§14: Van Kampen の定理

やること: Van Kampen の定理

と応用

様々な空間の基本群を計算する.

§14.1 : Van Kampen の定理

設定 : (X, a) : 基点付き空間

$(X$: 位相空間, $a \in X)$

U_1, U_2 : open sets in X

with $a \in U_1 \cap U_2$, $U_1 \cap U_2$: 弧状連結, $X = U_1 \cup U_2$.

記号 : $V := U_1 \cap U_2$

$i^k : V \hookrightarrow U_k$

$j^k : U_k \hookrightarrow X$ ・包含写像

$$
\begin{tikzcd}
 & X & \\
U_1 \arrow[ur, "j_1"] & a & U_2 \arrow[ul, "j_2"'] \\
 & V \arrow[ul, "i_1"] \arrow[ur, "i_2"'] &
\end{tikzcd}
$$

記号：　$i^k_* : \pi_1(V, a) \to \pi_1(U_k, a)$

$j^k_* : \pi_1(U_k, a) \to \pi_1(X, a)$

Thm 14.1.1 (Seifelt-Van Kampen の定理)：$(\pi_1(X, a), (j_1)_*, (j_2)_*)$ は

$(\pi_1(U_1, a), \pi_1(U_2, a), \pi_1(V, a), (i_1)_*, (i_2)_*)$

の融合積

証明のアイデア

$\left(Z', \{ j'_k : \pi_1(U_k, a) \to Z' \} \right)$ with

群

$$j'_1 \circ i^1_* = j'_2 \circ i^2_*$$

as $\pi_1(V, a) \to Z'$

を fix

示 $\exists ! \, \Phi : \pi_1(X, a) \to Z'$: 群 hom s.t.

$$\Phi \circ j^k_* = j'_k \quad (k = 1, 2)$$

@ 亜の一意性：以下を示せばよい ← 要解決

(示) $j^1_*(\pi_1(U_1,a)) \cup j^2_*(\pi_1(U_2,a))$ は $\pi_1(X,a)$ の生成系

$\forall \gamma \in \mathrm{Loop}(X,a)$ をとる.

(示) $\exists \alpha_1,\cdots,\alpha_n$ s.t. $\alpha_q \in \pi_1(U_{\varepsilon_q},a)$, $[\gamma]_b = j_*^{\varepsilon_n}(\alpha_n) * \cdots * j_*^{\varepsilon_1}(\alpha_1)$

$(q=1,\cdots,n,\ \varepsilon_q = 1 \text{ or } 2)$

ルベーグ数の議論を使うと以下が示せる.

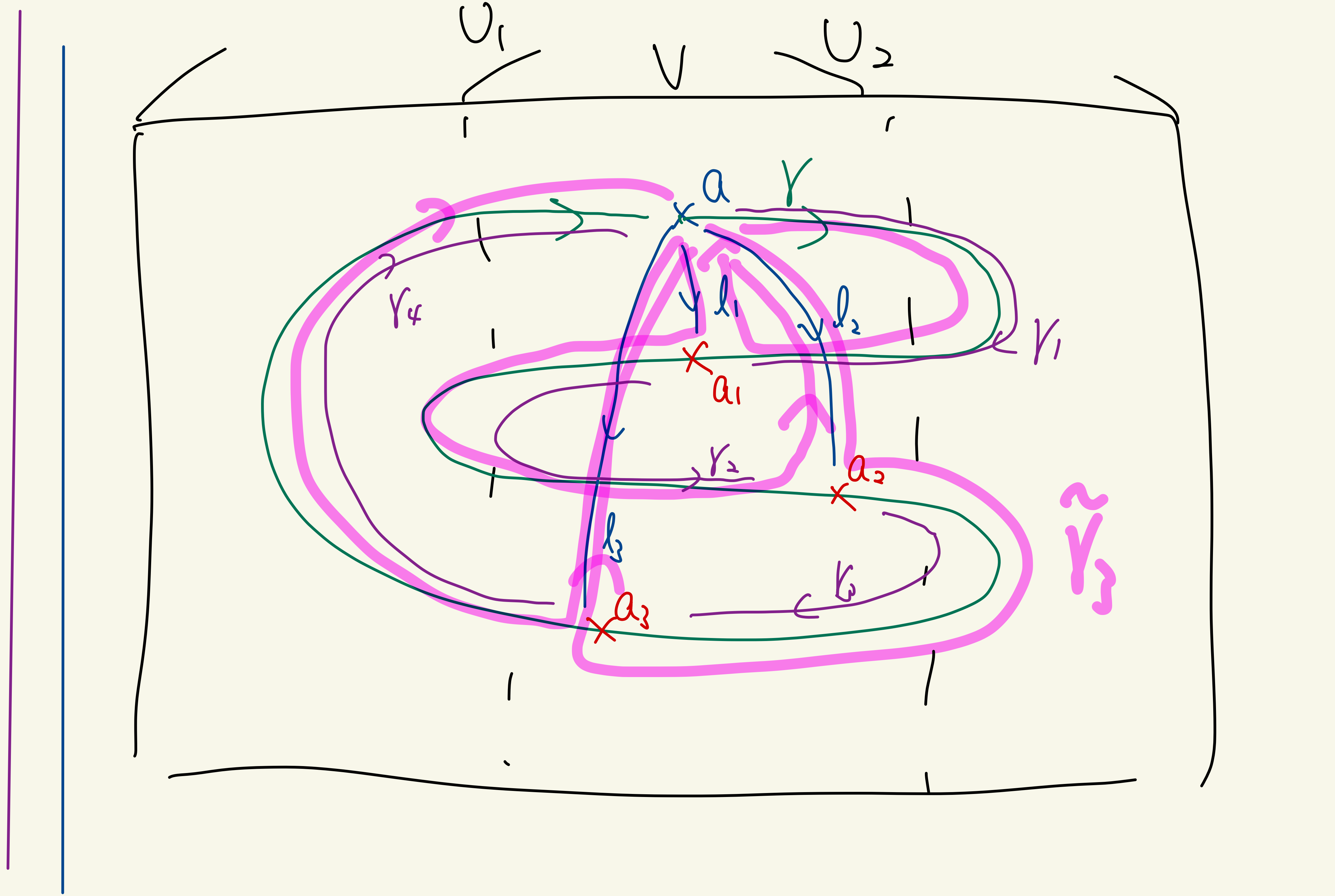
(※) ${}^\exists$ $0=t_0<t_1<\cdots<{}^\exists t_{n-1}<t_n=1$

s.t. $[t_{q-1},t_q]\subset U_{\varepsilon_q}$ $(\varepsilon_q=1 \text{ or } 2)$, $\gamma(t_q)\in V=U_1\cap U_2$

$(q=1,\cdots,n)$

$q=0,\cdots,n$ に対して $a_q := \gamma(t_q) \in V$ とおく $(a_0=a_n=a)$

(cf. V:弧状連結) → $l_q \in \mathrm{Path}(V,a,a_q)$ を fix. ただし $l_0=l_n=\gamma^a_{id}$ とする.



U1
V
U2
a
γ
l1
l2
γ1
γ4
a1
γ2
a2
γ̃3
l3
γ3
a3

各 $q=1,\cdots,n$ について

$$\gamma_q : I \to X,\ s \mapsto \gamma((1-s)t_{q-1} + st_q)$$ とおく.

$$(\gamma_q \in \mathrm{Path}(X, a_{q-1}, a_q))$$

また $\tilde{\gamma}_q := l_q * \gamma_q * l_{q-1} \in \mathrm{Loop}(U_{\varepsilon_q}, a)$ とおく.

このとき $[\gamma]_b = \hat{j}^{\varepsilon_n}_*(\underbrace{[\tilde{\gamma}_n]_b}_{\hat{\pi}_1(U_{\varepsilon_n}, a)}) * \cdots * \hat{j}^{\varepsilon_1}_*(\underbrace{[\tilde{\gamma}_1]_b}_{\pi_1(U_{\varepsilon_1}, a)})$ in $\pi_1(X, a)$

㊐

③ Φ の構成

各 $\gamma \in \mathrm{Loop}(X,a)$ について、一意性の証明のギロンで

$$[\gamma]_b = j_*^{\varepsilon_n}(\underbrace{[\tilde{\gamma}_n]_b}_{\in\, \pi_1(U_{\varepsilon_n},a)}) * \cdots * j_*^{\varepsilon_1}(\underbrace{[\tilde{\gamma}_1]_b}_{\in\, \pi_1(U_{\varepsilon_1},a)})$$ と表示し

$$\Phi([\gamma]_b) := j'_{\varepsilon_n}([\tilde{\gamma}_n]_b) \cdot \cdots \cdot j'_{\varepsilon_1}([\tilde{\gamma}_1]_b) \in Z'$$

とおけばよい

要チェック

- Φ の well-defined 性 ← ルベーグ数のギロンなどでがんばる (演習問題)
- Φ : 群 hom
- $\Phi \circ j_*^k = j'_k \quad (k=1,2)$

) easy

⑤

§14.2 : Van Kampen の定理の応用

応用①

Ex 14.2.1 : $S^2 := \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \}$, $* := (1,0,0) \in S^2$ とする.

Claim : $\pi_1(S^2, *)$: 自明 (特に S^2 は単連結)

$U_1 = S^2 \setminus \{(0,0,1)\}$, $U_2 = S^2 \setminus \{(0,0,-1)\}$ とおくと

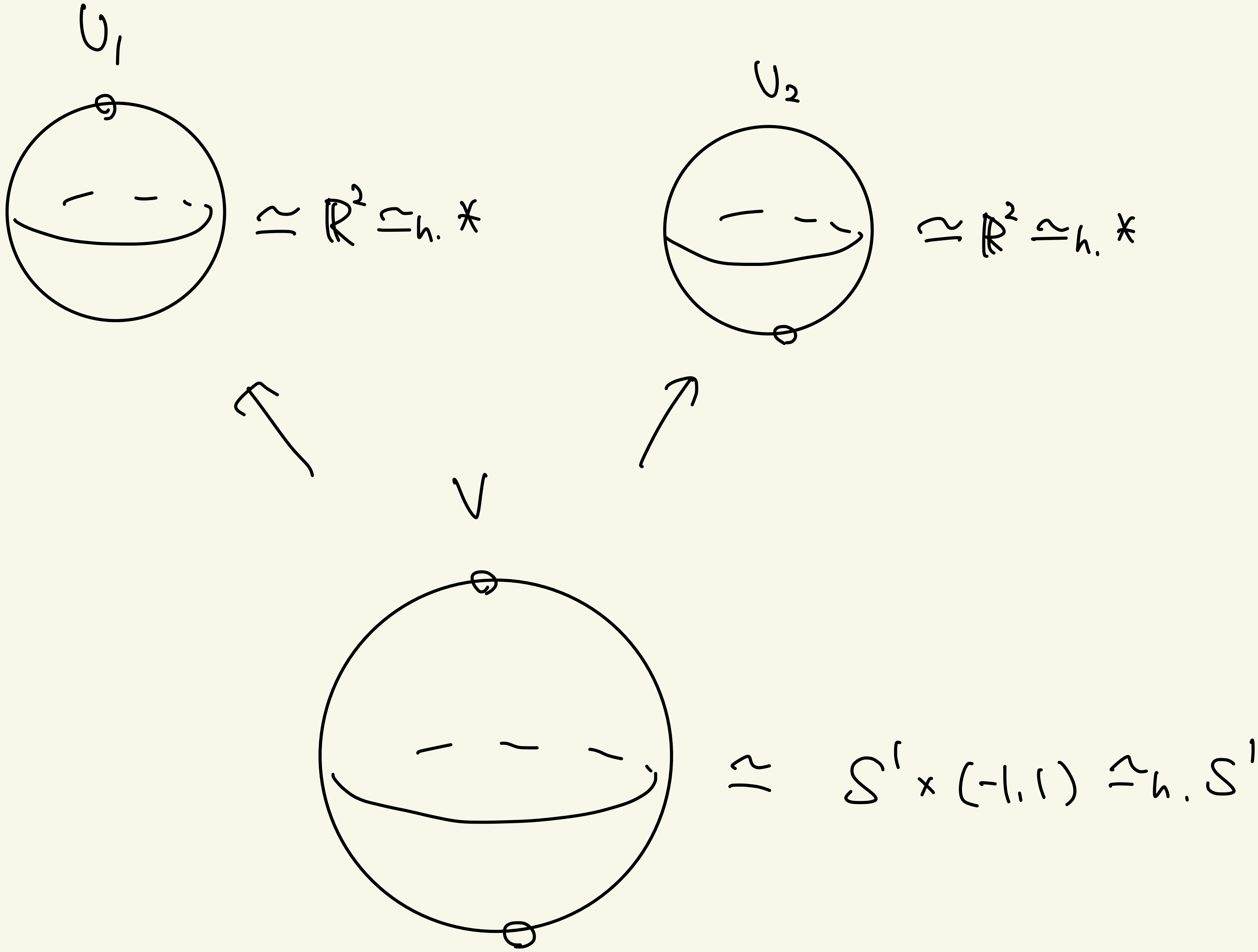
$* \in U_1, U_2 \underset{\text{open}}{\subset} S^2$　$S^2 = U_1 \cup U_2$

$V := U_1 \cap U_2$: 弧状連結

Thm 14.1.1 適用可!

特に

$$\pi_1(S^2, *) \cong \pi_1(U_1, a) *_{\pi_1(V, a)} \pi_1(U_2, a)$$


U_1
$\simeq \mathbb{R}^2 \simeq_{h.} *$
U_2
$\simeq \mathbb{R}^2 \simeq_{h.} *$
V
$\simeq S^1 \times (-1,1) \simeq_{h.} S^1$

U_1, U_2 : 可縮とする

$\pi_1(U_k, a)$: 自明 $(k=1,2)$

特に $\pi_1(S^2, a) \cong \pi_1(U_1, a) *_{\pi_1(V, a)} \pi_1(U_2, a)$ は自明群

Hint : G_1, G_2 : 自明群 ($\Rightarrow$ $G_1 * G_2$: 自明群)

$G_1 \supset H \subset G_2$

$\Rightarrow$ $G_1 *_H G_2$: 自明群

応用②

Ex 14.2.2 :

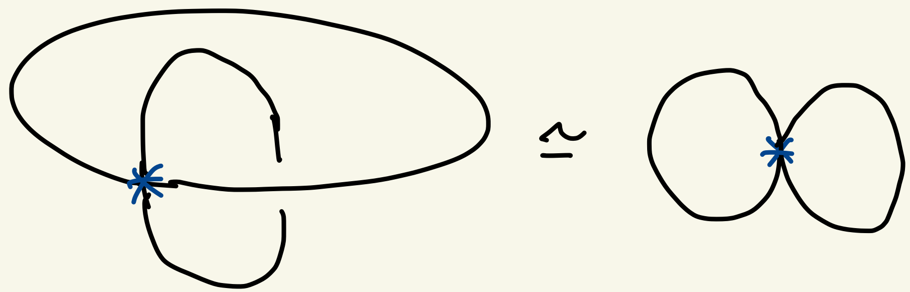
$S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$, $* := 1$ とおく.

$$(S^1, *) \vee (S^1, *) := (S^1 \vee S^1, *)$$

wedge 和

$$= \left(\{ (z_1, z_2) \in S^1 \times S^1 \mid z_1 = * \text{ or } z_2 = * \}, (1,1) \right)$$

とおく.

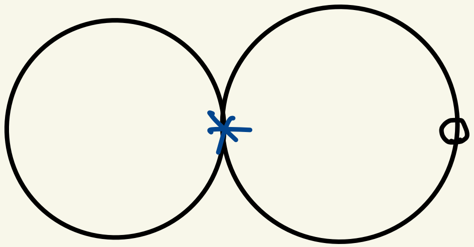
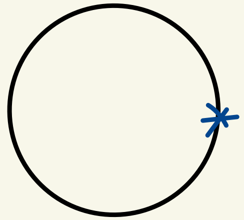
$(S^1 \vee S^1, *) =$ 

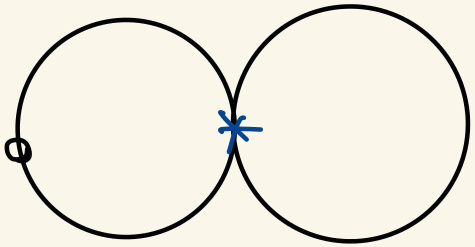
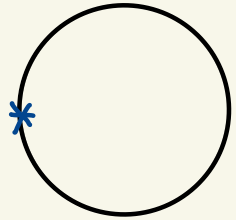
Claim $\pi_1(S^1 \vee S^1, *) \simeq \pi_1(S^1, *) * \pi_1(S^1, *)$ $(\simeq \mathbb{Z} * \mathbb{Z})$

(
$j^1 : S^1 \to S^1 \vee S^1,\ z_1 \mapsto (z_1, *)$

$j^2 : S^1 \to S^1 \vee S^2,\ z_2 \mapsto (*, z_2)$ とすると,

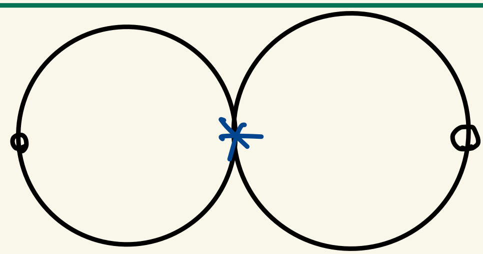
$(\pi_1(S^1 \vee S^1, *), j^1_*, j^2_*)$ は $\pi_1(S^1, *)$, $\pi_1(S^2, *)$ の自由積
)

$U_1 :=$  $\left(\simeq_h\right.$  $\left.\right)$

$U_2 :=$  $\left(\simeq_h\right.$  $\left.\right)$ とおけば

$U_1, U_2 \subset S^1 \vee S^1$: open

$* \in V := U_1 \cap U_2$: 弧状連結 かつ $U_1 \cup U_2 = S^1 \vee S^1$.

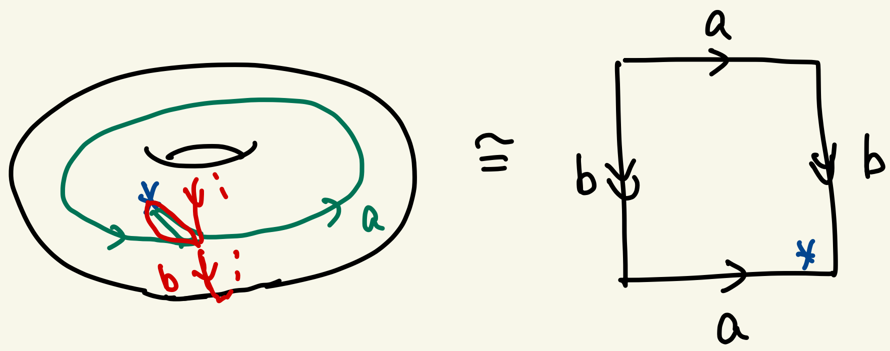
$V :=$  $(\simeq *)$

$\pi_1(V, *) \cong \pi_1(\{*\}, *)$: 自明群 より

$$\pi_1(S^1 \vee S^1, *) \underset{\text{Thm 14.1.1}}{\cong} \pi_1(S^1, *) *_{\pi_1(V,*)} \pi_1(S^1, *) \underset{*\text{演習問題}}{\cong} \pi_1(S^1, *) * \pi_1(S^1, *) \cong \mathbb{Z} * \mathbb{Z}. \quad \square$$

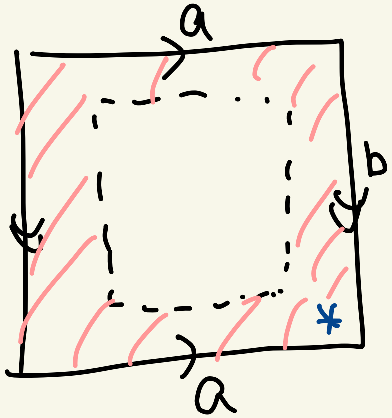
応用③

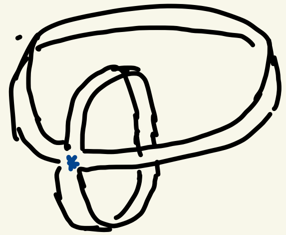
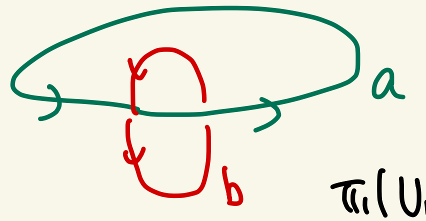
Ex 14.2.3: $T^2 := S^1 \times S^1$, $* = (1,1)$ とおく.

$(T^2, *) =$ 

Claim: $\pi_1(T^2, *) \cong \langle a, b \mid ab\bar{a}\bar{b} \rangle$

$\cong \mathbb{Z} \times \mathbb{Z}$ (Section 11 と別の証明を与える)

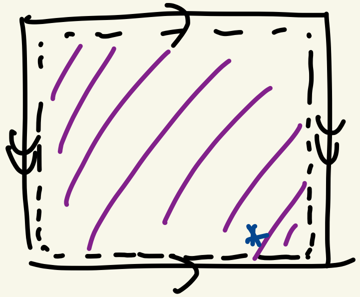
$U_1 :=$ 

$\Big(\simeq$  $\simeq_h$ 

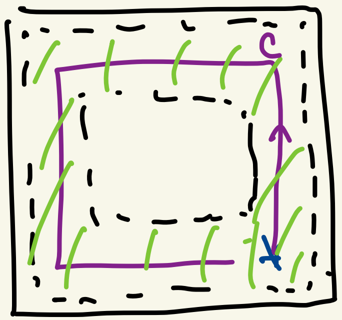
$\Big)$

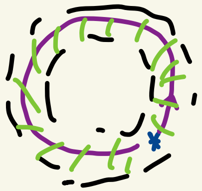
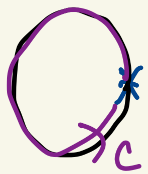
$= (S^1 \vee S^1, *)$

$\pi_1(U_1, *) \cong \langle a, b \rangle$

$U_2 :=$  $\Big(\simeq$  $\simeq_h * \Big)$ とおくと

$\pi_1(U_2, *)$: 自明群

$V =$ 

$\Big(\simeq$  $\simeq_h$ 

$= (S^1, *) \Big)$

弧状連結

$\pi_1(V, *) \cong \langle c \rangle \ (\cong \mathbb{Z})$

であり、 Thm 14.1.1 が適用可能

$\Big(\because U_1, U_2 \subset T^2$: open $U_1 \cup U_2 = T^2$, $* \in V$: 弧状連結$\Big)$

Thm 13.2.3 より,

$$\pi_1(T^2,*) \cong \pi_1(U_1,*) \ast_{\pi_1(V,*)} \pi_1(U_2,*)$$

$$\cong \langle a,b\rangle \ast_{\langle c\rangle} \langle \emptyset\rangle$$

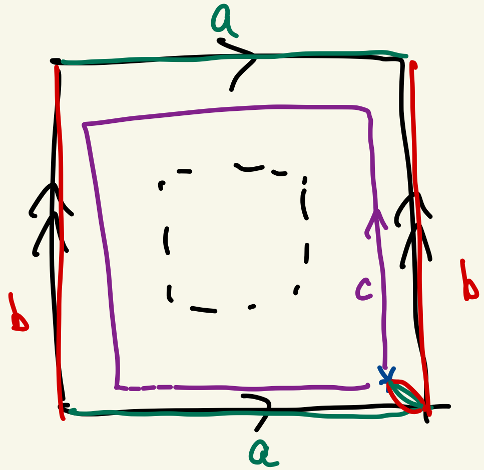
$$\cong \langle a,b \mid ab\bar{a}\bar{b}\rangle$$

$$\cong \mathbb{Z}\times\mathbb{Z}$$

要確認

$$\begin{array}{ccc} c & \longmapsto & ab\bar{a}\bar{b} \\ \cap & & \cap \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}\ast\mathbb{Z} = \langle a,b\rangle \end{array}$$

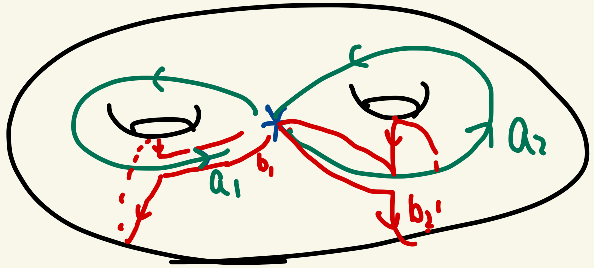
$$\left(\begin{array}{ccc} & \mathbb{Z} & \\ & \wr\wr & \\ \pi_1(S^1,*) & \longrightarrow & \pi_1(S^1\vee S^1,*) \\ \wr\wr & \circlearrowleft & \wr\wr \\ \pi_1(V,*) & \longrightarrow & \pi_1(U_1,*) \\ & \Uparrow & \\ V & \hookrightarrow & U_1 \end{array}\right.$$



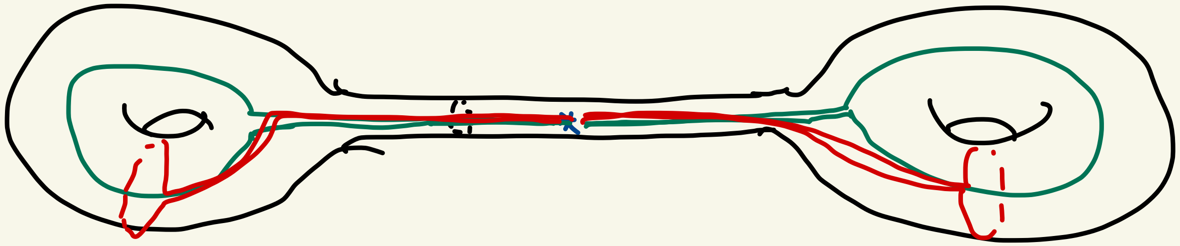
∎

応用④

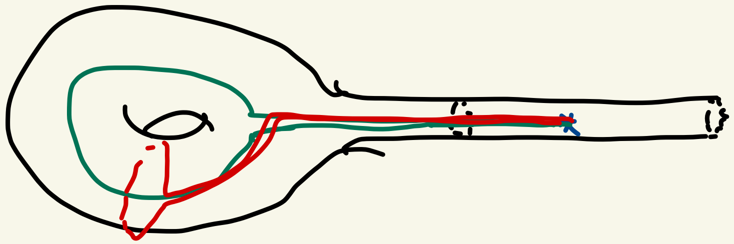
Ex 14.2.4

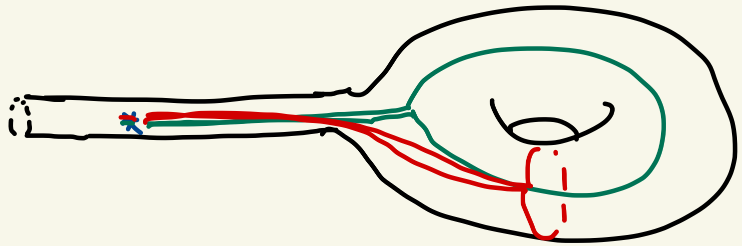
$(\Sigma_2, *) :=$  とおく.

Claim $\pi_1(\Sigma_2, *) = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 \bar{a}_1 \bar{b}_1 a_2 b_2 \bar{a}_2 \bar{b}_2 \rangle$

$(\Sigma_2, *) \cong$ 

($T^2 \# T^2$: T^2 と T^2 の連結和)

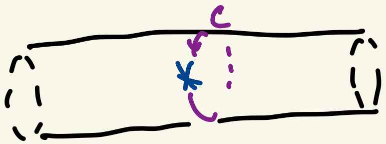
$U_1 :=$ 

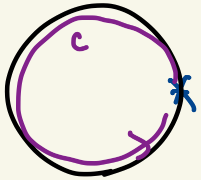
$U_2 :=$  とおく.

$U_1, U_2 \subset \Sigma_2$: open, $U_1 \cup U_2 = \Sigma_2$,

$* \in V := U_1 \cup U_2 =$  : 弧状連結

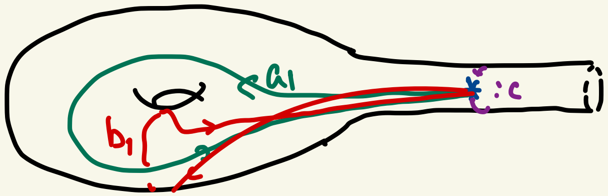
なので Thm 14.1.1 が 適用可能.



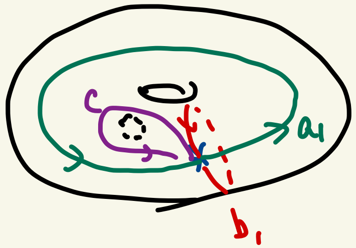
$\simeq_h$ 

とおくと $\{c\}$ は $\underset{\pi_1(S^1,*)\cong\mathbb{Z}}{\pi_1(V,*)}$ の生成系.

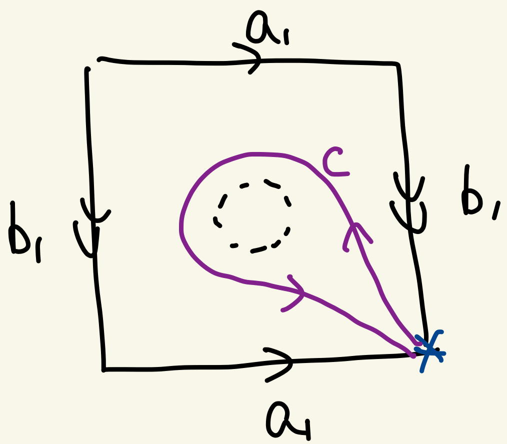
$(U_1, *) \cong$



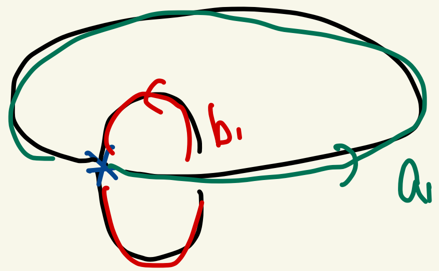
$\simeq_h$



$\cong$



$\cong$



$(c = a_1 b_1 \bar{a}_1 \bar{b}_1)$

$\simeq (S^1 \vee S^1, *)$

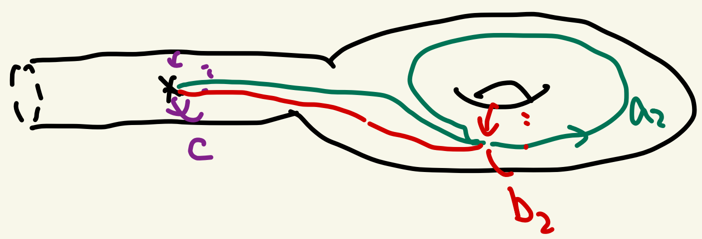
より $\pi_1(U_1, *) \cong \pi_1(S^1 \vee S^1, *) \cong \langle a_1, b_1 \rangle$ やつ

$\tilde{C}_1 = a_1 b_1 \bar{a}_1 \bar{b}_1$ とてよい.

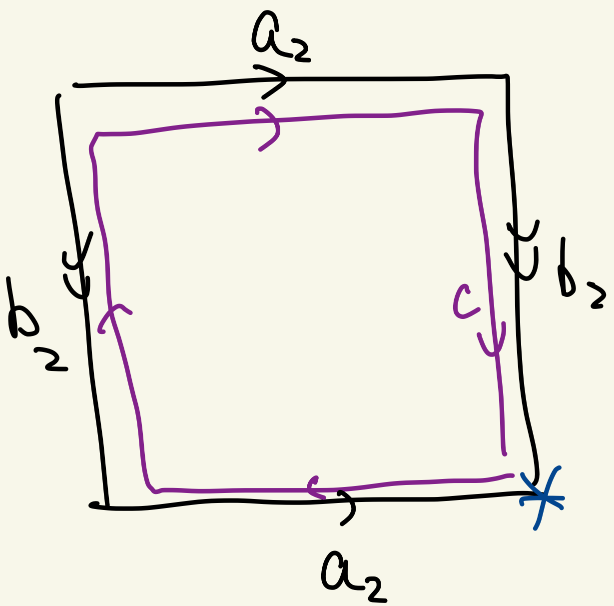
(Thm 13.2.3 の意味)

同様に考えると、 $\pi_1(U_2, *) = \langle a_2, b_2 \rangle$,

$\tilde{c}_2 = b_2 a_2 \bar{b}_2 \bar{a}_2$ としてよいことが分かる



$\simeq_h$



$(c = b_2 a_2 \bar{b}_2 \bar{a}_2)$

従って Thm 13.2.3 より

$$\pi_1(\Sigma_2, *) \cong \pi_1(U_1, *) \mathop{*}_{\pi_1(V,*)} \pi_1(U_2, *)$$

$$\cong \langle a_1, b_1 \rangle \mathop{*}_{\langle c \rangle} (a_2, b_2)$$

$$\cong \langle a_1, b_1, a_2, b_2 \mid \tilde{c}_1 \cdot \tilde{c}_2^{-1} \rangle$$

$$= \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 \bar{a}_1 \bar{b}_1 (b_2 a_2 \bar{b}_2 \bar{a}_2)^{-1} \rangle$$

$$= \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 \bar{a}_1 \bar{b}_1 a_2 b_2 \bar{a}_2 \bar{b}_2 \rangle$$

$\square$