i}$ for $i = 1, 2$. A smooth map $\varphi : U_1 \to U_2$ and a point $p \in U_1$ are also fixed.

We first recall that we have the $\mathbb{R}$-algebra homomorphism (cf. Proposition 5.1.2):

$$\varphi^* : C^\infty(U_2) \to C^\infty(U_1), \ f \mapsto f \circ \varphi.$$

**Proposition 5.2.1.** $\eta \circ \varphi^* \in T_{\varphi(p)} U_2$ for each $\eta \in T_p U_1$.

**Definition 5.2.2.** The map

$$(d\varphi)_p : T_p U_1 \to T_{\varphi(p)} U_2, \ \eta \mapsto \eta \circ \varphi^*$$

is called the total differentiation of $\varphi$ at $p \in U_1$.

**Proposition 5.2.3.** The total differentiation $(d\varphi)_p : T_p U_1 \to T_{\varphi(p)} U_2$ is a linear map.

The following theorem (a proof is omitted here: see Section 5.5 of the Japanese version) gives an analytic characterization of the total differentiation $(d\varphi)_p$:

**Theorem 5.2.4.** We use the symbol $\|v\| := \sqrt{\sum_i v_i^2}$ for $v \in \mathbb{R}^n$.

1. The equality below holds:

$$\lim_{v \to 0 \ (v \in \mathbb{R}^{n_1} \setminus \{0\})} \frac{\|\varphi(p+v) - \varphi(p) - (d\varphi)_p v\|}{\|v\|} = 0.$$

2. Let $A : T_p U_1 \to T_{\varphi(p)} U_2$ be a linear map. Suppose that

$$\lim_{v \to 0 \ (v \in \mathbb{R}^{n_1} \setminus \{0\})} \frac{\|\varphi(p+v) - \varphi(p) - Av\|}{\|v\|} = 0.$$

   Then $A = (d\varphi)_p$.

It should be emphasized that the total differentiation of $\varphi$ at $p$ gives "the linear approximation" of the map $\varphi$.

## 5.3 Matrix representations of Total differentiations

In this section, we study matrix representations of total differentiations of smooth maps.

As in the previous section, we fix $n_i \in \mathbb{Z}_{\geq 0}$ and open set $U_i \subset \mathbb{R}^{n_i}$ for $i = 1, 2$. A smooth map $\varphi : U_1 \to U_2$ and a point $p \in U_1$ are also fixed.

Throughout this section, we use the symbols

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n_1}}$$

for partial derivatives on $U_1 \subset \mathbb{R}^{n_1}$ and

$$\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n_2}}$$

for partial derivatives on $U_2 \subset \mathbb{R}^{n_2}$.

Recall that by Corollary 4.3.6, we have the following:

- $$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_{n_1}}\right)_p$$

  gives a basis of the tangent space $T_p U_1$.

- $$\left(\frac{\partial}{\partial y_1}\right)_{\varphi(p)}, \dots, \left(\frac{\partial}{\partial y_{n_2}}\right)_{\varphi(p)}$$

  gives a basis of the tangent space $T_{\varphi(p)} U_2$.

The purpose of this section is to give the matrix representation of the linear map

$$(d\varphi)_p : T_p U_1 \to T_{\varphi(p)} U_2.$$

Let us write

$$\varphi(x) = (\varphi_1(x), \dots, \varphi_{n_2}(x))$$

for $x \in U_1$. Then by Proposition 5.1.2, we have $\varphi_i \in C^\infty(U_1)$.

**Definition 5.3.1.** The matrix defined below is called the Jacobi matrix of $\varphi$ at $p$:

$$(J\varphi)_p := \left(\frac{\partial \varphi_i}{\partial x_j}(p)\right)_{i=1,\dots,n_1;j=1,\dots,n_1} \in M(n_2, n_1 : \mathbb{R}).$$

**Theorem 5.3.2.** The Jacobi matrix $(J\varphi)_p$ gives the matrix representation of the total differentiation

$$(d\varphi)_p : T_pU_1 \to T_{\varphi(p)}U_2.$$

with respect to the basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_{n_1}}\right)_p$$

of $T_pU_1$ and that

$$\left(\frac{\partial}{\partial y_1}\right)_{\varphi(p)}, \dots, \left(\frac{\partial}{\partial y_{n_2}}\right)_{\varphi(p)}$$

of $T_{\varphi(p)}U_2$, that is, for each vector

$$\eta = \sum_j a_j \left(\frac{\partial}{\partial x_j}\right)_p \in T_pU_1,$$

the vector $(d\varphi)_p(\eta) \in T_{\varphi(p)}U_2$ can be written as

$$(d\varphi)_p(\eta) = \sum_i b_i \left(\frac{\partial}{\partial y_i}\right)_{\varphi(p)}$$

by putting

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{n_2} \end{pmatrix} = (J\varphi)_p \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix}.$$

**Example 5.3.3.** Let $n_1 = 2$, $n_2 = 3$, $U_1 = \mathbb{R}^2$, $U_2 = \mathbb{R}^3$ and

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^3, \ (x_1, x_2) \mapsto (\cos x_1, \sin x_2, x_1 + x_2).$$

For each $p = (p_1, p_2) \in \mathbb{R}^2$, the Jacobi matrix $(J\varphi)_p$ can be computed as

$$(J\varphi)_p = \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1}(p) & \frac{\partial\varphi_1}{\partial x_2}(p) \\ \frac{\partial\varphi_2}{\partial x_1}(p) & \frac{\partial\varphi_2}{\partial x_2}(p) \\ \frac{\partial\varphi_3}{\partial x_1}(p) & \frac{\partial\varphi_3}{\partial x_2}(p) \end{pmatrix} = \begin{pmatrix} -\sin p_1 & 0 \\ 0 & \cos p_2 \\ 1 & 1 \end{pmatrix}.$$

Therefore, for each

$$\eta = a_1 \left(\frac{\partial}{\partial x_1}\right)_p + a_2 \left(\frac{\partial}{\partial x_2}\right)_p \in T_p\mathbb{R}^2,$$

we have

$$(d\varphi)_p(\eta) = -a_1(\sin p_1) \left(\frac{\partial}{\partial y_1}\right)_{\varphi(p)} + a_2(\cos p_2) \left(\frac{\partial}{\partial y_2}\right)_{\varphi(p)} + (a_1 + a_2) \left(\frac{\partial}{\partial y_3}\right)_{\varphi(p)}$$

since

$$(J\varphi)_p \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_1 \sin p_1 \\ a_2 \cos p_2 \\ a_1 + a_2 \end{pmatrix}.$$

Let us give a proof of Theorem 5.3.2. First, we observe the following lemma:

**Lemma 5.3.4.** For each $\eta \in T_{\varphi(p)}U_2$,

$$\eta = \sum_{i=1}^{n_2} \eta(\boldsymbol{y}_i) \left(\frac{\partial}{\partial y_i}\right)_{\varphi(p)},$$

where

$$\boldsymbol{y}_i : U_2 \to \mathbb{R},\ y \mapsto y_i.$$

*Proof of Theorem 5.3.2.* Take any $j = 1, \ldots, n_1$. Our goal is to show the following:

$$(d\varphi)_p \left(\frac{\partial}{\partial x_j}\right)_p = \sum_{i=1}^{n_2} ((J\varphi)_p)_{i,j} \left(\frac{\partial}{\partial y_i}\right)_{\varphi(p)}.$$

Recall that

$$((J\varphi)_p)_{i,j} = \frac{\partial \varphi_i}{\partial x_j}(p).$$

Take any $i = 1, \ldots, n_2$. By the lemma stated above, it suffices to show the following:

$$\left((d\varphi)_p \left(\frac{\partial}{\partial x_j}\right)_p\right)(\boldsymbol{y}_i) = \frac{\partial \varphi_i}{\partial x_j}(p).$$

One can see that

$$\begin{aligned}
(\text{LHS}) &= \left(\left(\frac{\partial}{\partial x_j}\right)_p \circ \varphi^*\right)(\boldsymbol{y}_i) \\
&= \left(\frac{\partial}{\partial x_j}\right)_p (\varphi^*(\boldsymbol{y}_i)) \\
&= \left(\frac{\partial}{\partial x_j}\right)_p (\varphi_i) \quad \text{(see Lemma 5.1.5)}, \\
&= (\text{RHS}).
\end{aligned}$$

□

## 5.4 Compositions of smooth maps and their total differentiations

In this section, we study the following two important properties of smooth maps and total differentiations:

- Compositions of smooth maps are also smooth.
- Total differentiations of compositions are compositions of total differentiations.

Throughout this section, we fix $n_i \in \mathbb{Z}_{\geq 0}$, an open set $U_i \subset \mathbb{R}^{n_i}$ for $i = 1, 2, 3$, and consider smooth maps $\varphi : U_1 \to U_2$ and $\psi : U_2 \to U_3$.

The following is the first main theorem of this section:

**Theorem 5.4.1.** The composition

$$\psi \circ \varphi : U_1 \to U_3$$

is smooth.

To prove the theorem above, we observe the lemma below:

**Lemma 5.4.2.**

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$

as maps from $C(U_3)$ to $C(U_1)$.

*Proof of Theorem 5.4.1.* Take any $f \in C^\infty(U_3)$. We only need to show that $(\psi \circ \varphi)^*(f) \in C^\infty(U_1)$. Note that $\varphi^*(\psi^*(f)) \in C^\infty(U_1)$ since $\psi$ and $\varphi$ are both smooth maps. By the lemma stated above, we have $(\psi \circ \varphi)^*(f) = \varphi^*(\psi^*(f))$. This completes the proof. □

The following is the second main theorem of this section:

**Theorem 5.4.3.** Fix $p \in U_1$. Then

$$(d(\psi \circ \varphi))_p = (d\psi)_{\varphi(p)} \circ (d\varphi)_p$$

as maps from $T_pU_1$ to $T_{(\psi\circ\varphi)(p)}U_3$.

*Proof of Theorem 5.4.3.* Take any $\eta \in T_pU_1$. Then our goal is to show the following:

$$(d(\psi \circ \varphi))_p(\eta) = ((d\psi)_{\varphi(p)} \circ (d\varphi)_p)(\eta)$$

as in $T_{(\psi\circ\varphi)(p)}U_3$. One can see that

$$\begin{aligned} \text{(LHS)} &= \eta \circ (\psi \circ \varphi)^* \\ &= \eta \circ (\varphi^* \circ \psi^*) \quad \text{(see Lemma 5.4.2)}, \\ &= ((d\varphi)_p(\eta)) \circ \psi^* \\ &= (d\psi)_{\varphi(p)}((d\varphi)_p(\eta)) \quad \text{(note that } (d\varphi)_p(\eta) \in T_{\varphi(p)}U_2), \\ &= \text{(RHS)}. \end{aligned}$$

□

For the Jacobi matrices, as a corollary to Theorems 5.3.2 and 5.1.3, we have the following:

**Corollary 5.4.4.**

$$(J(\psi \circ \varphi)_p) = (J\psi)_{\varphi(p)} \cdot (J\varphi)_p.$$

It should be noted that the corollary above claims the chain rule for partial derivatives of compositions of smooth maps.

## 5.5 Exercise

Let us fix $n_i \in \mathbb{Z}_{\geq 0}$ and an open subset $U_i$ of $\mathbb{R}^{n_i}$ for each $i = 1, 2$. We use the symbols $\{\boldsymbol{x}_j\}_{j=1,\dots,n_1}$, $\{\boldsymbol{y}_i\}_{i=1,\dots,n_2}$ for the system of coordinate functions on $U_1$ and that of $U_2$.

36. Suppose that $\varphi : U_1 \to U_2$ is a smooth map. Show that

$$\varphi^* : C^\infty(U_2) \to C^\infty(U_1)$$

is an $\mathbb{R}$-algebra homomorphism (Proposition 5.1.2).

37. (Topology) Let $X$ be a topological space, $\{Y_\lambda\}_{\lambda\in\Lambda}$ a family of topological space. We use the symbol $\prod_{\lambda \imath \Lambda} Y_\lambda$ for the direct product space of $\{Y_\lambda\}_{\lambda\in\Lambda}$ equipped with the product topology. Fix a map

$$\varphi : X \to \prod_{\lambda \imath \Lambda} Y_\lambda,\ x \mapsto (\varphi_\lambda(x))_{\lambda\in\Lambda}.$$

Show the following holds (Proposition 5.1.6):

(a) Assume that $\Lambda$ is a finite set. Show that the following two conditions on $\varphi$ are equivalent:

**Condition (i):** $\varphi : X \to \prod_{\lambda \imath \Lambda} Y_\lambda$ is continuous.

**Condition (ii):** For any $\lambda \in \Lambda$, the map$\varphi_\lambda : X \to Y_\lambda$ is continuous.

(b) What is the definition of the product topology on $\prod_{\lambda \imath \Lambda} Y_\lambda$ in the cases where $\Lambda$ is not finite? Show that the similar equivalence above for general $\Lambda$.

38. (Analysis; the Chain rule) Write

$$\varphi : U_1 \to U_2,\ x \mapsto (\varphi_1(x), \dots, \varphi_{n_2}(x))$$

and assume $\varphi_1, \dots, \varphi_{n_2} \in C^1(U_1)$. Show that for any $f \in C^1(U_2)$ について, $\varphi^*(f) \in C^1(U_1)$, furthermore, for each $j = 1, 2, \dots, n_1$, the equality below holds

$$\frac{\partial(\varphi^*(f))}{\partial x_j} = \sum_{i=1}^{n_2} \left(\varphi^*\left(\frac{\partial f}{\partial y_i}\right)\right) \cdot \frac{\partial \varphi_i}{\partial x_j}$$

on $U_1$ (Proposition 5.1.7).

39. Fix a map
$$\varphi : U_1 \to U_2,\ x \mapsto (\varphi_1(x), \ldots, \varphi_{n_2}(x)).$$
Show that the following two conditions on $\varphi$ are equivalent (Proposition 5.1.3):

**Condition (i):** $\varphi$ is smooth in the sense of Definition 5.1.1 (i.e. $\varphi$ is continuous and $\varphi^*(C^\infty(U_2)) \subset C^\infty(U_1)$).

**Condition (ii):** $\varphi_1, \ldots, \varphi_{n_2} \in C^\infty(U_1)$.

40. Let $n_1 = n_2 = 2$ and put
$$U_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1, x_1 > 0\}$$
$$U_2 = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 < 1, y_2 > 0\}.$$
   (a) Show that each $U_i$ is open in $\mathbb{R}^2$ respectively.
   (b) Show that the map
$$\varphi : U_1 \to U_2,\ (x_1, x_2) \mapsto \left(x_2, \sqrt{1 - (x_1^2 + x_2^2)}\right)$$
   is smooth (Example 5.1.4).

41. Let $\varphi : U_1 \to U_2$ be a smooth map and fix a point $p \in U_1$. Show $\eta \circ \varphi^* \in T_{\varphi(p)}U_2$ for each $\eta \in T_pU_1$ (Proposition 5.2.1).

42. Let $\varphi : U_1 \to U_2$ be a smooth map and fix a point $p \in U_1$. Show that the total differentiation
$$(d\varphi)_p : T_pU_1 \to T_{\varphi(p)}U_2,\ \eta \mapsto \eta \circ \varphi^*$$
of $\varphi$ at $p$ is a linear map (Proposition 5.2.3).

43. Fix $q \in U_2$. Show that for any $\eta \in T_qU_2$, the equality below holds
$$\eta = \sum_{i=1}^{n_2} \eta(\boldsymbol{y}_i) \left(\frac{\partial}{\partial y_i}\right)_q$$
by applying Corollary 4.3.6 (Lemma 5.3.4).

44. (Linear algebra) Let $V$, $W$ be real vector spaces with dimensions $n_1$ and $n_2$. We fix base $\{v_1, \dots, v_{n_1}\}$, $\{w_1, \dots, w_{n_2}\}$ of $V$, $W$. Let us also consider a linear map $\phi : V \to W$ and a matrix $A$ of size $n_2 \times n_1$. Show that the following two conditions on $(\varphi, A)$ are equivalent:

**Condition (i):** $A$ represents $\phi$ with respect to the base $\{v_1, \dots, v_{n_1}\}$ and $\{w_1, \dots, w_{n_2}\}$ i.e. for any

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix} \in \mathbb{R}^{n_1},$$

if we put

$$\phi(\sum_{j=1}^{n_1} a_j v_j) = \sum_{i=1}^{n_2} b_i w_i \text{ for } \begin{pmatrix} b_1 \\ \vdots \\ b_{n_2} \end{pmatrix} \in \mathbb{R}^{n_2},$$

then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{n_2} \end{pmatrix} = A \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix}.$$

**Condition (ii):** $\phi(v_j) = \sum_{i=1}^{n_2} A_{ij} w_i$ for any $j = 1, \dots, n_1$.

45. Let $\varphi : U_1 \to U_2$ be a smooth map and fix $p \in U_1$. Show that the Jacobi matrix

$$(J\varphi)_p := \left( \frac{\partial \varphi_i}{\partial x_j}(p) \right)_{i=1,\dots,n_2, j=1,\dots,n_1}$$

represents the linear map

$$(d\varphi)_p : T_p U_1 \to T_{\varphi(p)} U_2$$

with respect to the base $\left\{ \left( \frac{\partial}{\partial x_j} \right)_p \right\}_{j=1,\dots,n_1}$ and $\left\{ \left( \frac{\partial}{\partial y_i} \right)_{\varphi(p)} \right\}_{i=1,\dots,n_2}$ (Proposition 5.3.2).

46. Determine the Jacobi matrix $(J\varphi)_p$ for a smooth map $\varphi : U_1 \to U_2$ and a point $p \in U_1$ in the following situations:

(a) $n_1 = 2$, $n_2 = 3$, $U_1 = \mathbb{R}^2$, $U_2 = \mathbb{R}^3$,

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^3,\ (x_1, x_2) \mapsto (\cos x_1, \sin x_2, x_1 + x_2),$$

$p = (p_1, p_2) \in \mathbb{R}^2$ (Example 5.3.3).

(b) $n_1 = 1$, $n_2 = 2$, $U_1 = \mathbb{R}$, $U_2 = \mathbb{R}^2$,

$$\varphi : \mathbb{R} \to \mathbb{R}^2,\ x \mapsto (x^2, 2x),$$

$p \in \mathbb{R}$.

(c) $n_1 = 3$, $n_2 = 2$, $U_1 = \mathbb{R}^3$, $U_2 = \mathbb{R}^2$,

$$\varphi : \mathbb{R}^3 \to \mathbb{R}^2,\ (x_1, x_2, x_3) \mapsto (x_1^2, x_1 + x_2 + x_3),$$

$p = (p_1, p_2, p_3) \in \mathbb{R}^3$.

(d) $n_1 = 2$, $n_2 = 2$, $U_1 = \mathbb{R}^2 \setminus \{(0,0)\}$, $U_2 = \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\varphi : U_1 \to U_2,\ (x_1, x_2) \mapsto \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right),$$

$p = (p_1, p_2) \in U_1$.

**Setting:** For $i = 1, 2, 3$, let us fix $n_i \in \mathbb{Z}_{\geq 0}$ and an open set $U_i$ of $\mathbb{R}^{n_i}$. Furthermore, we also fix smooth maps $\varphi : U_1 \to U_2$ and $\psi : U_2 \to U_3$.

47. Show the equality

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \quad \text{as } C(U_3) \to C(U_1).$$

(Lemma 5.4.2).

48. Show that the map

$$\psi \circ \varphi : U_1 \to U_3$$

is smooth (Theorem 5.4.1).

49. Fix $p \in U_1$. Show the equality below:

$$(d(\psi \circ \varphi))_p = (d\psi)_{\varphi(p)} \circ (d\varphi)_p$$

(Theorem 5.4.3).

50. Can you find some relationship between "the chain rule" and the equality below?

$$(J(\psi \circ \varphi))_p = (J\psi)_{\varphi(p)} \cdot (J\varphi)_p$$

(Corollary 5.4.4).

# Chapter 6

# Local coordinate systems

In this section, we introduce the concept of local coordinate systems on topological spaces.

## 6.1 Some basic facts in Topology

In this section, we recall some basic facts in Topology which will be applied in this lecture course.

Throughout this section, topological spaces $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$ are fixed.

First, we recall the definition of relative topology:

**Definition 6.1.1.** For each subset $A$ of $X$, the topology

$$\mathcal{O}_X(A) := \{A \cap U \mid U \in \mathcal{O}_X\}$$

on $A$ is called the relative topology on $A$ induced from $\mathcal{O}_X$.

**Proposition 6.1.2.** Let us fix a subset $A$ of $X$. Then the relative topology on $A$ is the weakest topology on $A$ such that the inclusion map $A \hookrightarrow X$ is continuous.

**Proposition 6.1.3.** Fix subsets $A$ and $B$ of $X$ with $B \subset A$. Then

$$\mathcal{O}_X(B) = (\mathcal{O}_X(A))(B).$$

**Proposition 6.1.4.** Fix an open set $U$ of $X$ (i.e. $U \in \mathcal{O}_X$). For a subset $V$ of $U$, the following two conditions are equivalent:

1. $V$ is open in $U$ with respect to the relative topology (i.e. $V \in \mathcal{O}_X(U)$).

2. $V$ is open in $X$ (i.e. $V \in \mathcal{O}_X$).

It should be remarked that in the proposition above, the assumption that "$U$ is open in $X$" cannot be omitted. For example, if $U$ is NOT open in $X$, by putting $V = U$, we have an example of $V$ that $V$ is open in $U$ but not open in $X$.

Let us some basic facts for homeomorphisms:

**Proposition 6.1.5.** Let $\phi : X \to Y$ be a bijective continuous map. Then the following two conditions on $\phi$ are equivalent:

1. $\phi : X \to Y$ is a homeomorphism (i.e. the inverse map $\phi^{-1} : Y \to X$ is also continuous).

2. $\phi : X \to Y$ is an open map (i.e. for any open set $U$ of $X$, the image $\phi(U)$ is also open in $Y$).

**Proposition 6.1.6.** Fix a subset $A$ of $X$ and that $B$ of $Y$. We regard $A$ and $B$ are both topological spaces equipped with the relative topology.

1. Let $\phi : X \to Y$ be a continuous map such that $\phi(A) \subset B$. Then $\phi$ defines a continuous map from $A$ to $B$.

2. Let $\phi : X \to Y$ be a homeomorphism such that $\phi(A) = B$. Then $\phi$ defines a homeomorphism from $A$ to $B$.

## 6.2 Definition of local coordinate systems

The definition of local coordinate systems on a topological space is given in this section.

Throughout this section, we fix a topological space $M = (M, \mathcal{O}_M)$ and $n \in \mathbb{Z}_{\geq 0}$.

**Definition 6.2.1** (Local coordinate systems)**.** Fix an open set $O$ in $M$ and an open set $U$ of $\mathbb{R}^n$. We regard $O$ and $U$ are both topological spaces equipped with the relative topology. Let $\boldsymbol{u} : O \to U$ be a homeomorphism. Then the system $(O, U, \boldsymbol{u})$ is said to be an $n$-dimensional local coordinate system on $M$. Furthermore, $\boldsymbol{u}$ is called a coordinate on $O$.

It should be noted that sometimes the pair $(O, \boldsymbol{u})$ is called a local coordinate system on $M$ (an open set $U = \boldsymbol{u}(O)$ in $\mathbb{R}^n$ is omitted).

**Definition 6.2.2.** In this lecture course, we use the symbol

$$\mathcal{LC}(M; \mathbb{R}^n)$$

for the set of all $n$-dimensional local coordinate systems on $M$.

**Example 6.2.3.** Let us consider the situation that $M$ is an open set of $\mathbb{R}^n$. Then by putting $U = M$, we have a local coordinate system $(M, U = M, \mathrm{id}_M) \in \mathcal{LC}(M; \mathbb{R}^n)$.

**Example 6.2.4.** Suppose $n \in \mathbb{Z}_{\geq 1}$. Let us consider

$$S^n := \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}.$$

It should be noted that $S^n$ is not open in $\mathbb{R}^{n+1}$. We regard $S^n$ as a topological space equipped with the relative topology. Put

$$\begin{aligned} O &:= \{x \in S^n \mid x_{n+1} > 0\} \subset S^n, \\ U &:= \{u = (u_1, \dots, u_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i^2 < 1\} \subset \mathbb{R}^n, \\ \boldsymbol{u} &: O \to U,\ x \mapsto (x_1, \dots, x_n). \end{aligned}$$

Then $\boldsymbol{u}$ is well-defined and one can see that $(O, U, \boldsymbol{u}) \in \mathcal{LC}(S^n; \mathbb{R}^n)$ (with plenty of basic arguments).

The proposition below is useful:

**Proposition 6.2.5.** Let $(O, U, \boldsymbol{u}) \in \mathcal{LC}(M; \mathbb{R}^n)$ and $O_0$ is an open subset of $O$. Then

$$(O_0, \boldsymbol{u}(O_0), \boldsymbol{u}|_{O_0} : O_0 \to \boldsymbol{u}(O_0)) \in \mathcal{LC}(M; \mathbb{R}^n).$$

## 6.3 Exercise

**Setting on Problem 51 to 55:** We fix topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

51. (Topology) Let $A$ be a subset of $X$. The inclusion map is denoted by $\iota : A \hookrightarrow X$. Show that the relative topology $\mathcal{O}_X(A)$ is the weakest topology on $A$ such that the map $\iota : A \to X$ to be continuous (Proposition 6.1.2).

52. (Topology) Let $B \subset A \subset X$. Show that
$$\mathcal{O}_X(B) = (\mathcal{O}_X(A))(B)$$
(Proposition 6.1.3).

53. (Topology) Let $U \in \mathcal{O}_X$. For a subset $V$ of $U$, show that the following two conditions are equivalent (Proposition 6.1.4):

**Condition (i):** $V \in \mathcal{O}_X(U)$.

**Condition (ii):** $V \in \mathcal{O}_X$.

54. (Topology) Let $\phi : X \to Y$ be a bijective continuous map. Show that the following two conditions on $\phi$ are equivalent (Proposition 6.1.5):

**Condition (i):** $\phi : X \to Y$ is a homeomorphism (i.e. the inverse map $\phi^{-1} : Y \to X$ is also continuous).

**Condition (ii):** $\phi : X \to Y$ is an open map (i.e. for any open set $U$ of $X$, the image $\phi(U)$ is also open in $Y$).

55. (Topology) Let $A \subset X$ and $B \subset Y$. We regard $A$ and $B$ as topological spaces equipped with the relative topologies $\mathcal{O}_X(A)$ and $\mathcal{O}_Y(B)$. Show that the following holds (Proposition 6.1.6):

    (a) Fix a continuous map $\phi : X \to Y$ with $\phi(A) \subset B$. Then $\phi$ defines a continuous map from $A$ to $B$.

    (b) Fix a homeomorphism $\phi : X \to Y$ with $\phi(A) = B$. Then $\phi$ defines a homeomorphism from $A$ to $B$.

56. Let $M$ be a topological space, $n \in \mathbb{Z}_{\geq 0}$ and $(O, U, \boldsymbol{u}) \in \mathcal{LC}(M; \mathbb{R}^n)$. Fix an open set $O_0$ of $O$. Show that $(O_0, \boldsymbol{u}(O_0), \boldsymbol{u}|_{O_0} : O_0 \to \boldsymbol{u}(O_0)) \in \mathcal{LC}(M : \mathbb{R}^n)$ (Proposition 6.2.4).

**Setting in Exercise 57 to 61:** Fix $n \in \mathbb{Z}_{\geq 1}$ and put

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}.$$

We regard $S^n$ as a topological space equipped with the relative topology. Let us define

- $O = \{x \in S^n \mid x_{n+1} > 0\} \subset S^n$,
- $U = \{u \in \mathbb{R}^n \mid \sum_{i=1}^n u_i^2 < 1\} \subset \mathbb{R}^n$,
- $\boldsymbol{u} : O \to U,\ x \mapsto (x_1, \ldots, x_n)$.

57. (Topology) Show that $S^n$ is connected compact Hausdorff space.

58. Show that
$$\boldsymbol{u} : O \to U,\ x \mapsto (x_1, \ldots, x_n)$$
is well-defined as a map.

59. Show that $O$ is open in $S^n$. Furthermore, show that $U$ is open in $\mathbb{R}^n$ (Example 6.2.3).

60. Determine the inverse map $\boldsymbol{u} : O \to U$ (Example 6.2.3).

61. Show that $\boldsymbol{u} : O \to U$ is a homeomorphism (Example 6.2.3).

# Chapter 7

# Coordinate transformations and smooth atlas

In this section, we define the coordinate transformations between local coordinate systems. Furthermore, we shall introduce the concept of smooth atlas on a topological space.

## 7.1 Definition of Coordinate transformations

Throughout this section, we fix a topological space $M$ and $n \in \mathbb{Z}_{\geq 0}$. Furthermore, $n$-dimensional local coordinate systems $(O, U, \boldsymbol{u})$ and $(O'.V, \boldsymbol{v})$ on $M$ are also fixed.

Recall that we use the symbol $\mathcal{LC}(M; \mathbb{R}^n)$ for the set of all $n$-dimensional local coordinate systems.

First, we note that the intersection $O \cap O'$ is open in $M$, and thus $\boldsymbol{u}(O \cap O')$ and $\boldsymbol{v}(O \cap O')$ are both open in $\mathbb{R}^n$.

Let us define the coordinate transformation from $(O, U, \boldsymbol{u})$ to $(O', V, \boldsymbol{v})$ as a map from $\boldsymbol{u}(O \cap O')$ to $\boldsymbol{v}(O \cap O')$ as follows:

**Definition 7.1.1.** The map

$$\tau_{\boldsymbol{uv}} := \boldsymbol{v} \circ \boldsymbol{u}^{-1} : \boldsymbol{u}(O \cap O') \to \boldsymbol{v}(O \cap O'), u \mapsto \boldsymbol{v}(\boldsymbol{u}^{-1}(u))$$

is called the coordinate transformation from $(O, U, \boldsymbol{u})$ to $(O', V, \boldsymbol{v})$.

**Example 7.1.2.** Suppose $n = 2$ and $M = S^2 = \{x \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i^2 = 1\}$. Let us take $(O, U, \boldsymbol{u}), (O', V, \boldsymbol{v}) \in \mathcal{LC}(S^2; \mathbb{R}^2)$ as

$$
\begin{aligned}
O &:= \{x \in \mathcal{S}^2 \mid x_3 > 0\} \subset S^2, \\
U &:= \{u \in \mathbb{R}^2 \mid \sum_{i=1}^{2} u_i^2 < 1\} \subset \mathbb{R}^2, \\
\boldsymbol{u} &: O \to U,\ x \mapsto (x_1, x_2)
\end{aligned}
$$

and

$$
\begin{aligned}
O' &:= \{x \in \mathcal{S}^2 \mid x_2 > 0\} \subset S^2, \\
V &:= \{v \in \mathbb{R}^2 \mid \sum_{i=1}^{2} v_i^2 < 1\} \subset \mathbb{R}^2, \\
\boldsymbol{v} &: O' \to V,\ x \mapsto (x_1, x_3).
\end{aligned}
$$

We note that

$$
\begin{aligned}
\boldsymbol{u}^{-1} &: U \to O,\ u \mapsto (u_1, u_2, \sqrt{1 - u_1^2 - u_2^2}), \\
\boldsymbol{v}^{-1} &: V \to O',\ v \mapsto (v_1, \sqrt{1 - v_1^2 - v_2^2}, v_2),
\end{aligned}
$$

Then we have

$$
\begin{aligned}
O \cap O' &= \{x \in S^2 \mid x_2 > 0, x_3 > 0\}, \\
\boldsymbol{u}(O \cap O') &= \{u \in U \mid u_2 > 0\}, \\
\boldsymbol{v}(O \cap O') &= \{v \in V \mid v_2 > 0\}
\end{aligned}
$$

and the coordinate transformations $\tau_{\boldsymbol{uv}}$ and $\tau_{\boldsymbol{vu}}$ can be written as

$$
\begin{aligned}
\tau_{\boldsymbol{uv}} &: \{u \in U \mid u_2 > 0\} \to \{v \in V \mid v_2 > 0\},\ u \mapsto (u_1, \sqrt{1 - u_1^2 - u_2^2}), \\
\tau_{\boldsymbol{vu}} &: \{v \in V \mid v_2 > 0\} \to \{u \in U \mid u_2 > 0\},\ v \mapsto (v_1, \sqrt{1 - v_1^2 - v_2^2}).
\end{aligned}
$$

We will sometimes apply the following three propositions:

**Proposition 7.1.3.** $\tau_{\boldsymbol{uu}}$ is the identity map on $U$.

**Proposition 7.1.4.** $\tau_{\boldsymbol{uv}}$ and $\tau_{\boldsymbol{vu}}$ are inverse to each other.

**Proposition 7.1.5.** Let $(O_i, U_i, \boldsymbol{u}_i) \in \mathcal{LC}(M; \mathbb{R}^n)$ for $i = 1, 2, 3$. Then for any $u \in \boldsymbol{u}(O_1 \cap O_2 \cap O_3)$,

$$\tau_{\boldsymbol{u}_1\boldsymbol{u}_2}(u) \in \boldsymbol{u}_2(O_1 \cap O_2 \cap O_3),$$

and

$$\tau_{\boldsymbol{u}_1\boldsymbol{u}_3}(u) = \tau_{\boldsymbol{u}_2\boldsymbol{u}_3}(\tau_{\boldsymbol{u}_1\boldsymbol{u}_2}(u)).$$

## 7.2 Smoothness of coordinate transformations

Throughout this section, we fix a topological space $M$ and $n \in \mathbb{Z}_{\geq 0}$. Furthermore, $n$-dimensional local coordinate systems $(O, U, \boldsymbol{u})$ and $(O', V, \boldsymbol{v})$ are also fixed.

**Question:** Coordinate transformations

$$\begin{aligned} \tau_{\boldsymbol{uv}} &: \boldsymbol{u}(O \cap O') \to \boldsymbol{v}(O \cap O') \\ \tau_{\boldsymbol{vu}} &: \boldsymbol{v}(O \cap O') \to \boldsymbol{u}(O \cap O') \end{aligned}$$

are both smooth maps? It should be noted that since $\boldsymbol{u}(O \cap O')$ and $\boldsymbol{v}(O \cap O')$ are both open sets of Euclidean spaces, the smoothness of $\tau_{\boldsymbol{uv}}$ and $\tau_{\boldsymbol{vu}}$ are defined in the sense of Definition 5.1.1.

**Answer:** No! In general, coordinate transformations fail to be smooth!

**Example 7.2.1.** Let $M := \{x \in \mathbb{R}^2 \mid x_2 = x_1^3\}$. Take a pair of 1-dimensional local coordinate systems $(O, U, \boldsymbol{u})$ and $(O', V, \boldsymbol{v})$ on $M$ as below:

$$\begin{aligned} O &:= M, \\ U &:= \mathbb{R}, \\ \boldsymbol{u} &: O \to U,\ x \mapsto x_1, \\ (\boldsymbol{u}^{-1} &: U \to O,\ u \mapsto (u, u^3)). \end{aligned}$$

$$\begin{aligned} O' &:= M, \\ V &:= \mathbb{R}, \\ \boldsymbol{v} &: O' \to V,\ x \mapsto x_2, \\ (\boldsymbol{u}^{-1} &: V \to O',\ v \mapsto (v^{1/3}, v)). \end{aligned}$$

Then we have

$$
\begin{aligned}
O \cap O' &= M, \\
\boldsymbol{u}(O \cap O') &= \mathbb{R}, \\
\boldsymbol{v}(O \cap O') &= \mathbb{R}, \\
\tau_{\boldsymbol{uv}} : \mathbb{R} \to \mathbb{R}&, \; u \mapsto u^3, \\
\tau_{\boldsymbol{vu}} : \mathbb{R} \to \mathbb{R}&, \; v \mapsto u^{1/3}.
\end{aligned}
$$

Thus, $\tau_{\boldsymbol{uv}}$ is a smooth map but not $\tau_{\boldsymbol{vu}}$. (Note that

$$\mathbb{R} \to \mathbb{R}, \; u \mapsto u^{1/3}$$

is not differentiable at the origin.)

It should be emphasized that the question that "is $\boldsymbol{u} : O \to U$ smooth?" does not make sense!! Recall that now we have only the definition of smoothness of maps between open sets of Euclidean spaces (but our $O$ is not needed to be an open set of a Euclidean space).

**The key idea in this lecture:** Our goal is to define the algebra $C^\infty(M)$ of "smooth functions" on $M$. To this, it is suitable if coordinate transforms are smooth (we will discuss in Section 8).

**Example 7.2.2.** Let us consider the situation in Example 7.1.2. Recall that

$$
\begin{aligned}
\tau_{\boldsymbol{uv}} &: \{u \in U \mid u_2 > 0\} \to \{v \in V \mid v_2 > 0\}, \; u \mapsto (u_1, \sqrt{1 - u_1^2 - u_2^2}), \\
\tau_{\boldsymbol{vu}} &: \{v \in V \mid v_2 > 0\} \to \{u \in U \mid u_2 > 0\}, \; v \mapsto (v_1, \sqrt{1 - v_1^2 - v_2^2}).
\end{aligned}
$$

They are both smooth.

## 7.3 Smooth atlas

In this section, we introduce the concept of smooth atlas on topological spaces.

Throughout this section, we fix a topological space $M$ and $n \in \mathbb{Z}_{\geq 0}$. Recall that we use the symbol $\mathcal{LC}(M; \mathbb{R}^n)$ for the set of all $n$-dimensional local coordinate systems on $M$.

**Definition 7.3.1** (Smooth atlas)**.** A subset $\mathcal{A}_0$ of $\mathcal{LC}(M;\mathbb{R}^n)$ is said to be an $n$-dimensional smooth atlas ($C^\infty$-atlas) if the following two conditions hold:

1. $\bigcup_{(O,U,\boldsymbol{u})\in\mathcal{A}_0} O = M$,

2. For any $(O,U,\boldsymbol{u})$ and any $(O',V,\boldsymbol{v})$, the coordinate transformation
$$\tau_{\boldsymbol{uv}} : \boldsymbol{u}(O\cap O') \to \boldsymbol{v}(O\cap O'),\ u \mapsto \boldsymbol{v}(\boldsymbol{u}^{-1}(u))$$
is a smooth map.

**Example 7.3.2.** Suppose $n \in \mathbb{Z}_{\geq 1}$. For the topological space
$$S^n := \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1},$$
For each $k = 1,\dots,n{+}1$, let us define $n$-dimensional local coordinate systems $(O_k^+, U_k^+, \boldsymbol{u}_k^+)$ and $(O_k^-, U_k^-, \boldsymbol{u}_k^-)$ and as follows:
$$\begin{aligned}
O_k^+ &:= \{x \in S^n \mid x_k > 0\} \subset S^n,\\
U_k^+ &:= \{u \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i^2 < 1\} \subset \mathbb{R}^n,\\
\boldsymbol{u}_k^+ &: O_k^+ \to U_k^+,\ x(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1}),\\
(\boldsymbol{u}_k^+)^{-1} &: U_k^+ \to O_k^+,\ u(u_1,\dots,u_{k-1},\sqrt{1-\sum_{i=1}^{n} u_i^2},u_k,\dots,u_n).
\end{aligned}$$
$$\begin{aligned}
O_k^- &:= \{x \in S^n \mid x_k < 0\} \subset S^n,\\
U_k^- &:= \{u \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i^2 < 1\} \subset \mathbb{R}^n,\\
\boldsymbol{u}_k^- &: O_k^- \to U_k^-,\ x(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1}),\\
(\boldsymbol{u}_k^-)^{-1} &: U_k^- \to O_k^-,\ u(u_1,\dots,u_{k-1},-\sqrt{1-\sum_{i=1}^{n} u_i^2},u_k,\dots,u_n).
\end{aligned}$$
Then
$$\mathcal{A}_0 := \{(O_k^+, U_k^+, \boldsymbol{u}_k^+) \mid k = 1,\dots,n{+}1\} \bigsqcup \{(O_k^-, U_k^-, \boldsymbol{u}_k^-) \mid k = 1,\dots,n{+}1\} \subset \mathcal{LC}(S^n;\mathbb{R}^n)$$
is an $n$-dimensional smooth atlas on $S^n$.

## 7.4 Exercise

**Setting in Exercise 62 to 66:** Let $M$ be a topological space and $n \in \mathbb{Z}_{\geq 0}$. We also fix $n$-dimensional local coordinate systems $(O, U, \boldsymbol{u})$ and $(O', V, \boldsymbol{v})$ on $M$.

62. Explain the coordinate transformation by drawing a picture.

63. Show that the coordinate transformation

$$\tau_{\boldsymbol{u}\boldsymbol{u}} : \boldsymbol{u}(O) \to \boldsymbol{u}(O)$$

from $(O, U, \boldsymbol{u})$ $(O, U, \boldsymbol{u})$ itself is the identity (Proposition 7.1.3).

64. Show that the coordinate transformations

$$\tau_{\boldsymbol{u}\boldsymbol{v}} : \boldsymbol{u}(O \cap O') \to \boldsymbol{v}(O \cap O')$$

and

$$\tau_{\boldsymbol{v}\boldsymbol{u}} : \boldsymbol{v}(O \cap O') \to \boldsymbol{u}(O \cap O')$$

are inverse to each other (Proposition 7.1.4).

65. Fix $(O_1, U_1, \boldsymbol{u}_1), (O_2, U_2, \boldsymbol{u}_2), (O_3, U_3, \boldsymbol{u}_3) \in \mathcal{LC}(M; \mathbb{R}^n)$. Show that

$$\tau_{\boldsymbol{u}_1\boldsymbol{u}_3}|_{\boldsymbol{u}_1(O_1 \cap O_2 \cap O_3)} = (\tau_{\boldsymbol{u}_2\boldsymbol{u}_3}|_{\boldsymbol{u}_2(O_1 \cap O_2 \cap O_3)}) \circ (\tau_{\boldsymbol{u}_1\boldsymbol{u}_2}|_{\boldsymbol{u}_1(O_1 \cap O_2 \cap O_3)})$$

as maps from $\boldsymbol{u}_1(O_1 \cap O_2 \cap O_3)$ to $\boldsymbol{u}_3(O_1 \cap O_2 \cap O_3)$ (Proposition 7.1.5).

66. Put $M = \{x \in \mathbb{R}^2 \mid x_2 = x_1^3\} \subset \mathbb{R}^2$. We take $(O, U, \boldsymbol{u}), (O', V, \boldsymbol{v}) \in \mathcal{LC}(M; \mathbb{R})$ as

    - $O = O' = M$.
    - $U = V = \mathbb{R}$.
    - $\boldsymbol{u} : O \to U,\ x \mapsto x_1$.
    - $\boldsymbol{v} : O \to V,\ x \mapsto x_2$.

    Determine the coordinate transformations $\tau_{\boldsymbol{u}\boldsymbol{v}}$ and $\tau_{\boldsymbol{v}\boldsymbol{u}}$. Furthermore, discuss whether $\tau_{\boldsymbol{u}\boldsymbol{v}}$ and $\tau_{\boldsymbol{v}\boldsymbol{u}}$ are smooth or not (Example 7.2.1).

**Setting in Exercise 67 to 69:** We put

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}.$$

For each $k = 1, \ldots, n+1$, let us define $(O_k^{\pm}, U_k^{\pm}, \boldsymbol{u}_k^{\pm}) \in \mathcal{LC}(S^n; \mathbb{R}^n)$ as below:

- $O_k^+ = \{x \in S^n \mid x_k > 0\}, O_k^- = \{x \in S^n \mid x_k < 0\} \subset S^2$.
- $U_k^{\pm} = \{u \in \mathbb{R}^n \mid \sum_{i=1}^n u_i^2 < 1\} \subset \mathbb{R}^n$.
- $\boldsymbol{u}_k^{\pm} : O_k^{\pm} \to U_k^{\pm},\ x \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})$.

Furthermore we also define the subset $\mathcal{A}_0$ of $\mathcal{LC}(M; \mathbb{R}^n)$ by

$$\mathcal{A}_0 = \{(O_k^+, U_k^+, \boldsymbol{u}_k^+) \mid k = 1, \ldots, n{+}1\} \cup \{(O_k^-, U_k^-, \boldsymbol{u}_k^-) \mid k = 1, \ldots, n{+}1\}.$$

67. For each $k = 1, \ldots, n+1$, determine the inverse map of $\boldsymbol{u}_k^+$ and that of $\boldsymbol{u}_k^-$, respectively.

68. Take $1 \leq k_1 < k_2 \leq n{+}1$. Determine $\boldsymbol{u}_{k_1}^+(O_{k_1}^+ \cap O_{k_2}^-)$ and $\boldsymbol{u}_{k_2}^-(O_{k_1}^+ \cap O_{k_2}^-)$. Furthermore, determine the coordinate transformation

$$\tau_{\boldsymbol{u}_{k_1}^+ \boldsymbol{u}_{k_2}^-} : \boldsymbol{u}_{k_1}^+(O_{k_1}^+ \cap O_{k_2}^-) \to \boldsymbol{u}_{k_2}^-(O_{k_1}^+ \cap O_{k_2}^-),$$

and show that ${}_{\boldsymbol{u}_{k_1}^+ \boldsymbol{u}_{k_2}^-}$ is smooth (Example 7.1.2, Example 7.3.2).

69. Show that $\mathcal{A}_0$ is an $n$-dimensional smooth atlas on $S^n$ (Example 7.3.2).

# Chapter 8

# Smooth functions on a topological space equipped with a smooth atlas

In this section, we shall define the smooth functions on topological space equipped with a smooth atlas.

## 8.1 Smoothness on a local coordinate system

Throughout this section, we fix $M$ as a topological space, $n \in \mathbb{Z}_{\geq 0}$ and an $n$-dimensional local coordinate system $(O, U, \boldsymbol{u})$ on $M$.

Our goal is to give a definition that $f$ is smooth on $(O, U, \boldsymbol{u})$ for each function $f$ on $M$.

**Definition 8.1.1.** For each function $f : M \to \mathbb{R}$ on $M$, we define the function $f_{\boldsymbol{u}}$ on $U$ by

$$f_{\boldsymbol{u}} : U \to \mathbb{R},\ u \mapsto f(\boldsymbol{u}^{-1}(u)).$$

**Definition 8.1.2.** We say that $f$ is smooth (or $C^\infty$) on $(O, U, \boldsymbol{u})$ if $f_{\boldsymbol{u}} \in C^\infty(U)$.

**Example 8.1.3.** Let us consider $S^1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$ and a

1-dimensional local coordinate system $(O, U, \boldsymbol{u})$ defined by

$$\begin{aligned} O &:= \{x \in S^1 \mid x_2 > 0\} \subset S^1, \\ U &:= (-1, 1) \subset \mathbb{R}, \\ \boldsymbol{u} &: O \to U,\ x \mapsto x_1, \\ (\boldsymbol{u}^{-1} &: U \to O,\ u \mapsto (u, \sqrt{1-u^2})). \end{aligned}$$

For the function

$$f : S^1 \to \mathbb{R},\ x \mapsto x_2,$$

we have

$$f_{\boldsymbol{u}} : U \to \mathbb{R},\ u \mapsto f(\boldsymbol{u}^{-1}(u)) = f(u, \sqrt{1-u^2}) = \sqrt{1-u^2}.$$

Thus $f_{\boldsymbol{u}}$ is smooth on $U = (-1, 1)$, and thus $f$ is smooth on $(O, U, \boldsymbol{u})$.

## 8.2 Smooth functions on a topological space equipped with a smooth atlas

Throughout this section, we fix $M$ as a topological space, $n \in \mathbb{Z}_{\geq 0}$ and an $n$-dimensional smooth atlas $\mathcal{A}_0 \subset \mathcal{LC}(M; \mathbb{R}^n)$.

**Definition 8.2.1.** We say that a function $f$ on $M$ is smooth (or $C^\infty$) on $(M, \mathcal{A}_0)$ if $f_{\boldsymbol{u}} \in C^\infty(U)$ for any $(O, U, \boldsymbol{u}) \in \mathcal{A}_0$.

**Example 8.2.2.** For $S^1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$, let us consider the 1-dimensional smooth atlas $\mathcal{A}_0$ defined in Example 7.3.2. Then one can check that the function

$$f : S^1 \to \mathbb{R},\ x \mapsto x_2$$

is smooth on $(S^1, \mathcal{A}_0)$.

The following theorem is fundamental and useful:

**Theorem 8.2.3.** Let $f : M \to \mathbb{R}$ a function on $M$ (which might not be continuous). Then the following two conditions on $f$ are equivalent:

1. $f$ is smooth on $(M, \mathcal{A}_0)$.
2. For each $p \in M$, there exists $(O, U, \boldsymbol{u}) \in \mathcal{A}_0$ such that $p \in O$ and $f_{\boldsymbol{u}} \in C^\infty(U)$.

To prove Theorem 8.2.3, we will apply the following theorem:

**Theorem 8.2.4.** Let $U$ be an open set of $\mathbb{R}^n$ and fix an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of the topological space $U$. Then for each function $h : U \to \mathbb{R}$, the following two conditions are equivalent:

1. $h \in C^\infty(U)$.
2. $h|_{U_\lambda} \in C^\infty(U_\lambda)$ for any $\lambda \in \Lambda$.

*Proof of Theorem 8.2.3.* The implication (1) $\Rightarrow$ (2) is easy. Suppose (2) and we shall prove (1). Take any $(O, U, \boldsymbol{u}) \in \mathcal{A}_0$. Let us prove that $f_{\boldsymbol{u}} \in C^\infty(U)$. By (2), for each $p \in O$, one can choose $(O_p, U_p, \boldsymbol{u}_p) \in \mathcal{A}_0$ such that $p \in O_p$ and $f_{\boldsymbol{u}_p} \in C^\infty(U_p)$. Note that $\{\boldsymbol{u}(O \cap O_p)\}_{p\in O}$ is an open cover on $U$. Thus by Theorem 8.2.4, it suffices to show that

$$f_{\boldsymbol{u}}|_{\boldsymbol{u}(O\cap O_p)} \in C^\infty(\boldsymbol{u}(O \cap O_p))$$

for each $p \in O$. Let us fix $p \in O$. Put $(O', V, \boldsymbol{v}) := (O_p, U_p, \boldsymbol{u}_p)$. Then our goal is to show that

$$f_{\boldsymbol{u}}|_{\boldsymbol{u}(O\cap O')} \in C^\infty(\boldsymbol{u}(O \cap O')).$$

Since $(O, U, \boldsymbol{u}), (O', V, \boldsymbol{v}) \in \mathcal{A}_0$, the coordinate transformation

$$\tau_{\boldsymbol{uv}} : \boldsymbol{u}(O \cap O') \to \boldsymbol{v}(O \cap O'),\ u \mapsto \boldsymbol{v}(\boldsymbol{u}^{-1}(u))$$

is smooth. Recall that $f_{\boldsymbol{v}\in C^\infty(V)}$. Thus $f_{\boldsymbol{v}|_{\boldsymbol{v}(O\cap O')}\in C^\infty(\boldsymbol{v}(O\cap O'))}$ (cf. Proposition 3.1.5). Therefore

$$\tau^*_{\boldsymbol{uv}}(f_{\boldsymbol{v}}|_{\boldsymbol{v}(O\cap O')}) \in C^\infty(\boldsymbol{u}(O \cap O')).$$

One can easily check that

$$\tau^*_{\boldsymbol{uv}}(f_{\boldsymbol{v}})|_{\boldsymbol{v}(O\cap O')}) = f_{\boldsymbol{u}}|_{\boldsymbol{u}(O\cap O')}.$$

Thus we obtain

$$f_{\boldsymbol{u}}|_{\boldsymbol{u}(O\cap O')} \in C^\infty(\boldsymbol{u}(O \cap O')).$$

□

We note that the following also holds:

**Proposition 8.2.5.** If a function $f$ on $M$ is smooth on $(M, \mathcal{A}_0)$, then $f$ is continuous on $M$.

## 8.3 The $\mathbb{R}$-algebras of smooth functions on a smooth atlas

Throughout this section, we fix $M$ as a topological space, $n \in \mathbb{Z}_{\geq 0}$ and an $n$-dimensional smooth atlas $\mathcal{A}_0 \subset \mathcal{LC}(M; \mathbb{R}^n)$.

**Definition 8.3.1.** We put

$$C^\infty(M; \mathcal{A}_0) := \{f \in C(M) \mid f \text{ is smooth on } (M, \mathcal{A}_0)\}$$

The following theorem is one of the most important claims in this lecture course.

**Theorem 8.3.2.** $C^\infty(M; \mathcal{A}_0)$ is a sub $\mathbb{R}$-algebra of $C(M)$. In particular, $C^\infty(M; \mathcal{A}_0)$ itself is an $\mathbb{R}$-algebra.

Theorem 8.3.2 comes immediately from the following proposition and lemma:

**Proposition 8.3.3.** Let $V_1$, $V_2$ be both $\mathbb{R}$-algebras and $W$ a sub $\mathbb{R}$-algebra of $V_2$. We fix an $\mathbb{R}$-algebra homomorphism $\psi : V_1 \to V_2$. Then $\psi^{-1}(W)$ is also a sub $\mathbb{R}$-algebra of $V_1$.

**Lemma 8.3.4.** For each $(O, U, \boldsymbol{u}) \in \mathcal{A}_0$, the map

$$\psi_{\boldsymbol{u}} : C(M) \to C(U),\ f \mapsto f_{\boldsymbol{u}}$$

is an $\mathbb{R}$-algebra homomorphism.

## 8.4 Exercise

Preparing...

# Chapter 9

# Maximal smooth atlas

## 9.1 Maximal smooth atlas

Throughout this section, let $M$ be a topological space and $n \in \mathbb{Z}_{\geq 0}$.

**Definition 9.1.1.** We put

$C^\infty\text{-atlas}(M;\mathbb{R}^n) := \{\mathcal{A}_0 \subset \mathcal{LC}(M;\mathbb{R}^n) \mid \mathcal{A}_0 \text{ is an } n\text{-dimensional smooth atlas on } M\}.$

**Theorem 9.1.2.** Let $\mathcal{A}_0, \mathcal{A}_0' \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$ with $\mathcal{A}_0 \subset \mathcal{A}_0'$. Then $C^\infty(M;\mathcal{A}_0) = C^\infty(M;\mathcal{A}_0')$.

Hint: Theorem 8.2.4.

**Definition 9.1.3.** An $n$-dimensional $\mathcal{A} \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$ is called maximal if there does not exist $\mathcal{B} \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$ with $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$.

**Question:** For $\mathcal{A}_0 \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$, does there exist a maximal $\mathcal{A} \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$ with $\mathcal{A}_0 \subset \mathcal{A}$?

**Answer:** Yes!! Furthermore, such $\mathcal{A}$ is unique for each $\mathcal{A}_0$.

**Definition 9.1.4.** For each $\mathcal{A}_0 \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$, we define

$[\mathcal{A}_0] := \{(O,U,\boldsymbol{u}) \in \mathcal{LC}(M;\mathbb{R}^n) \mid \tau_{\boldsymbol{uv}}, \tau_{\boldsymbol{vu}} \text{ are both smooth for any } (O',V,\boldsymbol{v}) \in \mathcal{A}_0\}.$

**Example 9.1.5.** Preparing...

**Theorem 9.1.6.** Let $\mathcal{A}_0 \in C^\infty\text{-atlas}(M;\mathbb{R}^n)$. Then the following holds:

1. $\mathcal{A}_{/} \subset [\mathcal{A}_0]$.

2. $[\mathcal{A}_0] \in C^\infty$-atlas$(M; \mathbb{R}^n)$ and maximal.

3. $[\mathcal{A}_0]$ is the unique smooth atlas containing $\mathcal{A}_0$.

A proof of Theorem 9.1.6 can be found in Section 9.2 in the Japanese version of the lecture notes.

Note that by Theorem 9.1.2, the following holds:

**Proposition 9.1.7.** For each $\mathcal{A}_0 \in C^\infty$-atlas$(M; \mathbb{R}^n)$,

$$C^\infty(M; \mathcal{A}_0) = C^\infty(M; [\mathcal{A}_0]).$$

The next proposition is useful:

**Proposition 9.1.8.** Let $\mathcal{A}_0, \mathcal{B}_0 \in C^\infty$-atlas$(M; \mathbb{R}^n)$. Then the following two conditions on $\mathcal{A}_0, \mathcal{B}_0$ are equivalent:

1. $[\mathcal{A}_0] = [\mathcal{B}_0]$.

2. For any $(O, U, \boldsymbol{u}) \in \mathcal{A}_0$ and any $(O', V, \boldsymbol{v}) \in \mathcal{B}_0$, the coordinate transformations $\tau_{\boldsymbol{u}\boldsymbol{v}}$ and $\tau_{\boldsymbol{v}\boldsymbol{u}}$ are both smooth.

Hint: Theorem 9.1.6.

**Example 9.1.9.** Preparing...

## 9.2 Exercise

Preparing...

# Chapter 10

# Smooth manifolds

## 10.1 Hausdorff spaces

preparing...

## 10.2 Smooth manifolds

Let us fix $n \in \mathbb{Z}_{\geq 0}$. We shall define $n$-dimensional smooth manifold as below:

**Definition 10.2.1.** $(M, \mathcal{A})$ is an $n$-dimensional smooth manifold if $M$ is a Hausdorff topological space and $\mathcal{A}$ is a maximal $n$-dimensional smooth atlas on $M$.

Sometimes, we assume that $M$ is second countable (i.e. $M$ admits a countable base of the topology).

**Example 10.2.2.** The empty set $\emptyset$ is an $n$-dimensional smooth manifold for any $n \in \mathbb{Z}_{\geq 0}$ (with respect to $\mathcal{A} = \emptyset$).

**Example 10.2.3.** Let $U$ be an open set of $\mathbb{R}^n$. Then $\mathcal{A}_0 := \{(U, U, \mathrm{id}_U)\}$ is an $n$-dimensional smooth atlas on $U$. Thus $(U, [\mathcal{A}_0])$ is an $n$-dimensional smooth manifold. Note that in this situation,

$$C^\infty(U; [\mathcal{A}_0]) = C^\infty(U, \mathcal{A}_0) = C^\infty(U).$$

**Example 10.2.4.** preparing... ($S^n$ is an $n$-dimensional manifold.)

**Example 10.2.5.** preparing... (two types of smooth manifold structure on $M := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1^3\}$)

**Example 10.2.6.** $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$ is not a manifold because there does not exist a local coordinate system $(O, U, \boldsymbol{u})$ with $(0, 0) \in O$ (non-trivial fact).

## 10.3 Open submanifolds

preparing...

## 10.4 Exercise

Preparing...

# Chapter 11

# Direct products of smooth manifolds

preparing...

# Chapter 12

# Projective spaces

## 12.1 Projective spaces and their topology

Let us fix $n \in \mathbb{Z}_{\geq 0}$.

**Definition 12.1.1.** The $n$-dimensional projective space $\mathbb{R}\mathrm{P}^n$ is defined by

$$\mathbb{R}\mathrm{P}^n := \{\ell \subset \mathbb{R}^{n+1} \mid \ell \text{ is a one-dimensional linear subspace of } \mathbb{R}^{n+1}\}.$$

Let us give a definition of a topology of $\mathbb{R}\mathrm{P}^n$ as below: First, we consider the topological space $\mathbb{R}^{n+1} \setminus \{0\}$.

**Proposition 12.1.2.**

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^n, \ x \mapsto := \{rx \mid r \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$$

is well-defined as a map and surjective.

**Definition 12.1.3.** We consider $\mathbb{R}\mathrm{P}^n$ as a topological space with respect to the quotient topology on $\mathbb{R}\mathrm{P}^n$ by $\pi$. That is, a subset $O$ in $\mathbb{R}\mathrm{P}^n$ is said to be open if $\pi^{-1}(O)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.

The following proposition will be applied:

**Proposition 12.1.4.** Let $W$ be a linear subspace of $\mathbb{R}^{n+1}$. Then

$$O_W := \{\ell \in \mathbb{R}\mathrm{P}^n \mid \ell \subset W\}$$

is an open set of $\mathbb{R}\mathrm{P}^n$.

*Proof.* preparing... □

**Theorem 12.1.5.** $\mathbb{R}\mathrm{P}^n$ is compact and Hausdorff.

## 12.2 Smooth atlas on projective spaces

Let us fix $n \in \mathbb{Z}_{\geq 0}$. We shall give a smooth atlas on $\mathbb{R}\mathrm{P}^n$ as below:

For each $x \in \mathbb{R}^{n+1}\{0\}$, we write

$$[x_1 : x_2 : \cdots : x_{n+1}] := [x] := \pi(x) := \mathbb{R}x \in \mathbb{R}\mathrm{P}^n.$$

Note that for $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$[x_1 : \cdots : x_{n+1}] = [y_1 : \cdots : y_{n+1}]$$

holds if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$.

**Theorem 12.2.1.** For each $i = 1, \ldots, n+1$, we put $W_i := \{x \in \mathbb{R}^{n+1} \mid x_i = 0\}$ and define

$$\begin{aligned} &O_i := O_{W_i} \subset \mathbb{R}\mathrm{P}^n, \\ &U_i := \mathbb{R}^n, \\ &\boldsymbol{u}_i : O_i \to U_i,\ [x] \mapsto \frac{1}{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}). \end{aligned}$$

Then $(O_i, U_i, \boldsymbol{u}_i) \in \mathcal{LC}(\mathbb{R}\mathrm{P}^n; \mathbb{R}^n)$.

**Theorem 12.2.2.**

$$\mathcal{A}_0 := \{(O_i, U_i, \boldsymbol{u}_i) \mid i = 1, \ldots, n+1\} \in C^\infty\text{-atlas}.$$

Furthermore, $(\mathbb{R}\mathrm{P}^n, [\mathcal{A}_0])$ is an $n$-dimensional smooth manifold.

*Proof.* preparing... □

We also give an example of smooth function on $\mathbb{R}\mathrm{P}^n$:

**Theorem 12.2.3.** Let $k \in \mathbb{Z}_{\geq 0}$. Fix $f_1, f_2 \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$. Assume that $f_2$ has no zero on $\mathbb{R}^{n+1} \setminus \{0\}$, $f_1$ and $f_2$ are both homogeneous of degree $k$, that is, $f_i(\lambda x) = \lambda^k f_i(x)$ for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and any $\lambda \in \mathbb{R}$. Then

$$f : \mathbb{R}\mathrm{P}^n \to \mathbb{R},\ [x] \mapsto \frac{f_1(x)}{f_2(x)}$$

is well-defined and $f \in C^\infty(\mathbb{R}\mathrm{P}^n; [\mathcal{A}_0])$.

*Proof.* preparing... □

**Example 12.2.4.** Let

$$f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R},\ [x] \mapsto \frac{2x_1x_2}{x_1^2 + x_2^2 + x_3^2}.$$

Then $f$ is well-defined and smooth on $\mathbb{R}\mathrm{P}^2$.

# Chapter 13

# Tangent spaces

## 13.1 Definition of tangent space

Let $n \in \mathbb{Z}_{\geq 0}$. We fix an $n$-dimensional smooth manifold $M = (M, \mathcal{A})$ and a point $p \in M$. The $\mathbb{R}$-algebra of all smooth functions on $(M, \mathcal{A})$ will be denoted by $C^\infty(M) := C^\infty(M; \mathcal{A})$.

**Definition 13.1.1.** The tangent space $T_pM$ of $M$ at $p$ is defined by

$$T_pM := \{\eta : C^\infty(M) \to \mathbb{R} \mid \eta \text{ is linear and satisfies the Leibniz rule at } p\},$$

where we say that $\eta$ satisfies the Leibniz rule at $p$ if

$$\eta(f \cdot g) = \eta(f) \cdot g(p) + f(p) \cdot \eta(g)$$

holds for any $f, g \in C^\infty(M)$.

**Proposition 13.1.2.** $T_pM$ is a linear subspace of $\mathcal{L}(C^\infty(M), \mathbb{R})$.

## 13.2 Coordinate basis

Let $n \in \mathbb{Z}_{\geq 0}$. We fix an $n$-dimensional smooth manifold $M = (M, \mathcal{A})$ and a point $p \in M$. Furthermore, we also fix $(O, U, \boldsymbol{u}) \in \mathcal{A}$ with $p \in O$.

**Definition 13.2.1.** For each $i = 1, \ldots, n$, we define

$$\left(\frac{\partial}{\partial \boldsymbol{u}_i}\right)_p : C^\infty(M) \to \mathbb{R},\ f \mapsto \frac{\partial f_{\boldsymbol{u}}}{\partial u_i}(\boldsymbol{u}(p)).$$

**Proposition 13.2.2.** $\left(\frac{\partial}{\partial \boldsymbol{u}_i}\right)_p \in T_pM$ for $i = 1, \ldots, n$.

**Theorem 13.2.3.** $\left\{\left(\frac{\partial}{\partial \boldsymbol{u}_i}\right)_p\right\}_{i=1,\ldots,n}$ forms a basis of $T_pM$.

In our lectures, We call $\left\{\left(\frac{\partial}{\partial \boldsymbol{u}_i}\right)_p\right\}_{i=1,\ldots,n}$ the coordinate basis of $T_pM$ with respect to $(O, U, \boldsymbol{u})$.

**Corollary 13.2.4.** $\dim T_pM = n$.

## 13.3 Coordinate basis and change of basis

Let $n \in \mathbb{Z}_{\geq 0}$. We fix an $n$-dimensional smooth manifold $M = (M, \mathcal{A})$ and a point $p \in M$. Furthermore, we also fix $(O, U, \boldsymbol{u}), (O', V, \boldsymbol{v}) \in \mathcal{A}$ with $p \in O \cap O'$.

**Theorem 13.3.1.** Let us consider the Jacobi matrix

$$(J\tau_{\boldsymbol{uv}})_{\boldsymbol{u}(p)} := \left(\frac{\partial(\tau_{\boldsymbol{uv}})_i}{\partial u_j}(\boldsymbol{u}(p))\right)_{i,j=1,\ldots,n}.$$

Then $(J\tau_{\boldsymbol{uv}})_{\boldsymbol{u}(p)}$ gives the matrix of the change of basis $\left\{\left(\frac{\partial}{\partial \boldsymbol{v}_i}\right)_p\right\}_{i=1,\ldots,n}$ to $\left\{\left(\frac{\partial}{\partial \boldsymbol{u}_i}\right)_p\right\}_{i=1,\ldots,n}$ of $T_pM$, that is, for each

$$\eta = \sum_{i=1}^{n} a_i \left(\frac{\partial}{\partial \boldsymbol{u}_i}\right)_p = \sum_{i=1}^{n} b_i \left(\frac{\partial}{\partial \boldsymbol{v}_i}\right)_p \in T_pM,$$

we have

$$(J\tau_{\boldsymbol{uv}})_{\boldsymbol{u}(p)} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

**Example 13.3.2.** Let us consider the 2-dimensional smooth manifold $S^2 = (S^2, [\mathcal{A}_0])$ (cf. Ex 10.2.4). We put $p := \frac{1}{\sqrt{3}}(1, 1, 1) \in S^2$. Let us consider

$$(O, U, \boldsymbol{u}) = (O_1^+, U_1^+, \boldsymbol{u}_1^+) \in \mathcal{A}_0,$$
$$(O', V, \boldsymbol{v}) = (O_2^+, U_2^+, \boldsymbol{u}_2^+) \in \mathcal{A}_0.$$

Then $p \in O \cap O'$. One can compute that $\boldsymbol{u}(p) = \frac{1}{\sqrt{3}}(1,1)$,

$$\boldsymbol{u}(O \cap O') := \{u \in \mathbb{R}^2 \mid \|u\| < 1, u_1 > 0\},$$
$$\boldsymbol{v}(O \cap O') := \{v \in \mathbb{R}^2 \mid \|v\| < 1, v_1 > 0\},$$

and

$$\tau_{\boldsymbol{uv}} : \boldsymbol{u}(O \cap O') \to \boldsymbol{v}(O \cap O'),\ u \mapsto (\sqrt{1 - u_1^2 - u_2^2}, u_2).$$

Thus we have

$$(J\tau_{\boldsymbol{uv}})_{\boldsymbol{u}(p)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

In particular, we have that

$$\left(\frac{\partial}{\partial \boldsymbol{u}_1}\right)_p = -\left(\frac{\partial}{\partial \boldsymbol{v}_1}\right)_p$$
$$\left(\frac{\partial}{\partial \boldsymbol{u}_2}\right)_p = -\left(\frac{\partial}{\partial \boldsymbol{v}_1}\right)_p + \left(\frac{\partial}{\partial \boldsymbol{v}_2}\right)_p$$

# Chapter 14

# Smooth maps between smooth manifolds

preparing...

# Chapter 15

# Total differentiations of smooth maps between smooth manifolds

preparing...

# Chapter 16

# Regular submanifolds

preparing...

# Chapter 17

# Vector fields and flows

preparing...