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Nonintegrability of Hamiltonian system perturbed from integrable system with two singular points

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Abstract We give a Hamiltonian system which is nonintegrable in a domain containing two singular points and that is integrable in some neighborhood of a singular point. The system is an arbitrarily small nontrivial perturbation of an integrable Hamiltonian system given by confluence of regular singular points of a hypergeometric system.

Keywords nonintegrability · Hamiltonian system with two singular points · hypergeometric system · confluence · Okubo equation

1 Introduction

Let $n \geq 2$ be an integer, and consider the Hamiltonian system

$$\begin{cases} z^2 \dfrac{dq}{dz} = \ \nabla_p \mathcal{H}(z,q,p), \\ z^2 \dfrac{dp}{dz} = -\nabla_q \mathcal{H}(z,q,p), \end{cases} \tag{1}$$

where $q = (q_2, ..., q_n)$, $p = (p_2, ..., p_n)$. Here

$$\nabla_q := \left(\frac{\partial}{\partial q_2}, ..., \frac{\partial}{\partial q_n}\right), \quad \nabla_p := \left(\frac{\partial}{\partial p_2}, ..., \frac{\partial}{\partial p_n}\right).$$

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The system (1) is equivalent to an autonomous one

$$\begin{cases} \dot{q}_1 = \ H_{p_1}, & \dot{q} = \ \nabla_p H, \\ \dot{p}_1 = -H_{q_1}, & \dot{p} = -\nabla_q H, \end{cases} \tag{2}$$

where $q_1 = z$ and $H(q_1, q, p_1, p) := q_1^2 p_1 + \mathcal{H}(q_1, q, p)$ or $H(q_1, q, p_1, p) := p_1 + q_1^{-2}\mathcal{H}(q_1, q, p)$. We say that the Hamiltonian system (2) is C^ω-Liouville integrable if there exist first integrals $\phi_j \in C^\omega$ $(j = 1, \ldots, n)$ which are functionally independent on an open dense set and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. The Hamiltonian H is a first integral of this autonomous system. We abbreviate C^ω-Liouville integrable to C^ω- integrable or integrable if there is no fear of confusion.

In [2] Bolsinov and Taimanov showed a non C^ω-integrability of some Hamiltonian system related with geodesic flow on a Riemannian manifold. Then Gorni and Zampieri showed similar results in the local setting, namely for a Hamiltonian system being singular at the origin they showed the non C^ω-integrability. (cf. [3], [4]). In this paper we study the nonintegrability from a semi-global point of view. Namely we consider Hamiltonian system which is singular at the origin $q_1 = 0$ as well as $q_1 = 1$. We will show that the system is integrable near the origin, while it is not integrable in the domain containing $q_1 = 0$ and $q_1 = 1$. The Hamiltonian function is given by the arbitrary small non zero perturbation of an integrable Hamiltonian of the confluent hypergeometric system. (cf. §3).

More precisely, we consider

$$H = \sum_{j \in J'} \frac{\tau_j}{\lambda_j} q_j p_j + \frac{q_1^2}{(q_1 - 1)^2} \sum_{j \in J} \frac{\tau_j}{\lambda_j} q_j p_j + q_1^2 p_1, \tag{3}$$

where τ_j and $\lambda_j \neq 0$ are complex constants and J and J' are the sets of multi-indices such that

$$J \neq \emptyset, J' \neq \emptyset, J \cap J' = \emptyset, J \cup J' = \{2, ..., n\}. \tag{4}$$

The Hamiltonian is derived from the hypergeometric system by confluence of singularities. (cf. §3). The Hamiltonian system (2)- (3) determines the Hamiltonian vector field

$$\begin{aligned} \chi_H = \ & q_1^2 \frac{\partial}{\partial q_1} - 2 q_1 p_1 \frac{\partial}{\partial p_1} + \frac{2 q_1}{(q_1 - 1)^3} \left(\sum_{j \in J} \frac{\tau_j}{\lambda_j} q_j p_j \right) \frac{\partial}{\partial p_1} \\ & + \sum_{j \in J'} \frac{\tau_j}{\lambda_j} \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right) + \frac{q_1^2}{(q_1 - 1)^2} \sum_{j \in J} \frac{\tau_j}{\lambda_j} \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right). \end{aligned} \tag{5}$$

Let

$$H_1 := \sum_{j=2}^{n} p_j^2 B_j(q_1, p). \tag{6}$$

Note that H_1 does not depend on q. Suppose that the nonresonance condition (NRC) holds:

$$\forall \gamma = (\gamma_2, \dots, \gamma_n) \in \mathbb{Z}^{n-1} \setminus \{0\},\ \sum_{j=2}^{n} \frac{\tau_j}{\lambda_j}\gamma_j \neq 0, \tag{7}$$

i.e. τ_j/λ_j's are linearly independent over $\mathbb{Z}^{n-1}$. Moreover, assume

(TC): For $k \in J'$, the equation

$$q_1^2 \frac{d}{dq_1} v - 2\frac{\tau_k}{\lambda_k} v = B_k(q_1, 0) \tag{8}$$

has no solution v holomorphic at $q_1 = 0$, and for $k \in J$, the equation

$$q_1^2 \frac{d}{dq_1} w - 2\frac{\tau_k}{\lambda_k}\frac{q_1^2 w}{(q_1-1)^2} = B_k(q_1, 0) + \frac{\tau_k}{\lambda_k}\frac{q_1 B_k(0,0)}{(q_1-1)^2} + B_k(0,0) \tag{9}$$

has no solution w holomorphic at $q_1 = 1$.

Let $\Omega_1 \subset \mathbb{C}$ be a domain containing $\{q_1 = 0, 1\}$, and $\Omega_2 \subset \mathbb{C}^{2n-1}$ be a neighborhood of $(p_1, q, p) = (0, 0, 0)$ and define $\Omega := \Omega_1 \times \Omega_2$. Then we have

Theorem 1 *Assume that (NRC) and (TC) are satisfied. Then, there exists Ω such that the Hamiltonian system (2) is not C^ω-integrable in Ω. More precisely, for every first integral ϕ satisfying $\chi_{H+H_1}\phi = 0$ and holomorphic in Ω, there exists a holomorphic function $\psi(t)$ defined in some neighborhood of the origin $t = 0 \in \mathbb{C}$ such that $\phi(q_1, q, p_1, p) = \psi(H + H_1)$ in some neighborhood of the origin.*

Remark. (i) In §4 we will show that (TC) holds on an open dense set in the set of analytic functions. (TC) also implies that H_1 could be replaced by εH_1 with an arbitrary small $\varepsilon \neq 0$. On the other hand, it is necessary in Theorem 1 that H_1 does not vanish identically because H is integrable in view of Lemma 2. (cf. §3). Hence the non-integrability occurs by an arbitrary small non-zero generic perturbation. In Proposition 1 we also show that our class of Hamiltonians contains subclass for each of which the integrability at the origin holds. Hence the (non-) integrability in Theorem 1 is caused by the interference of singular points.

2 Proof of Theorem 1

Let $\phi =: u$ be a holomorphic first integral in Ω and expand u at $p = 0$

$$u = \sum_{\alpha} u_\alpha(q_1, q, p_1) p^\alpha. \tag{10}$$

Substitute (10) into $\chi_{H+H_1}u=0$ and compare the powers like $p^0=1$ of both sides. Then we have the equation of $u_0=u_0(q_1,q,p_1)$

$$\{q_1^2p_1,u_0\}+\sum_{j\in J'}\frac{\tau_j}{\lambda_j}q_j\frac{\partial}{\partial q_j}u_0+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}q_j\frac{\partial}{\partial q_j}u_0=0. \tag{11}$$

Indeed, no constant term in p appears from $\chi_{H_1}u$ in view of the definition of χ_{H_1}.

Substituting the expansion $u_0=\sum_\beta u_{0,\beta}(q_1,p_1)q^\beta$ into (11), we see that $U_0:=u_{0,0}$ satisfies $\{q_1^2p_1,U_0\}=0$, namely

$$\left(q_1\frac{\partial}{\partial q_1}-2p_1\frac{\partial}{\partial p_1}\right)U_0=0. \tag{12}$$

Substitute the expansion $U_0=\sum_{\nu,\mu}c_{\nu\mu}q_1^\mu p_1^\nu$ into (12). Then we have $\sum_{\nu,\mu}c_{\nu,\mu}(\mu-2\nu)q_1^\mu p_1^\nu=0$. It follows that $c_{\nu,\mu}=0$ for $\mu\neq 2\nu$. Hence we obtain

$$U_0=\sum_\nu c_{\nu,2\nu}q_1^{2\nu}p_1^\nu=\sum_\nu c_{\nu,2\nu}(q_1^2p_1)^\nu. \tag{13}$$

It follows that there exists a function of one variable t, $\phi_0(t)$ holomorphic in some neighborhood of $t=0$ such that $U_0=\phi_0(q_1^2p_1)$.

Next, we focus on the equation of $u_{0,\beta}$ with $\beta\neq 0$

$$\{q_1^2p_1,u_{0,\beta}\}+\sum_{j\in J'}\frac{\tau_j}{\lambda_j}\beta_ju_{0,\beta}+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\beta_ju_{0,\beta}=0.$$

Expand

$$u_{0,\beta}=\sum_\nu \omega_{\beta,\nu}(q_1)p_1^\nu, \tag{14}$$

and consider the equation of $\omega_{\beta,\nu}$. If $\nu=0$, then, by comparing the coefficients of $p_1^0=1$, we have

$$q_1^2\frac{d}{dq_1}\omega_{\beta,0}+\left(\sum_{j\in J'}\frac{\tau_j}{\lambda_j}\beta_j+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\beta_j\right)\omega_{\beta,0}=0. \tag{15}$$

Since $\beta\neq 0$, it follows from (NRC), (7), that either $A':=\sum_{j\in J'}\frac{\tau_j}{\lambda_j}\beta_j\neq 0$ or $A:=\sum_{j\in J}\frac{\tau_j}{\lambda_j}\beta_j\neq 0$ is valid. If $A'\neq 0$, then we have $\omega_{\beta,0}=0$ in some neighborhood of $q_1=0$. Indeed, by subsituting the expansion $\omega_{\beta,0}=\sum_{l=0}^\infty C_lq_1^l$ into (15) and by using the relations

$$q_1^2\frac{d}{dq_1}\omega_{\beta,0}=\sum_{l=0}^\infty C_llq_1^{l+1}$$

and

$$\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\beta_j\omega_{\beta,0}=\sum_{l=0}^{\infty}C_l'\ {q_1}^{l+2}$$

for some C_l', we obtain

$$\begin{aligned}&C_0A'=0 \quad \text{i.e. } C_0=0,\\&C_1A'+C_0\cdot 0=0 \quad \text{i.e. } C_1=0,\\&C_2A'+C_0'+C_1=0 \quad \text{i.e. } C_2=0,\\&\qquad\cdots\end{aligned}$$

Note that $C_0'=0$ since $C_0=0$. Hence we have $\omega_{\beta,0}=0$.

In the case where $A'=0$ and $A\neq 0$, (15) is written in

$$(q_1-1)^2\frac{d}{dq_1}\omega_{\beta,0}+A\omega_{\beta,0}=0. \tag{16}$$

Similarly to the case $A'\neq 0$, we obtain $\omega_{\beta,0}=0$ in some neighborhood of $q_1=1$. Therefore, we have $\omega_{\beta,0}=0$ in Ω_1.

Next, by comparing the coefficients of $p_1^1=p_1$, we have the equation of $\omega_{\beta,1}(q_1)$

$$\left(q_1^2\frac{d}{dq_1}-2q_1\right)\omega_{\beta,1}+\left(A'+\frac{q_1^2}{(q_1-1)^2}A\right)\omega_{\beta,1}=0. \tag{17}$$

Similarly to the above, $A'\neq 0$ implies $\omega_{\beta,1}=0$ near $q_1=0$, while $A'=0$ and $A\neq 0$ imply $\omega_{\beta,1}=0$ near $q_1=1$. Hence we have $\omega_{\beta,1}=0$ in Ω_1. By the same argument we obtain $\omega_{\beta,\nu}=0$ in Ω_1 for all $\nu\in\mathbb{N}\cup\{0\}$. It follows that $u_{0,\beta}=0$ for all $\beta\neq 0$.

Therefore, we have

$$u_0=u_{0,0}(q_1^2p_1)+\sum_{\beta\neq 0}u_{0,\beta}(q_1^2p_1)q^\beta=\phi_0(q_1^2p_1) \tag{18}$$

for some $\phi_0(t)$ of one variable being analytic at $t=0$. Note that

$$\begin{aligned}u|_{p=0}-\phi_0(H+H_1)|_{p=0}&=u_0(q_1,p_1)-\phi_0(H|_{p=0})\\&=\phi_0(q_1^2p_1)-\phi_0(q_1^2p_1)\equiv 0.\end{aligned}$$

Hence, without loss of generality, we may assume $u|_{p=0}=0$.

Next we consider $u_\alpha=u_\alpha(q_1,p_1,q)$ for $|\alpha|=1$. Write $\alpha=e_k$ $(2\leq k\leq n)$ where $e_k:=(0,...,0,1,0,...,0)$ is the k-th unit vector. Then, u_α satisfies

$$\begin{aligned}&\{q_1^2p_1,u_\alpha\}+\sum_{j\in J'}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j}-\delta_{k,j}\right)u_\alpha\\&\qquad+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j}-\delta_{k,j}\right)u_\alpha=0,\end{aligned} \tag{19}$$

where $\delta_{k,j}$ is the Kronecker's delta, $\delta_{k,j} = 1$ if $k = j$, and $=0$ if otherwise. Note that, because $u_0 = 0$, χ_{H_1} gives no term.

Substitute the expansion $u_\alpha = \sum_\beta u_{\alpha,\beta}(q_1, p_1)q^\beta$ into (19), and compare the powers like $q^0 = 1$. Then we have the equation of $u_{\alpha,0}$

$$\{q_1^2 p_1, u_{\alpha,0}\} - \frac{\tau_k}{\lambda_k}\left(\sum_{j\in J'} \delta_{k,j}\right) u_{\alpha,0} - \frac{q_1^2}{(q_1-1)^2}\left(\sum_{j\in J} \frac{\tau_j}{\lambda_j}\delta_{k,j}\right) u_{\alpha,0} = 0. \quad (20)$$

If $k \in J'$, then

$$\{q_1^2 p_1, u_{\alpha,0}\} - \frac{\tau_k}{\lambda_k} u_{\alpha,0} = 0.$$

Because $\tau_k/\lambda_k \neq 0$ by (NRC) condition, we have $u_{\alpha,0} = 0$.

On the other hand, if $k \in J$, then

$$\{q_1^2 p_1, u_{\alpha,0}\} - \frac{q_1^2}{(q_1-1)^2}\frac{\tau_k}{\lambda_k} u_{\alpha,0} = 0.$$

By considering the equation around $q_1 = 1$ together with (NRC) condition we obtain $u_{\alpha,0} = 0$.

Next we consider $u_{\alpha,\beta}(\beta \neq 0)$ $(\alpha = (\alpha_2, ..., \alpha_n)$, $\alpha_j = \delta_{j,k})$.

$$\begin{aligned} \{q_1^2 p_1, u_{\alpha,\beta}\} &+ \sum_{j\in J'} \frac{\tau_j}{\lambda_j}(\beta_j - \alpha_j) u_{\alpha,\beta} \\ &+ \frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}(\beta_j - \alpha_j) u_{\alpha,\beta} = 0. \end{aligned} \quad (21)$$

If $\beta \neq \alpha$, then (NRC) condition yields $u_{\alpha,\beta} = 0$, by the similar argument as in the above. If $\beta = \alpha$, then we have $\{q_1^2 p_1, u_{\alpha,\alpha}\} = 0$. Hence, there exists $\phi_\alpha(t)$ of one variable t such that $u_{\alpha,\alpha} = \phi_\alpha(q_1^2 p_1)$. Therefore we obtain

$$u = \sum_{|\alpha|=1} \phi_\alpha(q_1^2 p_1) q^\alpha p^\alpha + O(|p|^2). \quad (22)$$

Now we consider the equation for u_α when $|\alpha| = 2$. We substitute (10) and (22) into the equation $\chi_{H+H_1} u = 0$ and compare the powers like p^α $(|\alpha| = 2)$. In order to get the expressions of the powers like p^α, we note that the following terms appear from $\chi_H u$:

$$\begin{aligned} &\{q_1^2 p_1, u_\alpha\} + \sum_{j\in J'}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j} - \alpha_j\right) u_\alpha + \frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j} - \alpha_j\right) u_\alpha \\ &+ \frac{2q_1}{(q_1-1)^3}\sum_{j\in J}\frac{\tau_j}{\lambda_j} q^\alpha \frac{\partial}{\partial p_1}\phi_{\alpha - e_j}. \end{aligned} \quad (23)$$

On the other hand, the following terms appear from $\chi_{H_1}u$.

$$\sum_{\nu}\frac{\partial}{\partial p_\nu}\left(\sum_j p_j^2 B_j(q_1,p)\right)\frac{\partial}{\partial q_\nu}(\phi_{e_\nu}q_\nu p_\nu) \tag{24}$$
$$-\frac{\partial}{\partial q_1}\left(\sum_j p_j^2 B_j(q_1,p)\right)\frac{\partial}{\partial p_1}(\sum_{|\alpha|=1}\phi_\alpha q^\alpha p^\alpha).$$

Note that the second term in (24) is $O(|p|^3)$. Hence it does not appear in the recurrence formula because $|\alpha|=2$. Moreover, since we consider terms of $O(|p|^2)$, the first term yields

$$2\sum_\nu \phi_{e_\nu}B_\nu(q_1,0)\delta_{\alpha,2e_\nu}. \tag{25}$$

Therefore, by comparing the powers like p^α in $\chi_{H+H_1}u=0$ we have

$$\begin{aligned}&\{q_1^2p_1,u_\alpha\}+\sum_{j\in J'}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j}-\alpha_j\right)u_\alpha \\ +\;&\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j}-\alpha_j\right)u_\alpha \\ +\;&\frac{2q_1}{(q_1-1)^3}q^\alpha\sum_{j\in J}\frac{\tau_j}{\lambda_j}\frac{\partial}{\partial p_1}\phi_{\alpha-e_j}+2\sum_\nu\phi_{e_\nu}B_\nu(q_1,0)\delta_{\alpha,2e_\nu}=0.\end{aligned} \tag{26}$$

Expand u_α with respect to q, $u_\alpha=\sum_\beta u_{\alpha,\beta}(q_1,p_1)q^\beta$ and insert the expansion into (26). By comparing the power of q^β we obtain the recurrence relation for $u_{\alpha,\beta}(q_1,p_1)$. We consider 4 cases:

(i) $\alpha\neq 2e_\nu$ for every ν and $\beta\neq\alpha$.
(ii) $\alpha=2e_k$ for some k and $\beta\neq\alpha,0$.
(iii) $\alpha=2e_k$ for some k and $\beta=0$.
(iv) $\beta=\alpha$.

Case (i): We note that the fourth and the fifth terms of the left-hand side of (26) yield no term in the recurrence relation for $u_{\alpha,\beta}$. Indeed, the fourth term is a monomial of q^α. Hence, $u_{\alpha,\beta}$ satisfies

$$\{q_1^2p_1,u_{\alpha,\beta}\}+\sum_{j\in J'}\frac{\tau_j}{\lambda_j}(\beta_j-\alpha_j)u_{\alpha,\beta}+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}(\beta_j-\alpha_j)u_{\alpha,\beta}=0. \tag{27}$$

By virtue of (NRC) and $\beta\neq\alpha$, either $\sum_{j\in J'}\tau_j\lambda_j^{-1}(\beta_j-\alpha_j)\neq 0$ or $\sum_{j\in J}\tau_j\lambda_j^{-1}(\beta_j-\alpha_j)\neq 0$ holds. One can easily show that $u_{\alpha,\beta}=0$ by the holomorphy of $u_{\alpha,\beta}$.

Case (ii): Because the fourth and fifth terms of the left-hand side of (26) do not yield terms by the assumption $\beta \neq \alpha, 0$, we see that $u_{\alpha,\beta}$ satisfies (27). Therefore, we have $u_{\alpha,\beta} = 0$.

Case (iii): Let $k \in J'$. Because the fourth term of the left-hand side of (26) is a monomial q^α, $u_{\alpha,0}$ satisfies

$$\{q_1^2 p_1, u_{\alpha,0}\} - 2\frac{\tau_k}{\lambda_k} u_{\alpha,0} + 2\phi_{e_k}(q_1^2 p_1) B_k(q_1, 0) = 0. \tag{28}$$

Expand $u_{\alpha,0}(q_1, p_1) = \sum_\nu u_{\alpha,0,\nu}(q_1) p_1^\nu$ and compare the constant terms in p_1 of both sides of (28). Then we have

$$q_1^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\frac{\tau_k}{\lambda_k} u_{\alpha,0,0} + 2\phi_{e_k}(0) B_k(q_1, 0) = 0. \tag{29}$$

If $\phi_{e_k}(0) \neq 0$, then $v := u_{\alpha,0,0}/(-2\phi_{e_k}(0))$ satisfies

$$q_1^2 \frac{d}{dq_1} v - 2\frac{\tau_k}{\lambda_k} v = B_k(q_1, 0),$$

which contradicts (TC). Hence, $\phi_{e_k}(0) = 0$ and (29) reduces to

$$q_1^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\frac{\tau_k}{\lambda_k} u_{\alpha,0,0} = 0.$$

(NRC) condition implies $2\tau_k/\lambda_k \neq 0$, and the holomorphcity of $u_{\alpha,0,0}$ at $q_1 = 0$ tells us $u_{\alpha,0,0} = 0$.

Next, $u_{\alpha,0,1}$ satisfies

$$(q_1^2 \frac{d}{dq_1} - 2q_1) u_{\alpha,0,1} - 2\frac{\tau_k}{\lambda_k} u_{\alpha,0,1} + 2B_k(q_1, 0)\phi'_{e_k}(0) q_1^2 = 0. \tag{30}$$

Since $u_{\alpha,0,1}(q_1) = O(q_1^2)$, we put $u_{\alpha,0,1}(q_1) = q_1^2 \tilde{u}_{\alpha,0,1}(q_1)$ with $\tilde{u} := \tilde{u}_{\alpha,0,1}(q_1)$ satisfying

$$q_1^2 \frac{d}{dq_1} \tilde{u} - 2\frac{\tau_k}{\lambda_k} \tilde{u} = -2B_k(q_1, 0)\phi'_{e_k}(0).$$

If $\phi'_{e_k}(0) \neq 0$, then, by putting $v = \tilde{u}/(-2\phi'_{e_k}(0))$, we have a contradiction to (TC). Therefore, $\phi'_{e_k}(0) = 0$ and $\tilde{u} = 0$.

Similarly we can show $u_{\alpha,0,\nu} = 0$ and $\phi_{e_k}^{(\nu)}(0) = 0$ for $\nu \in \mathbb{N} \cup \{0\}$, which implies $u_{\alpha,0} = 0$ and $\phi_{e_k} = 0$ for every $k \in J'$.

Let $k \in J$. Then $u_{\alpha,0}$ satisfies

$$\{q_1^2 p_1, u_{\alpha,0}\} - 2\frac{\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} u_{\alpha,0} + 2\phi_{e_k}(q_1^2 p_1) B_k(q_1, 0) = 0.$$

Expand $u_{\alpha,0}(q_1, p_1) = \sum_\nu u_{\alpha,0,\nu}(q_1) p_1^\nu$. Then $u_{\alpha,0,0}$ satisfies

$$q_1^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\frac{\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} u_{\alpha,0,0} + 2\phi_{e_k}(0) B_k(q_1, 0) = 0. \tag{31}$$

If $\phi_{e_k}(0) \neq 0$, then, by (31) we have $B_k(0,0) = 0$. On the other hand, $v := u_{\alpha,0,0}/(-2\phi_{e_k}(0))$ satisfies

$$q_1^2 \frac{d}{dq_1} v - 2\frac{\tau_k}{\lambda_k} \frac{q_1^2}{(q_1-1)^2} v = B_k(q_1, 0),$$

which contradicts (TC). So, $\phi_{e_k}(0) = 0$ and (31) reduces to

$$(q_1-1)^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\frac{\tau_k}{\lambda_k} u_{\alpha,0,0} = 0.$$

Again we have $u_{\alpha,0,0} = 0$.

Next, consider the equation of $u_{\alpha,0,1}$

$$(q_1^2 \frac{d}{dq_1} - 2q_1) u_{\alpha,0,1} - 2\frac{\tau_k}{\lambda_k} \frac{q_1^2}{(q_1-1)^2} u_{\alpha,0,1} = -2\phi'_{e_k}(0) q_1^2 B_k(q_1, 0). \tag{32}$$

Observing $u_{\alpha,0,1}(0) = 0$, we put $u_{\alpha,0,1}(q_1) = cq_1 + q_1^2 v$. Substituting it into (32), we have $c = -2\phi'_{e_k}(0) B_k(0,0)$ and v satisfies

$$\begin{aligned} -2\phi'_{e_k}(0) &\left\{ B_k(q_1,0) + B_k(0,0) + B_k(0,0) \frac{2\tau_k}{\lambda_k} \frac{q_1}{(q_1-1)^2} \right\} \\ &= \left(q_1^2 \frac{d}{dq_1} - \frac{2\tau_k}{\lambda_k} \frac{q_1}{(q_1-1)^2} \right) v. \end{aligned}$$

By use of (TC), we obtain $\phi'_{e_k}(0) = 0$ and $u_{\alpha,0,1} = 0$.

In general, $u_{\alpha,0,\nu}$ $(\nu \geq 2)$ satisfies

$$(q_1^2 \frac{d}{dq_1} - 2\nu q_1) u_{\alpha,0,\nu} - 2\frac{\tau_k}{\lambda_k} \frac{q_1^2}{(q_1-1)^2} u_{\alpha,0,\nu} = -2\frac{\phi_{e_k}^{(\nu)}(0)}{\nu!} q_1^{2\nu} B_k(q_1, 0). \tag{33}$$

Since we easily see $u_{\alpha,0,\nu} = O(q^{2\nu-1})$, we put $u_{\alpha,0,\nu} = cq_1^{2\nu-1} + q_1^{2\nu} w$. Then we have $c = -2\phi_{e_k}^{(\nu)}(0) B_k(0,0)/\nu!$ and w satisfies

$$\begin{aligned} -\frac{2\phi'_{e_k}(0)}{\nu!} &\left\{ B_k(q_1,0) + B_k(0,0) + B_k(0,0) \frac{2\tau_k}{\lambda_k} \frac{q_1}{(q_1-1)^2} \right\} \\ &= \left(q_1^2 \frac{d}{dq_1} - \frac{2\tau_k}{\lambda_k} \frac{q_1^2}{(q_1-1)^2} \right) w. \end{aligned}$$

By virtue of (TC), we obtain $\phi_{e_k}^{(\nu)}(0) = 0$ and $w = 0$. Therefore, $u_{\alpha,0,\nu} = 0$ for all $\nu \in \mathbb{N} \cup \{0\}$. Because of analyticity, we have $u_{\alpha,0} = 0$ and $\phi_{e_k} = 0$ for every $k \in J$. Consequently, $\phi_{e_k} = 0$ holds for all $k \in J' \cup J$.

Case (iv): Because $\phi_{e_k} = 0$ for every k by what we have proved in the above, the fourth and fifth terms of the left-hand side of (26) do not yield terms in the recurrence relation. Hence, $u_{\alpha,\alpha}$ satisfies $\{q_1^2 p_1, u_{\alpha,\alpha}\} = 0$. It follows that there exists a function of one variable $\phi_\alpha(t)$ such that $u_{\alpha,\alpha} = \phi_\alpha(q_1^2 p_1)$.

Therefore we have proved

$$u = \sum_{|\alpha|=2} \phi_\alpha(q_1^2 p_1)\ q^\alpha p^\alpha + O(|p|^3).$$

Finally we will prove

Lemma 1 *Suppose*

$$u = \sum_{|\alpha|=\nu} \phi_\alpha(q_1^2 p_1) q^\alpha p^\alpha + O(|p|^{\nu+1}) \tag{34}$$

for some $\nu \geq 1$. Then we have
(i) $\phi_\alpha = 0$ for all α satisfying $|\alpha| = \nu$.
(ii) for every α satisfying $|\alpha| = \nu + 1$, there exists a holomorphic function ϕ_α of one variable such that

$$u = \sum_{|\alpha|=\nu+1} \phi_\alpha(q_1^2 p_1) q^\alpha p^\alpha + O(|p|^{\nu+2}). \tag{35}$$

We have already proved (34) for $\nu = 1, 2$. Note that the lemma ends the proof of Theorem 1 because we have $u = 0$ as an analytic function of q and p.

Proof of Lemma 1. By comparing the coefficients of p^α in $\chi_{H+H_1} u = 0$ we have

$$\begin{aligned} \{q_1^2 p_1, u_\alpha\} &+ \sum_{J'} \frac{\tau_j}{\lambda_j} \left(q_j \frac{\partial}{\partial q_j} - \alpha_j \right) u_\alpha \\ &+ \frac{q_1^2}{(q_1-1)^2} \sum_J \frac{\tau_j}{\lambda_j} \left(q_j \frac{\partial}{\partial q_j} - \alpha_j \right) u_\alpha \\ &+ \frac{2q_1}{(q_1-1)^3} \left(\sum_J \frac{\tau_j}{\lambda_j} q_j p_j \right) \frac{\partial}{\partial p_1} u_\gamma + \sum_{j,\gamma} \frac{\partial H_1}{\partial p_j} \frac{\partial}{\partial q_j} u_\gamma = 0, \end{aligned} \tag{36}$$

where $|\gamma| < |\alpha|$ and $\alpha = \gamma + e_j$.

Let $|\alpha| = \nu + 1$. Substituting the expansion $u_\alpha = \sum_\beta u_{\alpha,\beta}(q_1, p_1) q^\beta$ into (36) and by using (34), we obtain the relation for $u_{\alpha,\beta}$

$$\begin{aligned} &\{q_1^2 p_1, u_{\alpha,\beta}\} + \sum_{J'} \frac{\tau_j}{\lambda_j} (\beta_j - \alpha_j) u_{\alpha,\beta} + \frac{q_1^2}{(q_1-1)^2} \sum_J \frac{\tau_j}{\lambda_j} (\beta_j - \alpha_j) u_{\alpha,\beta} \\ &+ 2 \frac{q_1}{(q_1-1)^3} \sum_J \frac{\tau_j}{\lambda_j} \frac{\partial}{\partial p_1} \phi_{\alpha - e_j}(q_1^2 p_1) \delta_{\alpha,\beta} \\ &+ 2 \sum_{j \in J' \cup J} \delta_{\alpha - 2e_j, \beta} B_j(q_1, 0) \phi_{\alpha - e_j} (\alpha_j - 1) = 0. \end{aligned} \tag{37}$$

Indeed, because it is easy to show the expressions up to the fourth term in the left-hand side of (37), we consider the fifth term, which corresponds to the fifth term in the left-hand side of (36). In view of (34) we may consider $2\sum_j p_j B_j(q_1, 0)$ in $\frac{\partial H_1}{\partial p_j}$ because other terms have no effect to (36). Hence we may consider terms containing $p^{\alpha - e_j}$ in $\frac{\partial}{\partial q_j} u_\gamma$. By (34) the coefficient of the term containing $p^{\alpha - e_j}$ is $(\alpha_j - 1) q^{\alpha - 2e_j} B_j(q_1, 0) \phi_{\alpha - e_j}$. Hence we have the desired expression.

Set $B' := \sum_{\in J'} \frac{\tau_j}{\lambda_j}(\beta_j - \alpha_j)$ and $B := \sum_{j \in J} \frac{\tau_j}{\lambda_j}(\beta_j - \alpha_j)$. We consider 4 cases.

Case (1). The case where $\alpha - 2e_j \neq \beta$ for $j = 2, ..., n$ and $B' \neq 0$. Clearly we have $\beta \neq \alpha$. It follows that the fourth and the fifth terms in the left-hand side of (37) vanish. Hence we have $u_{\alpha,\beta} = 0$ by considering (37) at $q_1 = 0$.

Case (2). The case where $\alpha - 2e_j \neq \beta$ for $j = 2, ..., n$, $\beta \neq \alpha$ and $B' = 0$. By (NRC) we have $B \neq 0$. Hence the fourth and the fifth terms in the left-hand side of (37) vanish. We have $u_{\alpha,\beta} = 0$ by considering (37) at $q_1 = 1$.

Case (3). The case where $\alpha - 2e_k = \beta$ for some k. Clearly, we have $\beta \neq \alpha$. Assume $k \in J$. Then, for every $j \in J'$ we have $j \neq k$, and hence $\alpha_j = \beta_j$, which implies $B' = 0$. Equation (37) is reduced to

$$\{q_1^2 p_1, u_{\alpha,\beta}\} - \frac{2\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} u_{\alpha,\beta} + 2(\alpha_k - 1)\phi_{\alpha - e_k} B_k(q_1, 0) = 0.$$

Expand $u_{\alpha,\beta} = \sum_{\nu=0}^{\infty} u_{\alpha,\beta,\nu}(q_1) p_1^\nu$. We will show that $\phi_{\alpha-e_k}$ vanishes.

Indeed, $v := u_{\alpha,\beta,0}$ satisfies

$$q_1^2 \frac{dv}{dq_1} - \frac{2\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} v = -2(\alpha_k - 1)\phi_{\alpha-e_k}(0) B_k(q_1, 0).$$

Note that $\alpha_k = 2 + \beta_k \geq 2$. If $\phi_{\alpha-e_k}(0) \neq 0$, then $w := v/(-2(\alpha_k - 1)\phi_{\alpha-e_k}(0))$ is a holomorphic solution at $q_1 = 0$ of the equation

$$q_1^2 \frac{dw}{dq_1} - \frac{2\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} w = B_k(q_1, 0).$$

Because one can verify $B_k(0, 0) = 0$, we have a contradiction to (TC). Hence we have $\phi_{\alpha-e_k}(0) = 0$ and $u_{\alpha,\beta,0} = 0$.

Next, $v = u_{\alpha,\beta,1}$ satisfies

$$q_1^2 \frac{dv}{dq_1} - \frac{2\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} v - 2q_1 v = -2(\alpha_k - 1)\phi'_{\alpha-e_k}(0) q_1^2 B_k(q_1, 0).$$

By comparing the coefficients of q_1^2 of both sides we see that $v = O(q_1^2)$. Similarly to the above, $w := v q_1^{-2}$ leads to a contradiction to (TC). Hence, we have $\phi'_{\alpha-e_k}(0) = 0$ and $u_{\alpha,\beta,1} = 0$.

In general, $v = u_{\alpha,\beta,\nu} (\nu \geq 2)$ satisfies

$$q_1^2 \frac{dv}{dq_1} - \frac{2\tau_k}{\lambda_k} \frac{q_1^2}{(q_1 - 1)^2} v - 2q_1 \nu v = -\frac{2(\alpha_k - 1)}{\nu!} \phi^{(\nu)}_{\alpha-e_k}(0) q_1^{2\nu} B_k(q_1, 0).$$

Similarly to the above, we have $\phi^{(\nu)}_{\alpha-e_k}(0) = 0$ and $u_{\alpha,\beta,\nu} = 0$. Therefore, $\phi_{\alpha-e_k} = 0$ and $u_{\alpha,\beta} = 0$ for $k \in J$.

Let $k \in J'$. Equation (37) is reduced to

$$\{q_1^2 p_1, u_{\alpha,\beta}\} - \frac{2\tau_k}{\lambda_k} u_{\alpha,\beta} + 2(\alpha_k - 1)\phi_{\alpha-e_k} B_k(q_1, 0) = 0.$$

The holomorphicity of $u_{\alpha,\beta}$ at $q_1 = 0$ and (TC) implies $\phi_{\alpha-e_k}(0) = 0$ and $u_{\alpha,\beta} = 0$ for $k \in J'$. Therefore, $\phi_\alpha = 0$ for $k \in J'$. Because $\phi_\alpha = 0$ for $k \in J$, we have $\phi_\alpha = 0$ for all α with $|\alpha| = \nu$.

Case (4). The case $\beta = \alpha$. We have $\{q_1^2 p_1, u_{\alpha,\alpha}\} = 0$, since we have proved $\phi_\gamma = 0$ for $|\gamma| = \nu$. Hence, there exists ϕ_α such that $u_{\alpha,\alpha} = \phi_\alpha(q_1^2 p_1)$.

Consequently, we have proved the lemma.

3 Confluence of singularities

In this section we deduce (3) from the hypergeometric system

$$(z - C)\frac{dv}{dz} = Av, \tag{38}$$

where $C = \mathrm{diag}(\Lambda_1, {}^t\Lambda_1)$, Λ_1 being $(n-1)\times(n-1)$ matrix with eigenvalues $\lambda_2, ..., \lambda_n$ such that $\lambda_j \neq 0$ for all j (cf. [1]). For the sake of simplicity, we assume $\Lambda_1 = \mathrm{diag}(\lambda_2, ..., \lambda_n)$. We assume $A = \mathrm{diag}(A_1, {}^tA_1)$, where A_1 is an $(n-1)\times(n-1)$ constant matrix satisfying $\Lambda_1 A_1 = A_1 \Lambda_1$. For simplicity, we further assume $A_1 = \mathrm{diag}(\tau_2, ..., \tau_n)$.

Let $v = {}^t(q, p) \in \mathbb{C}^{2(n-1)}$. Define

$$H = \langle (z - \Lambda_1)^{-1} p, A_1 q \rangle, \tag{39}$$

where $\langle (x_2, ..., x_n), {}^t(y_2, ..., y_n) \rangle := \sum_{2\le k\le n} x_k y_k$. Then, (38) is written in the Hamiltonian system

$$\frac{dq}{dz} = H_p(z, q, p), \ \frac{dp}{dz} = -H_q(z, q, p). \tag{40}$$

Now we operate the confluence of regular singularities. Let v_ν and $(Av)_\nu$ denote the ν-th entry of v and Av, respectively. Then we can write (40) in the form

$$(z - \lambda_\nu)\frac{dv_\nu}{dz} = (Av)_\nu.$$

Substituting $z = 1/\zeta$, we have

$$-\zeta^2 \frac{dv_\nu}{d\zeta} = (\zeta^{-1} - \lambda_\nu)^{-1}(Av)_\nu. \tag{41}$$

In the following, $a \mapsto b$ denotes the replacement of a by b.

Let $\zeta \mapsto \epsilon^{-1}\eta$; and $\lambda_\nu \mapsto \epsilon\lambda_\nu$ for $\nu \in J$, $\lambda_\nu \mapsto \lambda_\nu$ for $\nu \in J'$. Multiply the ν-th row of A in (41) by ϵ^{-1} if $\nu \in J'$ and take the limit $\epsilon \to 0$. Then (40) is reduced to the Hamiltonian system

$$-\eta^2 \frac{dq}{d\eta} = \mathfrak{A} A_1 q, \ -\eta^2 \frac{dp}{d\eta} = -{}^tA_1 \mathfrak{A} p, \tag{42}$$

where $\mathfrak{A} = \mathrm{diag}(\mathfrak{A}_2, ..., \mathfrak{A}_n)$ and

$$\mathfrak{A}_\nu := \begin{cases} -\lambda_\nu^{-1} & (\nu \in J'), \\ (\eta^{-1} - \lambda_\nu)^{-1} & (\nu \in J). \end{cases} \tag{43}$$

Note that (42) is irregular singular at $\eta = 0$.

In order to introduce another singular point, choose any $a \neq 0$ such that $a \neq \lambda_j^{-1}$ for all j and put $\zeta = \eta - a$. Let $\zeta \mapsto \epsilon^{-1}\zeta$ and $(A)_\nu \mapsto \epsilon^{-1}(A)_\nu$. Make substitution $a \mapsto \epsilon^{-1}a$ for $j \in J'$ and $a \mapsto a$ for $j \in J$ and take the limit $\epsilon \to 0$. Then (40) is reduced to a Hamiltonian system with irregular points at 0 and $-a$. Set $a = -1$. Finally, by transforming to the autonomous system, we obtain (3).

4 Integrability at singular point

Let H and H_1 be given by (3) and (47), respectively. We will show the integrability about a singular point of χ_{H+H_1}. Note that the Hamiltonian system corresponding to $H + H_1$ has irregular singularity at $q_1 = 0$ and $q_1 = 1$. First we show

Lemma 2 *If $k \in J$, then χ_H has first integrals*

$$q_k \exp\left(\frac{\tau_k}{\lambda_k}\frac{1}{q_1 - 1}\right), \quad p_k \exp\left(-\frac{\tau_k}{\lambda_k}\frac{1}{q_1 - 1}\right), \tag{44}$$

while, for $k \in J'$ it has

$$q_k \exp\left(\frac{\tau_k}{\lambda_k}\frac{1}{q_1}\right), \quad p_k \exp\left(-\frac{\tau_k}{\lambda_k}\frac{1}{q_1}\right). \tag{45}$$

Note that χ_H is analytically integrable at $q_1 = 0$ or $q_1 = 1$, because $q_k p_k$ is an analytic first integral about the singular point $q_1 = 0$ or $q_1 = 1$.

Proof of Lemma 2. The assertion is easily verified in view of the definition of first integrals.

Next we consider χ_{H+H_1}. In view of the definition of χ_{H_1}, the following functions are annihilated by χ_{H_1}.

$$p_k \exp\left(-\frac{\tau_k}{\lambda_k}\frac{1}{q_1 - 1}\right), \ (k \in J) \quad p_k \exp\left(-\frac{\tau_k}{\lambda_k}\frac{1}{q_1}\right), \ (k \in J'). \tag{46}$$

Hence they are also first integrals of χ_{H+H_1}. Note that we have $|J|$ analytic first integrals near the origin. We will construct other first integrals which areanalytic near the origin. Let H_1 be given by

$$H_1 := \sum_{j \in J} p_j^2 B_j(q_1, q). \tag{47}$$

Then we have

Proposition 1 *Suppose that there exists an analytic function $\tilde{H}_1(q_1, p)$ at $q_1 = 0$, $p = 0$ which is independent of p_ν for every $\nu \in J'$ such that $H_1(q_1, p) = q_1^2 \tilde{H}_1(q_1, p)$. Then, for every $k \in J$, χ_{H+H_1} has the first integral of the form $q_k u_k(q_1) + W_k(q_1, p)$, where $W_k(q_1, p)$ is analytic at $q_1 = 0$, $p = 0$, and*

$$u_k(q_1) = \exp\left(\frac{\tau_k}{\lambda_k}\frac{1}{q_1 - 1}\right). \tag{48}$$

If $2|J| + 1 \geq n$, then χ_{H+H_1} is analytically integrable in some neighborhood of the origin.

Proof. By definition we have

$$\chi_{H_1} = \sum_{j=2}^{n}\left(2p_j B_j \frac{\partial}{\partial q_j} + p_j^2 \sum_{\nu=2}^{n} \partial_{p_\nu} B_j \frac{\partial}{\partial q_\nu} - p_j^2 (\partial_{q_1} B_j)\frac{\partial}{\partial p_1}\right). \tag{49}$$

Let $k \in J$. We will determine u_k and W_k by the equation

$$(\chi_H + \chi_{H_1})(q_k u_k + W_k) = 0. \tag{50}$$

First we note that $\chi_{H_1} W_k = 0$ by definition. $\chi_{H_1}(q_k u_k)$ and $\chi_H W_k$ do not contain q. Therefore, the terms containing powers of q with degree 1 in the left-hand side of (50) come from $\chi_H(q_k u_k)$. By (50) we have $\chi_H(q_k u_k) = 0$. This is an ordinary differential equation for u_k studied in Lemma 2. The solution is given by (48).

Next, by comparing terms of degree 0 in q of (50) we have

$$\chi_H W_k + \chi_{H_1}(q_k u_k) = 0. \tag{51}$$

Note

$$\chi_{H_1}(q_k u_k) = u_k\left(2p_k B_k + \sum_{j\in J} p_j^2 (\partial_{p_k} B_j)\right). \tag{52}$$

Expand

$$W_k(q_1, p) = \sum_\ell W_k^{(\ell)}(q_1)p^\ell, \quad B_j(q_1, p) = \sum_\ell B_j^{(\ell)}(q_1)p^\ell, \tag{53}$$

and insert (52) and (53) into (51). Then, by comparing the powers like p^ℓ we obtain

$$\left(q_1^2 \frac{d}{dq_1} - \sum_{j\in J'}\frac{\tau_j \ell_j}{\lambda_j} - \frac{q_1^2}{(q_1 - 1)^2}\sum_{j\in J}\frac{\tau_j \ell_j}{\lambda_j}\right) W_k^{(\ell)}(q_1) = R^{(\ell)}(q_1) u_k(q_1), \tag{54}$$

where

$$R^{(\ell)}(q_1) := \left(-2B_k^{(\ell - e_k)}(q_1) - \sum_{j\in J}(\ell + e_k - 2e_j) B_j^{(\ell + e_k - 2e_j)}(q_1)\right). \tag{55}$$

One easily verifies that $\prod_{j=2}^{n} u_j^{\ell_j}$ is a solution of the homogeneous equation of (54), where u_j is given by (48). Therefore we take the solution $W_k^{(\ell)}$ of (54) as follows.

$$W_k^{(\ell)} = \left(\prod_{j=2}^{n} u_j^{\ell_j}\right) \int_a^{q_1} t^{-2} u_k(t) R^{(\ell)}(t) \prod_{j=2}^{n} u_j^{-\ell_j}(t) dt \tag{56}$$

where $a \neq 0$ is a constant sufficiently close to the origin.

We note that, if $j \in J$, then $u_j(q_1)$ is holomorphic at $q_1 = 0$. Moreover, by assumption on H_1 we see that $R^{(\ell)}$ vanish if $\ell_\nu \neq 0$ for $\nu \in J'$. It follows from (56) that $W_k^{(\ell)}(q_1)$ is analytic at $q_1 = 0$. The convergence of the sum $W := \sum_\ell W_k^{(\ell)} p^\ell$ is almost clear from the recurrence formula because $t^{-2}R^{(\ell)}(t)$ is holomorphic at $t = 0$ by assumption. Hence we have an analytic first integral at the origin. The last statement is clear from the definition of integrability. This ends the proof.

5 Properties of (TC)

We will show that (TC) holds for almost all $B_k(q_1, 0)$. Set $q_1 = t$, $B_k(t, 0) =: a(t)$ and $c := \tau_k/\lambda_k$, and write (8) in the form

$$t^2 \frac{d}{dt} v - 2cv = a(t). \tag{57}$$

Clearly, if $a(t)$ is a constant function, then (TC) does not hold since (57) has a constant solution $v = -a(0)/(2c)$. We first prove

Proposition 2 *Suppose that $a(t)$ is a polynomial of degree $\ell \geq 1$. Then (57) has an analytic solution at $t = 0$ if and only if (57) has a polynomial solution v of degree $\ell - 1$. The set of $a(t)$ for which (57) has a polynomial solution is contained in the set of codimension one of the set of polynomials of degree ℓ.*

Remark. For a given polynomial v of degree $\ell - 1$, define $a(t)$ by (57). Clearly the set of a's such that (57) has a polynomial solution is an infinite set.

Proof of Proposition 2 . Let $a(t) = \sum_{j=0}^{\ell} a_j t^j$ ($a_\ell \neq 0$) and let $v(t) = \sum_{j=0}^{\infty} v_j t^j$ be the analytic solution of (57). By inserting the expansions into (57) and by comparing the powers of t we obtain

$$v_0 = -a_0/(2c), \quad v_n = (n-1)v_{n-1}/(2c) - a_n/(2c), \ n = 1, 2, \ldots \tag{58}$$

If $n > \ell$, then we have $v_n = (n-1)v_{n-1}/(2c)$. Therefore, if $v_\ell = 0$, then $v_n = 0$ for $n > \ell$. Hence v is a polynomial. On the other hand, if $v_\ell \neq 0$, then $v_n = (2c)^{\ell-n}(n-1)(n-2)\cdots \ell v_\ell$. It follows that $v(t)$ is not analytic in any neighborhood of the origin, which contradicts to the assumption. Hence v is a polynomial of degree $\ell - 1$. The converse statement is trivial.

We will show the latter half. By the recurrence formula (58), one easily sees that v_ℓ is a nontrivial linear function of $a_0, \ldots, a_\ell$. Hence the condition $v_\ell = 0$

is satisfied for a polynomial $a(t)$ on the set of codimension 1. This completes the proof.

Next we study (TC) when $a(t)$ is an analytic function. By replacing $v(t)$ and $a(t)$ with $v(t) - v(0)$ and $a(t) - a(0)$, $(2cv(0) = -a(0))$, respectively, we may assume that $v(0) = 0$ and $a(0) = 0$ in (57). Then we have

Proposition 3 *The set of analytic functions $a(t)$'s at the origin such that (57) has an analytic solution v is contained in the set of codimension 1 of the set of germs of analytic functions at $t = 0$.*

Proof. Let v be the analytic solution of (57) at $t = 0$. Set $v(t) = t\tilde{v}(t)$ and $a(t) = t\tilde{a}(t)$. Then

$$t^2 \frac{d}{dt}\tilde{v} + t\tilde{v} - 2c\tilde{v} = \tilde{a}(t). \tag{59}$$

We make the (formal) Borel transform $\mathcal{B}(\tilde{v})$ to (59)

$$\mathcal{B}(\tilde{v})(z) \equiv \widehat{\tilde{v}}(z) := \sum_{n=1}^{\infty} v_n \frac{z^{n-1}}{(n-1)!}. \tag{60}$$

Because $\tilde{v}(t)$ and $\tilde{a}(t)$ are analytic at $t = 0$, it follows that $\mathcal{B}(\tilde{v})(z)$ and $\mathcal{B}(\tilde{a})(z)$ are entire functions of exponential type of order 1. Recalling that $\mathcal{B}\left((t^2 \frac{d}{dt} + t)\tilde{v}\right)(z) = z\mathcal{B}(\tilde{v})(z)$ we have

$$(z - 2c)\mathcal{B}(\tilde{v}) = \mathcal{B}(\tilde{a})(z). \tag{61}$$

It follows that

$$\mathcal{B}(\tilde{a})(2c) = 0. \tag{62}$$

This shows that the germ $\{a_n\}_{n=1}^{\infty}$ of $a(t)$ at $t = 0$ is contained in the hyperplane. This ends the proof.

Next we consider (9) in (TC). We set $t = q_1 - 1$, $a(t+1) := B_k(t+1, 0)$, $c = \tau_k/\lambda_k$ and $a(0) = B_k(0, 0)$. Then (9) can be written in

$$\left(t^2 \frac{d}{dt} - 2c\right) w = \frac{t^2}{(t+1)^2} a(t+1) + \frac{a(0)}{(t+1)^2}(t^2 + c(t+1)) =: b(t). \tag{63}$$

This equation has the same form as (57). We determine $w(0)$ by $-2cw(0) = b(0)$. If we make the appropriate change of unknown functions w and b as before, one may assume that $w(0) = 0$ and $b(0) = 0$. In view of the definition of $b(t)$ we have $ca(0) = 0$. Hence we have $a(0) = 0$. It follows that $b(t) = t^2 a(t+1)/(t+1)^2$. In the following we assume $w(0) = 0$ and $a(0) = 0$. Then we have

Proposition 4 *Suppose that $a(t)$ is holomorphic in a connected domain containing $t = 0$ and $t = 1$. Then the set of $a(t)$ for which (63) has an analytic solution is contained in the set of codimension one of the set of germs of analytic functions at $t = 0$.*

Proof. Let $w(t)$ be an analytic solution of (63) at $t=0$. We set $\alpha := a'(0)$ and $a(z) = \alpha z + A(z)z^2$ for some analytic function $A(z)$. Then, by the general formula w is given by

$$w = \exp\left(-\frac{2c}{t}\right)\left(K + \int_{\tau}^{t} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds\right), \tag{64}$$

where K and $\tau \neq 0$ are some constants. We take a smooth curve γ which connects τ and the origin such that it stays in the half space, $\Re\,(c/t) < 0$ near the origin. Then the limit

$$\begin{aligned}&\int_{\tau}^{0} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds \\ := &\lim_{t\in\gamma, t\to 0} \int_{\tau}^{t} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds\end{aligned} \tag{65}$$

exists and it is a non-constant analytic function of τ. If the condition

$$K + \int_{\tau}^{0} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds \neq 0 \tag{66}$$

holds, then, by taking the limit $t \to 0$, $\Re\,(c/t) < 0$ in (64) we see that $w(t)$ tends to infinity, which contradicts to the analyticity of w at the origin. Hence we have

$$K = \int_{0}^{\tau} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds. \tag{67}$$

By substituting (67) to (64) we have

$$w(t) = \exp\left(-\frac{2c}{t}\right)\int_{0}^{t} \left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds. \tag{68}$$

We take t sufficiently close to the origin such that the Taylor expansion $A(s+1) = \sum_{n=0}^{\infty} a_n s^n$ converges for $|s| \leq |t|$. Because $w(te^{2\pi i}) = w(t)$ holds by the analyticity of w, it follows that

$$\int_{t}^{te^{2\pi i}} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds = 0. \tag{69}$$

By calculating the residue we have $\int_t^{te^{2\pi i}} \exp\left(\frac{2c}{s}\right)\frac{\alpha}{s+1} ds = 2\pi i\alpha(1-e^{-2c})$. The non-resonance condition implies $c = \tau_k/\lambda_k \neq 0$, and hence $1 - e^{-2c} \neq 0$. Hence, by (69) the germ of $A(z)/\alpha$ at $z=1$(in case $\alpha \neq 0$) or that of $A(z)$ at $z = 1$(in case $\alpha = 0$) is contained in some hyperplane of the set of germs of analytic functions.

We recall that $A(z)$ is analytic in some domain containing $z=0$ and $z=1$. We will show that by the analytic continuation from $z=1$ to $z=0$ the germ of $A(z)$ at $z=1$ is transformed to that of $A(z)$ at $z=0$ by an infinite matrix. If we can prove this, then the germ of $A(z)$ or $A(z)/\alpha$ at $z=0$ is contained in some hyperplane. In view of $a(z) = \alpha z + A(z)z^2$, the germ of $a(z)$ at $z = 0$ is contained in some hyperplane.

We take a rectifiable curve which connects $z = 1$ and $z = 0$. First we consider the analytic continuation from $z = 1$ to $z = z_0$, where z_0 is contained in the disk centered at $z = 1$ in which $A(z)$ is analytic. Let $A(z) = \sum_{n=0}^{\infty} a_n(z-1)^n$ be the expansion at $z = 1$. Then the Taylor expansion of $A(z)$ at $z = z_0$ is given by

$$\sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} \sum_{n=k}^{\infty} a_n(z_0-1)^{n-k} \frac{n!}{(n-k)!}. \tag{70}$$

It follows that the germ at $z = z_0$ is given by

$$\left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_0-1)^{n-k} \right)_{k=0}^{\infty}. \tag{71}$$

Hence the germ at $z = 1$ is transformed to the one in (70) by the infinite matrix

$$\mathcal{A} := \left((z_0-1)^{n-k} \binom{n}{k} \right)_{k\downarrow 0,1,\ldots;n\to 0,1,\ldots}, \tag{72}$$

where we set the (k, n)-component $(k > n)$ to be zero. Note that if $|z_0 - 1|$ is sufficiently small, then $\mathcal{A}$ defines a continuous linear operator on the space of sequences with an appropriate norm. Therefore, if the germ of $A(z)$ at $z = 1$ is contained in the hyperplane, then the germ of $A(z)$ at $z = z_0$ is contained in some hyperplane. By finite times of analytic continuation we see that the germ of $A(z)$ at $z = 0$ is contained in some hyperplane. This completes the proof.

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