Integrality of Dynkin indices for totally geodesic submanifolds of compact symmetric spaces

奥田隆幸（広島大学）

joint work with 久保亮（広島大学），
田丸博士（大阪市立大学）

広島幾何学研究集会 2018（板書用ノート：修正版）

1. Main results and Examples

2. Symmetric spaces

3. Dynkin indices for homomorphisms between non-compact simple Lie algebras

4. Key ideas of proofs

§1 Main results and Examples

M, N : isotropy-irreducible compact connected symmetric spaces of dimension ≥ 2

$\iota : M \to N$: a totally geodesic immersion.

Theorem A (Main theorem)

$$\mathrm{Dgn}(\iota) := K_{(N,g)} \Big/ K_{(M,\iota^* g)} \in \mathbb{Z}_{\geq 1}.$$

where g is an invariant Riemannian metric on N.

<u>Notation</u> : For each compact Riemannian mfd (S, g), we denote by $K_{(S,g)}$ the maximum value of sectional curvature on (S, g), that is,

$$K_{(S,g)} := \max \left\{ \begin{array}{c} \text{the sectional curvature of } V \\ \text{w.r.t. } g \end{array} \middle| \begin{array}{l} p \in S, \\ V \subset T_pS : \\ \text{2-dim'l} \end{array} \right\}.$$

Notation

M, N : symmetric spaces.

$$\mathrm{Hom}(M, N) := \{ \iota : M \to N \mid \text{totally geodesic immersions} \}.$$

Example A : $M = S^m$, $N = S^n$ $(n \geq m \geq 2)$.

$$\# \left(\mathrm{Hom}(S^m, S^n) / O(n+1) \right) = 1.$$

$$\forall z \in \mathrm{Hom}(S^m, S^n), \quad \mathrm{Dyn}(z) = 1.$$

In otherwords, for $r_1, r_2 > 0$, we have

(i) $S^m(r_1)$ has totally geodesic isometric immersion into $S^n(r_2)$.

$\Updownarrow$

(ii) $K_{S^m(r_1)} = K_{S^n(r_2)}$ $(\Leftrightarrow r_1 = r_2)$.

Example B

$$M = S^2, \quad N = Gr_2(\mathbb{R}^5).$$

$$\#\left(\mathrm{Hom}(S^2, Gr_2(\mathbb{R}^5)) / O(5) \right) \leq 7.$$

$$\mathrm{Dyn}(\mathrm{Hom}(S^2, Gr_2(\mathbb{R}^5))) = \{1, 2, 10\}.$$

Example e

$$M = S^2, \quad N = Gr_3(\mathbb{R}^6).$$

$$\#\left(\mathrm{Hom}(S^2, Gr_3(\mathbb{R}^6)) \big/ O(6) \right) \leq 6.$$

$$\mathrm{Dyn}\left(\mathrm{Hom}(S^2, Gr_3(\mathbb{R}^6)) \right) = \{1, 2, 4, 10\}.$$

Observation

For $L \xrightarrow{y} M \xrightarrow{z} N$,

we have

$$\mathrm{Dyn}(z \circ y) = \mathrm{Dyn}(y) \cdot \mathrm{Dyn}(z).$$

More precisely, "Dyn" gives a covariant functor
from
the category of isotropy-irred. cpt comm. symm. sp of dim ≥ 2
& tot. geod. immersions
to
the monoid $\mathbb{Z}_{\geq 1}$ $\left(\begin{array}{l} \mathrm{Obj} = \{*\} \\ \mathrm{Hom}(*,*) = \mathbb{Z}_{\geq 1} \end{array} \right)$

Example D

We observe that

$$\mathrm{Dyn}\left(\mathrm{Hom}\left(Gr_2(\mathbb{R}^5),\ Gr_3(\mathbb{R}^6)\right)\right) = \{1\}$$

In particular, for any $\iota \in \mathrm{Hom}\left(Gr_2(\mathbb{R}^5),\ Gr_3(\mathbb{R}^6)\right)$

and any Helgason sphere S of $Gr_2(\mathbb{R}^5)$,

there exists a Helgason sphere S' of $Gr_3(\mathbb{R}^6)$

s.t. $\iota(S) \subset S'$ (totally geodesic).

<u>Rem</u> : "Dyn" is not "complete invariant" in general

<u>Example E</u> Let $N := SU(8)/SO(8)$

$\exists\ \iota_1, \iota_2 : S^2 \to N$

s.t.

$\mathrm{Dyn}(\iota_1) = \mathrm{Dyn}(\iota_2)$ but

$$\iota_1(S^2) \underset{\mathrm{Aff}(N)}{\not\sim} \iota_2(S^2)$$

Theorem B :

Suppose that

N is a Hermitian symmetric space

(isotropy-irred semisimple compact connected)

We put

$$\mathrm{Hom}_{hol}(\mathbb{C}P^1, N) := \left\{ z : \mathbb{C}P^1 \to N \;\middle|\; \begin{array}{l} \text{totally geodesic} \\ \text{holomorphic} \\ \text{immersions} \end{array} \right\}$$

Then

$$\#\left(\mathrm{Hom}_{hol}(\mathbb{CP}^1, N) \big/ A(N) \right) = \mathrm{rank}\, N$$

and

$$\mathrm{Dyn} : \mathrm{Hom}_{hol}(\mathbb{CP}^1, N) \big/ A(N) \xrightarrow{1:1} \{1, 2, \cdots, \mathrm{rank}\, N\}$$

$$\left(\text{Ex} : N = Gr_k(\mathbb{C}^n) \right)$$

§2 Symmetric spaces

<u>Def</u> (Symmetric spaces [cf. 長野, RIMS 1206 (2001)])

M : finite dim'l C^∞-mfd.

$s : M \times M \to M$: C^∞-map
$(x, y) \mapsto s_x(y)$

(M, s) is a symmetric space if
the following three conditions hold :

Condition (i) $s_x : M \to M$ is an involutive (i.e. $s_x^2 = id_M$) diffeomorphic for any $x \in M$

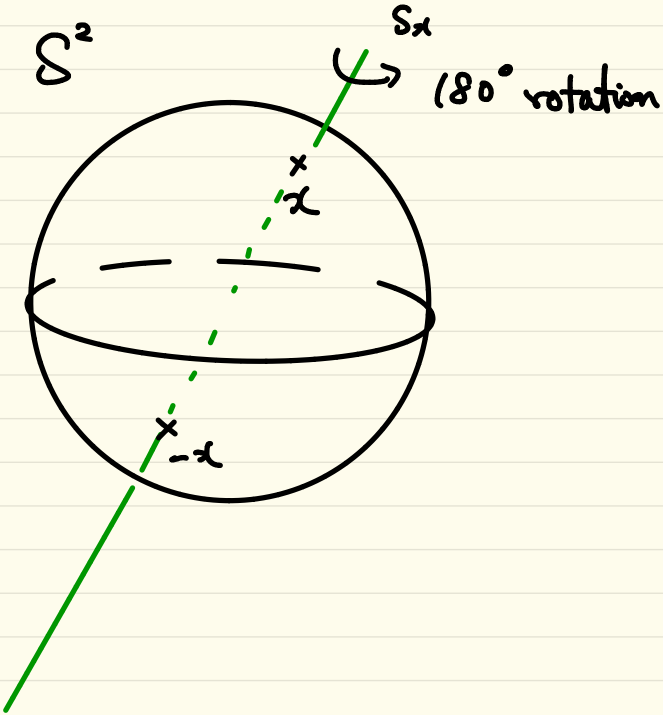
(ii) x is an isolated fixed point of s_x for any $x \in M$

(iii) For any x, y

$$\begin{tikzcd} M \arrow[r, "s_x"] \arrow[d, "s_y"'] & M \arrow[d, "s_{s_x(y)}"] \\ M \arrow[r, "s_x"'] & M \end{tikzcd}$$

is commutative

Ex S^2



M : a connected symmetric space

$U = U_M := \mathrm{Aut}(M)$: the group of automorphisms of M

For each $x \in M$, we put

$$U^x := \{ f \in U \mid f(x) = x \}$$

(the isotropy subgp of U at x)

Fact ⓐ U is a Lie gp with respect to the compact open topology

ⓑ $U \curvearrowright M$ is transitive

ⓒ (U, U^x) is a symmetric pair

ⓓ $M \simeq U/U^x$ as symmetric spaces

Fact: $\exists! \; \nabla_M$: U-invariant affine connection on M

and $\mathrm{Aut}\, M = \mathrm{Aff}\, M$.

Observation

$(M, s_M), (N, s_N)$: symmetric spaces

∇_M, ∇_N : invariant affine connections

$f : M \to N$: an immersion

Then

(i) f : totally geodesic w.r.t. ∇_M and ∇_N

$\Updownarrow$

(ii) f : homomorphism w.r.t. s_M and s_N

M : a connected symmetric space.

We call

M is isotropy-irreducible if

$U^x \curvearrowright T_x M$ is irreducible

as real linear representation

for any $x \in M$

<u>Fact</u> : M : an isotropy-irreducible compact connected symmetric space

Then U-invariant Riemannian metrics on M exist uniquely up to scalar

Therefore for

M, N : isotropy-irreducible compact connected symmetric spaces of $\dim \geq 2$

$\iota : M \to N$: a totally geodesic immersion

$K(N,g) / K(M, \iota^* g)$ does not depend on the choice of U-invariant Riemannian metrics g on N.

Recall (Theorem A)

$$\lfloor \mathrm{Dyn}(c) := K_{(N,g)} \big/ K_{(M,2^*g)} \in \mathbb{Z}_{\geq 1}$$

§3 Dynkin indices for homomorphisms between non-compact simple Lie algebras

M : compact connected symmetric space

$U := \mathrm{Aut}(M)$

We put $\mathfrak{u} := \mathrm{Lie}\, U$

$\mathfrak{u}^x := \mathrm{Lie}\, U^x$ for each $x \in M$

Then $(\mathfrak{u}, \mathfrak{u}^x)$ is a compact symmetric pair

That is, $\exists\, \theta_x : \mathfrak{u} \to \mathfrak{u}$: an involutive automorphism on $\mathfrak{u}$

s.t. $\mathfrak{k} := \mathfrak{u}^x = \{ X \in \mathfrak{u} \mid \theta_x X = X \}$

We put $\mathfrak{p} := \{ X \in \mathfrak{u} \mid \theta_x X = -X \}$

Then $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$.

<u>Def</u>

$\mathfrak{g} := \mathfrak{k} + \sqrt{-1}\,\mathfrak{p} \quad (\subset \mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C})$

$\mathfrak{g}_M = \mathfrak{g}$ is a non-compact reductive Lie algebra

<u>Fact</u> M : isotropy-irreducible with $\dim M \geq 2$

$\Leftrightarrow$ $\mathfrak{g}_M$: simple$/\mathbb{R}$

Assume M is isotropy-irreducible with $\dim M \geq 2$

Fact (cf. Helgason (1966)) Fix an U-inv. Riem. met. g on M.

Then $K_{(M,g)} = \dfrac{2}{\| \lambda^{\vee}_{\mathfrak{g}_M} \|_g^2}$,

where $\| \cdot \|_g$ denotes the norm on $\mathfrak{p}$

induced by the Riem. metric g on M

and $\lambda^{\vee}_{\mathfrak{g}_M}$ is the coroot of a highest root

of $(\mathfrak{g}_M, \mathfrak{a})$

where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$

<u>Proposition</u> : M, N : compact connected symmetric spaces

$$\{\, \iota : M \to N : \text{totally geodesic immersions} \,\} / \mathrm{Aut}(N)$$

$$\exists \int$$

$$\{\, \iota : \mathfrak{g}_M \to \mathfrak{g}_N : \text{non-zero Lie algebra homomorphisms} \,\} / G_N$$

Here (G_N, U_N^σ) is

the non-compact dual of (U_N, U_N^σ).

$$U_N = \mathrm{Aut}(N)$$

In particular, if M, N : isotropy-irreducible with $\dim M, N \geq 2$

Then $$D_{\varphi n}(\iota) = \frac{\| \iota(\lambda^\vee_{g_M}) \|^2_g}{\| \lambda^\vee_{g_N} \|^2_g}$$ (g : U-inv Riem. met. on N)

Ex : $M = S^2$, $\quad N = Gr_2(\mathbb{R}^5)$

$$\mathfrak{g}_M = \mathfrak{sl}_2(\mathbb{R}) \qquad \mathfrak{g}_N = \mathfrak{so}(3,2)$$

We can take

$$\lambda^\vee_{\mathfrak{sl}_2(\mathbb{R})} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

$$\lambda^\vee_{\mathfrak{so}(3,2)} = \left(\begin{array}{ccc|cc} & & & 1 & \\ & & & & 1 \\ & & & & \\ \hline -1 & & & & \\ & -1 & & & \end{array}\right)$$

$$\iota : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{so}(3,2) \quad : \quad \overset{\text{non-zero}}{\text{Lie alg hom.}}$$

then

$$\iota(\lambda^\vee_{\mathfrak{sl}_2(\mathbb{R})}) \simeq \left(\begin{array}{c|c} & \begin{matrix}1 & \\ & 1\end{matrix} \\ \hline \begin{matrix}-1 & \\ & -1\end{matrix} & \end{array}\right) : \text{Dyn} = 1$$

$$\left(\begin{array}{c|c} & \begin{matrix}2 & \\ & 0\end{matrix} \\ \hline \begin{matrix}-2 & \\ & 0\end{matrix} & \end{array}\right) : \text{Dyn} = 2$$

or

$$\left(\begin{array}{c|c} & \begin{matrix}4 & \\ & 2\end{matrix} \\ \hline \begin{matrix}-4 & \\ & -2\end{matrix} & \end{array}\right) : \text{Dyn} = 10$$

Theorem C $\mathfrak{f}, \mathfrak{g}$: non-compact simple Lie algebras

$\iota : \mathfrak{f} \to \mathfrak{g}$: a Lie algebra hom.

Then $$D_{\mathrm{syn}}(\iota) := \frac{\| \iota(\lambda_{\mathfrak{f}}^{\vee}) \|^2}{\| \lambda_{\mathfrak{g}}^{\vee} \|^2} \in \mathbb{Z}_{\geq 1}$$

($\| \cdot \|$ is the norm induced from a non-deg. invariant bilinear form on $\mathfrak{g}$)

Theorem A (Main theorem) follows from Theorem C

<u>Remark</u> In the cases where

$\mathfrak{f}, \mathfrak{g}$ are both complex simple

and

$\iota : \mathfrak{f} \to \mathfrak{g}$: complex Lie alg. hom.

Theorem C is proved by Dynkin ('52) by using some classifications.

§4 Key ideas for proofs

$(V, \langle , \rangle)$: a fin-dim vector sp$/\mathbb{R}$ with an inner-product.

$\Delta \subset V$: a root system with $\operatorname{span} \Delta = V$

$W(\Delta) \curvearrowright V$: the Weyl group

$\Delta^+ \subset \Delta$: a positive system

$w_0 \in W(\Delta)$: the longest element of $W(\Delta)$ w.r.t. $\Delta^+ \subset \Delta$

We put

$$\tau := w_0 \circ -\mathrm{id}_V : V \to V \quad : \text{the Tits involution}$$

$$V^\tau := \{ v \in V \mid \tau(v) = v \}$$

Key lemma for Theorem C

$$\exists \{\beta_1, \cdots, \beta_m\} \subset \Delta \text{ s.t.}$$

(i) $\beta_1, \cdots, \beta_m$ are strongly-orthogonal to each other (i.e. $\beta_i \pm \beta_j \notin \Delta$)

(ii) $\operatorname{span}\{\beta_1 \cdots \beta_m\} = V^{\tau}$

Example $V = \{ (a_1 \cdots a_n) \in \mathbb{R}^n \mid \Sigma a_i = 0 \}$

$$\Delta = \{ \varepsilon_i - \varepsilon_j \mid i \neq j \}$$

where $\varepsilon_i = (0, \cdots 0, \underset{\hat{i}}{1}, 0, \cdots 0)$

$$\Delta^+ = \{ \varepsilon_i - \varepsilon_j \mid i > j \}$$

$$W(\Delta) = S_n \underset{\text{permutation}}{\curvearrowright} \mathbb{R}^n \supset V$$

$$w_0 \cdot (a_1 \cdots a_n) = (a_n, \cdots, a_1)$$

$$\tau (a_1 \cdots a_n) = (-a_n, \cdots, -a_1)$$

Thus we can take

$\{\beta_1 \cdots \beta_m\}$ as $\{\varepsilon_1 - \varepsilon_n, \varepsilon_2 - \varepsilon_{n-1}, \cdots \}$

Rem: Key lemma above can be proved without any classifications (but by induction)

Cf. Agaoka - Kaneda (2002)

classified maximal strongly-othogonal subsets

in each root systems

Idea for the proof of Theorem C :

$\mathfrak{f}$, $\mathfrak{g}$: non-cpt simple Lie algebras

$\iota : \mathfrak{f} \to \mathfrak{g}$: Lie alg. hom.

<u>Claim</u> $\dfrac{\| \iota(\lambda_{\mathfrak{f}}^{\vee}) \|^2}{\| \lambda_{\mathfrak{g}} \|^2} \in \mathbb{Z}$

<u>Step 1</u> $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: Cartan decomp

$$\begin{array}{ll} \mathfrak{p} & \\ \cup & \\ \mathfrak{a} & : \text{maximal abelian} \\ \in & \\ \lambda_{\mathfrak{g}}^{\vee} & \left(\lambda_{\mathfrak{g}} \in \Sigma^{+} \subset \Sigma(\mathfrak{g}, \mathfrak{a}) \right) \end{array}$$

<u>Def</u> $\Sigma_{\text{even}} := \{ \alpha \in \Sigma \mid \alpha(\lambda_{\mathfrak{g}}^{\vee}) : \text{even} \}$

Then Σ_{even} is a root system

$\Sigma_{\text{even}}^{+} := \Sigma_{\text{even}} \cap \Sigma^{+} \subset \Sigma_{\text{even}}$: positive system

Step 2 τ : the Tits involution for

$$\Sigma^{f}_{even} \subset \Sigma_{even}$$

Then $z(\lambda^\vee_f)$ is conjugate to an element in $\operatorname{span}(\Sigma^\vee_{even})^\tau$

Thus we assume

$$z(\lambda^\vee_f) \in \operatorname{Span}(\Sigma^\vee_{even})^\tau$$

<u>Step 3</u> Take $\{\beta_1 \cdots \beta_m\}$ as the key lemma for $\Sigma_{even}^{+} \subset \Sigma_{even}$

Then

$$2(\lambda_f^\vee) = \sum_i c_i \beta_i^\vee$$

with $c_i \in \mathbb{Z}$

Step 4

$$\frac{\| \alpha^\vee \|^2}{\| \lambda_g^\vee \|^2} = 1,\ 2 \text{ or } 4$$

for any $\alpha \in \Sigma_{\text{even}}$

In particular

$$\frac{\| \beta_i^\vee \|^2}{\| \lambda_g^\vee \|^2} \in \mathbb{Z}$$

Step 5

$$\frac{\| z(\lambda_f^\vee) \|^2}{\| \lambda_g^\vee \|^2} = \sum_i c_i^2 \frac{\| \beta_i^\vee \|^2}{\| \lambda_g^\vee \|^2} \quad (\{\beta_i^\vee\}_i \text{ is orthogonal})$$

?

$$\geq$$

Thank you for your attention!