

PERTURBATIONS OF VECTOR FIELDS ON TORI: RESONANT NORMAL FORMS AND DIOPHANTINE PHENOMENA

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ABSTRACT. This paper is about perturbations of smooth vector fields on \mathbb{T}^n (constant if $n \geq 3$) with zero'th order C^∞ and Gevrey G^σ , $\sigma \geq 1$ pseudodifferential operators. Simultaneous resonance is introduced and simultaneous resonant normal forms are shown (via conjugation with an elliptic pseudodifferential operator) under optimal simultaneous Diophantine conditions outside the resonances. In the C^∞ category the results are complete while in the Gevrey category the effect of the loss of the Gevrey regularity of the conjugating operators due to Diophantine conditions is encountered. The normal forms are used to study the global hypoellipticity in C^∞ and Gevrey G^σ . Finally, the exceptional sets associated with the simultaneous Diophantine conditions are studied. Generalized Hausdorff dimension is used to give precise estimates of the "size" of different exceptional sets, including some inhomogeneous examples.

1. INTRODUCTION

The purpose of this paper is to study resonant normal forms and the Diophantine phenomena of perturbations of commuting resonant vector fields X_k on the n -dimensional torus $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$. Here

$$X_k = \langle \omega^k, \partial_x \rangle = \sum_{j=1}^n \omega_j^k \partial_{x_j}, \quad 1 \leq k \leq d,$$

where $x = (x_1, \dots, x_n) \in \mathbb{T}^n$ and $\omega^k = (\omega_1^k, \dots, \omega_n^k) \in \mathbb{R}^n$, and $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j} = \partial / \partial x_j$. If $n = 2$ we also allow the X_k to be nonconstant vector fields.

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The X_k are perturbed with zero'th order classical pseudodifferential operators so that the operators

$$(1.1) \quad b^k(x, D) := -iX_k + a^k(x, D), \quad 1 \leq k \leq d,$$

are commuting, where $D = (D_{x_1}, \dots, D_{x_n})$, $D_{x_j} = -i\partial_{x_j}$, $j = 1, \dots, n$ and $a^k(x, D)$ is a zero'th order pseudodifferential operator defined by $a^k(x, D)f = \sum_{\xi \in \mathbb{Z}^n} e^{ix\xi} a^k(x, \xi) \hat{f}(\xi)$, where $\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{T}^n} e^{-ix\xi} f(x) dx$ is the Fourier transform defined on a discrete group and $a^k(x, \xi) \in S^0(\mathbb{T}^n \times \mathbb{R}^n)$ is a classical symbol of pseudodifferential operators. As a special case of (1.1), we also consider the following first order commuting differential operators

$$(1.2) \quad \mathcal{L}_j = \langle \omega^j, D_x \rangle + a_j(x), \quad a_j \in C^\infty(\mathbb{T}^n), \quad j = 1, \dots, d.$$

We will actually consider perturbations with functions in the Gevrey class $G^\sigma(\mathbb{T}^n)$. Recall that $G^\sigma(\mathbb{T}^n)$ is the set of all $f \in C^\infty(\mathbb{T}^n)$ such that there exists $C > 0$ satisfying

$$\max_{x \in \mathbb{T}^n} |\partial^\beta f(x)| \leq C^{|\beta|+1} (\beta!)^\sigma, \quad \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n,$$

where $\beta! = \beta_1! \dots \beta_n!$ and $|\beta| = \beta_1 + \dots + \beta_n$. Clearly $G^1(\mathbb{T}^n)$ coincides with the set of all analytic functions on \mathbb{T}^n .

For a single vector field (respectively, map) with an isolated singular (respectively, fixed) point the formal reduction to its linear part requires a nonresonance condition on the eigenvalues of the linear part, while in order to show the convergence of formal transformations, arithmetic conditions are imposed (see the surveys [27], [42] and the references therein). If commuting vector fields or maps are considered it is not necessary for each vector field or map to satisfy the aforementioned conditions. More precisely, simultaneous arithmetic conditions are required (see [10], [13], [24], [36], [41], [42]). Certain exceptional sets will arise in connection with some of the parameters in these simultaneous arithmetic conditions (cf. [18]) and these will be discussed later.

This paper deals with four closely related problems. First, after introducing the concept of simultaneous (non)resonance for $\omega^1, \dots, \omega^d$, we are interested in the simultaneous resonant normal forms of $\{b^k(x, D)\}_1^d$ and/or $\{\mathcal{L}_k\}_1^d$ via conjugation with an elliptic p.d.o. (which in the case of differential operators reduces to a change of variables on \mathbb{T}^n) in the framework of the C^∞ and the Gevrey spaces. Roughly speaking, resonant normal forms mean that lower order terms depend only on resonant variables. For this purpose the geometrical properties of a simultaneously resonant set associated with $\omega := \{\omega^k\}_1^d$ and defined by

$$\Gamma_{\mathbb{Z}}^\omega := \{z \in \mathbb{Z}^n; \langle \omega^1, z \rangle = \dots = \langle \omega^d, z \rangle = 0\}.$$

will be considered. We say that $\omega^1, \dots, \omega^d$ are simultaneously nonresonant (respectively resonant) if $\Gamma_{\mathbb{Z}}^\omega = \{0\}$ (respectively, $\Gamma_{\mathbb{Z}}^\omega \neq \{0\}$). Using theorems on finitely generated free abelian groups and exact sequences the existence of suitable bases for $\Gamma_{\mathbb{Z}}^\omega$ will be obtained and lead naturally to the introduction

of resonant variables on the torus. These constructions are invariant under the action of the group of linear automorphisms of the lattice \mathbb{Z}^n . A simultaneous resonant normal form will be exhibited for every perturbed system under simultaneous Diophantine conditions for $\omega^1, \dots, \omega^k$ on the set $\mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^{\omega}$ in the class C^∞ or Gevrey G^σ . In particular, if $\Gamma_{\mathbb{Z}}^{\omega} = \{0\}$ it is possible to transform $\{b^k\}_1^d$ (respectively, $\{\mathcal{L}_k\}_1^d$) simultaneously into a constant p.d.o. (respectively, differential operator). It will also be shown that in general the Diophantine condition is necessary to transform the perturbed systems to their normal forms - constant pseudodifferential operators on \mathbb{T}^n . For results on normal forms in dynamical systems by means of KAM type methods and Diophantine phenomena we refer, for example, to [1], [6], [8], [12], [19], [31], [37], [42] and the references therein.

We then consider the simultaneous reduction of nonconstant vector fields to constant ones for the case $n = 2$. There are very few results regarding normal forms of vector fields and differential operators on \mathbb{T}^n except for those vector fields on \mathbb{T}^2 of the form $\partial_{x_1} + \lambda(x)\partial_{x_2}$, where the rotation number of the Poincaré map plays an essential role (cf. [1]). For $n \geq 3$, there is a recent result of Chen Wenyi and M.Y. Chi [11] for the reduction of a smooth vector field \mathcal{L} on \mathbb{T}^n to a constant one provided that the adjoint operator \mathcal{L}^* is globally C^∞ hypoelliptic. We stress that the classical results of Arnold and Moser in KAM theory are not applied to the original vector field, but to a small perturbation of it (cf. [42] and the references therein).

In the second part of this paper we estimate the loss of the Gevrey regularity of the conjugating p.d.o.'s if the lower order perturbations are Gevrey G^σ . Although we are not able to prove that our Gevrey estimates are best possible there is some evidence to suggest this from the sharp loss of Sobolev regularity in each inductive step. This phenomenon resembles a similar one in the effective stability (Nekhoroshev estimates) of normal forms in Dynamical systems and their applications (cf. [25], [34], [30], [21], [2]; see also [23] for Nekhoroshev estimates for billiard ball maps in \mathbb{R}^n , $n \geq 3$, by means of Gevrey techniques).

In the third part an application of the resonant normal forms to the study of the global hypoellipticity of commuting systems of pseudodifferential operators is presented. Indeed, assuming $\Gamma_{\mathbb{Z}}^{\omega} = \{0\}$, we show that certain inhomogeneous Diophantine conditions completely characterize the global properties in C^∞ and in G^θ for large values of θ by virtue of the Nekhoroshev type estimates. The case $\Gamma_{\mathbb{Z}}^{\omega} \neq \{0\}$ is more difficult. A natural extension of completely resonant systems is introduced and it will be shown that the discrete condition is necessary and sufficient for the global hypoellipticity of the perturbations of such systems. We will also consider systems where no discrete phenomena appear.

The final question considered in this paper is that of the “size” of the exceptional sets of ω associated with G^σ Siegel conditions, for which perturbations of the vector fields X_k are not reducible to normal forms. The inhomogeneous Diophantine conditions also play an essential role when studying

the global properties of the reduced operators on the torus. Here the problem is twofold: firstly, ordinary Hausdorff dimension gives little information about the exceptional sets in the Gevrey category and so we use logarithmic Hausdorff dimension; secondly, the inhomogeneity of the Diophantine conditions causes some difficulties.

The paper is organized in the following way. Section 2 deals with the geometry of the resonant set and Section 3 is devoted to simultaneous normal forms. The Nekhoroshev type estimates are shown in section 4 and global hypoellipticity and solvability are discussed in section 5. Finally, estimates for the exceptional sets using a generalized concept of Hausdorff dimension are proved in section 6.

2. SIMULTANEOUS RESONANCE

The resonant set $\Gamma_{\mathbb{Z}}^{\omega}$ will now be investigated. We observe that if $P \in SL(n; \mathbb{Z})$ where $SL(n; \mathbb{Z})$ is the set of all $n \times n$ integer matrices with determinant ± 1 (i.e., the group of linear automorphisms of the lattice \mathbb{Z}^n), and $y = {}^t r P x$ then $\langle \omega^j, \partial_x \rangle \rightarrow \langle P \omega^j, \partial_y \rangle$, which implies the invariance property $P(\Gamma_{\mathbb{Z}}^{\omega}) = \Gamma_{\mathbb{Z}}^{P\omega}$, $P\omega = \{P\omega^1, \dots, P\omega^k\}$. Also note that if the \mathbb{R} -span of $\tilde{\omega}^1, \dots, \tilde{\omega}^{\ell} \in \mathbb{R}^n$ coincides with the \mathbb{R} -span of $\omega^1, \dots, \omega^k \in \mathbb{R}^n$, then $\Gamma_{\mathbb{Z}}^{\omega} = \Gamma_{\mathbb{Z}}^{\tilde{\omega}}$.

Theorem 2.1. *Assume $\Gamma_{\mathbb{Z}}^{\omega} \neq \{0\}$. Then*

i) there exist $\kappa^1, \dots, \kappa^r \in \mathbb{Z}^n \setminus \{0\}$ linearly independent over \mathbb{R} such that $\{\kappa^{\nu}\}_{\nu=1}^r$ form a basis of $\Gamma_{\mathbb{Z}}^{\omega}$, i.e.

$$\Gamma_{\mathbb{Z}}^{\omega} = \{z \in \mathbb{Z}^n; z = \sum_{\nu=1}^r t_{\nu} \kappa^{\nu}, \text{ with } t_{\nu} \in \mathbb{Z} \text{ uniquely determined}\}.$$

The number $r = r(\omega) = \dim_{\mathbb{Z}} \Gamma_{\mathbb{Z}}^{\omega}$ is invariant under the action of $SL(n; \mathbb{Z})$ i.e. $r(\omega) = r(P\omega)$ for every $P \in SL(n; \mathbb{Z})$. It is always possible to find κ^j (after changes of indices $x_{\mu} \rightarrow x_{\nu}$) to satisfy

$$(2.1) \quad \kappa^j = (\kappa_1^j, \kappa_2^j, \dots, \kappa_n^j), \quad \kappa_j^j \in \mathbb{N}, \kappa_{\nu}^j = 0, 1 \leq \nu \leq j-1.$$

Every other basis $\{\tilde{\kappa}^{\nu}\}_{\nu=1}^r$ is represented uniquely in the form $(\tilde{\kappa}^1, \dots, \tilde{\kappa}^r) = (\kappa^1, \dots, \kappa^r)Q$ for some $Q \in SL(r; \mathbb{Z})$.

ii) Let $\kappa^1, \dots, \kappa^r$ be a basis of $\Gamma_{\mathbb{Z}}^{\omega}$. Then there exists $P \in SL(n; \mathbb{Z})$ such that its first r columns coincide with $\kappa^1, \dots, \kappa^r$, i.e. the basis of $\Gamma_{\mathbb{Z}}^{\omega}$ is extended into a basis of \mathbb{Z}^n . In particular, if $y = (y', y'') = {}^t r P x$ with $y' = (y_1, \dots, y_r)$ and $y'' = (y_{r+1}, \dots, y_n)$, then

$$(2.2) \quad P\omega^k = (0', \theta^k), \quad \theta^k \in \mathbb{R}^{n-r}, 1 \leq k \leq d$$

with $\Gamma_{\mathbb{Z}}^{\theta} = \{0\}$, i.e. $\theta^1, \dots, \theta^k$ are nonresonant in \mathbb{R}^{n-r} . In the y variables the resonant set $\Gamma_{\mathbb{Z}}^{P\omega}$ coincides with $\mathbb{Z}_{y'}^r$ where $\mathbb{Z}_{y'}^r$ is the lattice in \mathbb{Z}^r with basis y' . The set of all matrices P from $SL(n; \mathbb{Z})$ with such a property is isomorphic to $\mathcal{M}_{n-r \times r}(\mathbb{Z}) \oplus SL(n-r; \mathbb{Z})$, where $\mathcal{M}_{n-r \times r}(\mathbb{Z})$ denotes the group of $n-r \times r$ integer matrices.

In order to prove the theorem we recall the following well known assertion (see [35] for more general statements).

Lemma 2.2. *Let $A \subset B := \mathbb{Z}^n$ be a subgroup. Set $C = A^\perp = \{\xi \in \mathbb{Z}^n : \xi \perp A\}$. Then*

$$(2.3) \quad B = A \oplus C$$

holds if and only if the sequence

$$(2.4) \quad 0 \rightarrow A \xrightarrow{id} B \xrightarrow{\pi} C \rightarrow 0$$

is exact (i.e. $\text{Ker}\pi = id(A)$) where π (respectively, id) stands for the orthogonal projection on C (respectively, the identity map).

Proof of Theorem 2.1 The existence of a basis follows from the well known fact that any subgroup A (in our case $A = \Gamma_{\mathbb{Z}}^\omega$) of a finitely generated abelian group B (in our case $B = \mathbb{Z}^n$) is also a finitely generated abelian group [35]. Define $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\pi_j(\xi) = \langle \xi, \vec{e}_j \rangle \vec{e}_j = \xi_j \vec{e}_j$, where \vec{e}_j is the j -th unit vector. Without loss of generality we may assume that for each $j \in \{1, \dots, n\}$ we have $\pi_j(\omega^k) \neq 0$ for some k , otherwise we are reduced to a space of dimension $n - 1$. Because $\Gamma_{\mathbb{Z}}^\omega$ is a linear \mathbb{Z} -submodule of \mathbb{Z}^n , $\pi_1(\Gamma_{\mathbb{Z}}^\omega)$ is closed under summation, and therefore there exists $p_1 \in \mathbb{N}$ such that $\pi_1(\Gamma_{\mathbb{Z}}^\omega) = p_1 \mathbb{Z} \vec{e}_1$. Let $\kappa^1 \in \Gamma_{\mathbb{Z}}^\omega$ be such that $\pi_1(\kappa^1) = p_1 \vec{e}_1$. Let $\Lambda_2 \subset \Gamma_{\mathbb{Z}}^\omega$ be the \mathbb{Z} -submodule such that $\pi_1(\Lambda_2) = \{0\}$. In fact, Λ_2 is uniquely determined: $\Lambda_2 = \{(\xi_1, \dots, \xi_n) \in \Gamma_{\mathbb{Z}}^\omega : \xi_1 = 0\}$. If $\Lambda_2 = \{0\}$ the set $\Gamma_{\mathbb{Z}}^\omega$ is generated by κ^1 and $r = 1$. If $\Lambda_2 \neq \{0\}$, this procedure is continued and sequences $\{\kappa^k\} \subset \Gamma_{\mathbb{Z}}^\omega$, $\{p_k\} \subset \mathbb{N}$ are constructed together with a nested family of \mathbb{Z} -submodules $\{\Lambda_k\}$, $\Lambda_k \subset \Lambda_{k-1}$ such that $\pi_k(\Lambda_k) = p_k \mathbb{Z} \vec{e}_k$, $\pi_k(\kappa^k) = p_k \vec{e}_k$. Clearly, this procedure ends after a finite number of steps. Let $r \leq n - 1$ be the smallest integer such that $\Lambda_{r+1} = \{0\}$, then the vectors κ^j form a basis of $\Gamma_{\mathbb{Z}}^\omega$ and satisfy (2.1).

Let now $\{\tilde{\kappa}^j\}_1^r \subset \Gamma_{\mathbb{Z}}^\omega$ be another basis of $\Gamma_{\mathbb{Z}}^\omega$. Then there exist two $r \times r$ integer matrices Q and \tilde{Q} such that $(\kappa^1 \cdots \kappa^r)Q = (\tilde{\kappa}^1 \cdots \tilde{\kappa}^r)$ and $(\tilde{\kappa}^1 \cdots \tilde{\kappa}^r)\tilde{Q} = (\kappa^1 \cdots \kappa^r)$. Hence $\tilde{Q} = Q^{-1}$ and $\det Q \det \tilde{Q} = 1$. Because $\det Q$ and $\det \tilde{Q}$ are integers it follows that $\det Q = \det \tilde{Q} = \pm 1$, i.e. $Q, \tilde{Q} \in SL(r; \mathbb{Z})$. Conversely, for any $Q \in SL(r; \mathbb{Z})$, $(\kappa^1 \cdots \kappa^r)Q$ gives a basis of $\Gamma_{\mathbb{Z}}^\omega$. This proves (i).

Set $A = \Gamma_{\mathbb{Z}}^\omega$ and $C = A^\perp = \{\xi \in \mathbb{Z}^n : \xi \perp A\}$. Evidently $A \subset C^\perp$. If we show the exactness of (2.4), namely $\rho \in \mathbb{Z}^n$ and $\rho \perp C$ imply $\rho \in A$, then Lemma 2.2 yields the first assertion of ii) since by (2.3) we can always extend a basis of A with a basis of C and obtain a basis of \mathbb{Z}^n .

Let $\rho \in \mathbb{Z}^n$, $\rho \perp C$. Set $C_{\mathbb{R}}^\perp = \{\xi \in \mathbb{R}^n; \xi \perp C\}$. Fix a basis as in (2.1). Since $\kappa^j \perp C$ for $j = 1, \dots, r$, and $\dim_{\mathbb{R}} C_{\mathbb{R}}^\perp = r$ we obtain that $\kappa^1, \dots, \kappa^r$ is a basis of $C_{\mathbb{R}}^\perp$ and ρ is a linear combination of $\kappa^1, \dots, \kappa^r$ implying that $\langle \omega^k, \rho \rangle = 0$ for all $1 \leq k \leq d$. Since $\rho \in \mathbb{Z}^n$, this implies $\rho \in A$.

Next, note that $\kappa^1, \dots, \kappa^r$ become the first r unit vectors in \mathbb{R}_y^n . Thus the invariance of the resonant set implies that $\Gamma_{\mathbb{Z}}^{P\omega} = \{\zeta \in \mathbb{Z}^n : \zeta_{r+1} =$

$\dots = \zeta_n = 0\}$ and (2.2) is therefore true. If $\theta^1, \dots, \theta^{n-r}$ are resonant in $\mathbb{R}_{y'}^{n-r}$ there will exist a vector $\zeta^0 \in \mathbb{Z}^{n-d} \setminus \{0\}$ orthogonal to $\theta^1, \dots, \theta^{n-r}$, so that $(0', \zeta^0) \in \Gamma_{\mathbb{Z}}^{P\omega}$, which is impossible by $\zeta^0 \neq 0''$.

The proof is concluded by observing that $P \in SL(n; \mathbb{Z})$ preserves the basis $\vec{e}_1, \dots, \vec{e}_r$ of $\Gamma_{\mathbb{Z}}^{P\omega} = \mathbb{Z}^r \times 0$ if and only if $P = \begin{pmatrix} I_r & 0_{n-r \times r} \\ M & P'' \end{pmatrix}$ with $P'' \in SL(n-r; \mathbb{Z})$ and $M \in \mathcal{M}_{n-r \times r}(\mathbb{Z})$.

The basis $\kappa^1, \dots, \kappa^r$ is called a canonical basis of $\Gamma_{\mathbb{Z}}^{\omega}$, and r is a $(\mathbb{Z} -)$ dimension of $\Gamma_{\mathbb{Z}}^{\omega}$ over \mathbb{Z} so that we write $\dim_{\mathbb{Z}} \Gamma_{\mathbb{Z}}^{\omega} = r$. We stress that $\mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^{\omega}$ is not a linear \mathbb{Z} submodule of \mathbb{Z}^n . \square

Remark 2.3. Take $A = \{(2t, -2t); t \in \mathbb{Z}\}$, $C = A_{\mathbb{Z}}^{\perp} = \{(t, t); t \in \mathbb{Z}\}$. Then although $A, B = \mathbb{Z}^2$ and C are finitely generated abelian groups, the decomposition (2.3) is not true (the sequence (2.4) is not exact because $C_{\mathbb{Z}}^{\perp} \neq A$).

The vectors $\omega^k \in \mathbb{R}^n$, $(1 \leq k \leq d)$ are said to satisfy the simultaneous resonant σ -Siegel condition if, for every $\varepsilon > 0$

$$\liminf_{|\xi| \rightarrow \infty, \xi \in \mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^{\omega}} \left(e^{\varepsilon |\xi|^{1/\sigma}} \max_{1 \leq k \leq d} |\langle \omega^k, \xi \rangle| \right) > 0, \quad (Si)_{\sigma}$$

for $1 \leq \sigma < \infty$, and

$$\liminf_{|\xi| \rightarrow \infty, \xi \in \mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^{\omega}} \left(|\xi|^{\tau} \max_{1 \leq k \leq d} |\langle \omega^k, \xi \rangle| \right) > 0, \quad (Si)_{\infty}$$

when $\sigma = \infty$ for some $\tau \in \mathbb{R}$.

Remark 2.4. Observe that $(Si)_{\sigma}$ and $(Si)_{\infty}$ are invariant under the action of $SL(n; \mathbb{Z})$. In particular, extending to a basis of \mathbb{Z}^n and in view of (2.2), we get that $(Si)_{\sigma}$ (respectively $(Si)_{\infty}$) becomes

$$\liminf_{|\xi''| \rightarrow \infty, \xi'' \in \mathbb{Z}^{n-r}} \left(e^{\varepsilon |\xi''|^{1/\sigma}} \max_{1 \leq k \leq d} |\langle \theta^k, \xi'' \rangle| \right) > 0, \quad \forall \varepsilon > 0 \quad (Si)_{\sigma}'$$

(respectively,

$$\liminf_{|\xi''| \rightarrow \infty, \xi'' \in \mathbb{Z}^{n-r}} \left(|\xi''|^{\tau} \max_{1 \leq k \leq d} |\langle \theta^k, \xi'' \rangle| \right) > 0, \quad (Si)_{\infty}'$$

for the nonresonant vectors $\theta^1, \dots, \theta^k$ in $\mathbb{R}_{y'}^{n-k}$).

If $d = 1$ and $\omega^1 \in \mathbb{R}^n$ is nonresonant, i.e. $\Gamma_{\mathbb{Z}}^{\omega} = \{0\}$, then $(Si)_{\infty}$ (respectively, $(Si)_{\sigma}$, $1 \leq \sigma < \infty$) is the well known small divisor condition for ω^1 and the number τ in $(Si)_{\infty}$ is necessarily not less than $n - 1$ [40] (respectively, coincides with the Gevrey type Siegel condition used in [22]). See also [39], [4] for $\sigma = 1$). Furthermore, if $n \geq 3$ and $d = n - 1$, $(Si)_{\infty}$ in the nonresonance case is the same as the simultaneous Diophantine condition used in [5].

We exhibit examples related to the above notions. In the completely resonant case (i.e., $r = n - 1$), according to Theorem 2.1, after a linear automorphism of the torus \mathbb{T}^n we can reduce the original system to the one consisting of vector fields proportional to ∂_{y_n} . Next, consider two vector fields

$$L_1 = \partial_{x_1} + \alpha_1 \partial_{x_2}, \quad L_2 = \partial_{x_1} + \alpha_2 \partial_{x_3}$$

on \mathbb{T}^3 , where α_1 and α_2 are irrational. Then the resonant set of L_j has dimension 1, $j = 1, 2$, and L_j satisfies $(Si)_\sigma$ if and only if α_j is not Liouville, for $\sigma = \infty$, and α_j is not σ -Liouville, i.e., for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left| \alpha_j - \frac{p}{q} \right| \geq C_\varepsilon \exp(-\varepsilon q^{1/\sigma}), \quad p \in \mathbb{Z}, q \in \mathbb{N}.$$

Furthermore, the system $\chi = \{L_1, L_2\}$ is simultaneously nonresonant if and only if α_1/α_2 is irrational. Finally, if α_1 and α_2 are rationally independent σ -Liouville numbers, then the simultaneous σ -Siegel condition $(Si)_\sigma$ is satisfied if and only if for $\sigma = \infty$ α_1 and α_2 satisfy the simultaneous Diophantine condition in [36] (see also [5], [28]), while for $1 \leq \sigma < \infty$ for every $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that

$$\max_{j=1,2} \left\{ \left| \alpha_j - \frac{p}{q} \right| \right\} \geq C_\varepsilon \exp(-\varepsilon q^{1/\sigma}), \quad p \in \mathbb{Z}, q \in \mathbb{N}.$$

(cf. [4] for $\sigma = 1$, see also [24] for other simultaneous Gevrey arithmetic conditions).

Let $C_{2\pi}^\infty(\mathbb{R}_x^n)$ be the space of all 2π periodic C^∞ functions (i.e. the lift on \mathbb{R}^n of all functions from $C^\infty(\mathbb{T}^n)$) where \mathbb{R}_x^n is the lattice in \mathbb{R}^n with basis x . It is readily verified that this space is invariant under the action of $SL(n; \mathbb{Z})$, namely $f(y) \in C_{2\pi}^\infty(\mathbb{R}_y^n)$ if and only if $P^* f(x) := f(Px) \in C_{2\pi}^\infty(\mathbb{R}_x^n)$ for some $P \in SL(n; \mathbb{Z})$. We will identify $C^\infty(\mathbb{T}^n)$ and $G^\sigma(\mathbb{T}^n)$ with $C_{2\pi}^\infty(\mathbb{R}^n)$ and $G_{2\pi}^\sigma(\mathbb{T}^n)$, respectively.

3. GLOBAL CANONICAL FORM OF A RESONANT SYSTEM

First we recall the basic properties of G^σ pseudodifferential operators (see [38], [15]). In the C^∞ case, the constants in the estimates below may depend on the multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ and the index of homogeneity j . Let $FS_\sigma^m = FS_\sigma^m(\mathbb{T}^n \times \mathbb{R}^n)$ be the set of all formal sums $\sum_{j=0}^\infty a_{m-j}(x, \xi)$ such that $a_{m-j} \in G^\sigma(\mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\}))$, $\text{ord}_\xi a_{m-j} = m - j$ (i.e. a_{m-j} is (positively) homogeneous of order $m - j$) and that there exists $A > 0$ satisfying

$$(3.1) \quad \sup_{x \in \mathbb{T}^n} |\partial_x^\beta \partial_\xi^\gamma a_{m-j}(x, \xi)| \leq A^{j+|\gamma|+|\beta|+1} (\beta! j!)^\sigma \gamma! |\xi|^{m-j-|\gamma|}$$

for all $\beta \in \mathbb{Z}_+^n$, $\gamma \in \mathbb{Z}_+^n$, $j \in \mathbb{Z}_+$, $\xi \in \mathbb{R}^n$. By Cauchy's integral formula and the homogeneity of a_{m-j} (3.1) is equivalent to the existence of a neighbourhood \tilde{S} in \mathbb{C}^n of S^{n-1} such that

$$\sup_{x \in \mathbb{T}^n, \xi \in \tilde{S}} |\partial_x^\beta a_{m-j}(x, \xi)| \leq A^{j+|\beta|+1} (\beta! j!)^\sigma, \quad \forall \beta \in \mathbb{Z}_+^n, j \in \mathbb{Z}_+.$$

A function $a(x, \xi) \in G^\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ is said to be a G^σ realization of $\sum_{j=0}^{\infty} a_{m-j}(x, \xi) \in FS_\sigma^m$, written $a(x, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi)$, if for every $\gamma \in \mathbb{Z}_+$ one can find $C = C_\gamma > 0$ such that

$$\left| \partial_x^\beta \partial_\xi^\gamma (a(x, \xi) - \sum_{j=0}^N a_{m-j}(x, \xi)) \right| \leq C^{N+|\gamma|+|\beta|+1} (\beta! N!)^\sigma |\xi|^{m-N-|\gamma|-1}$$

for all $\beta \in \mathbb{Z}_+^n$, $N \in \mathbb{Z}_+$, $x \in \mathbb{T}^n$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$. The set of G^σ realizations of order m is denoted by $S_\sigma^m(\mathbb{T}^n \times \mathbb{R}^n)$ and the set of pseudodifferential operators with symbol in $S_\sigma^m(\mathbb{T}^n \times \mathbb{R}^n)$ by $OS_\sigma^m(\mathbb{T}^n)$. We also recall that for each $\tilde{a} \in FS_\sigma^m$ there exists a realization $a \in S_\sigma^m$ such that $a \sim \tilde{a}$. If $b(x, \xi)$ is another G^σ realization of \tilde{a} , then $R(x, \xi) = a(x, \xi) - b(x, \xi) \in S_\sigma^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$, i.e. R is a G^σ smoothing symbol, so that for some $c > 0$

$$\sup_{x \in \mathbb{T}^n} |\partial_x^\beta R(x, \xi)| \leq c^{-|\beta|-1} (\beta!)^\sigma e^{-c|\xi|^{1/\sigma}}, \quad \xi \in \mathbb{R}^n, \beta \in \mathbb{Z}_+^n.$$

Note that the composition rule for p.d.o.s on \mathbb{T}^n is valid as in the local case, that is if $a(x, \xi) \sim \sum_{j=0}^{\infty} a_{\mu-j} \in S_\sigma^\mu(\mathbb{T}^n \times \mathbb{R}^n)$ and $b(x, \xi) \sim \sum_{j=0}^{\infty} b_{\nu-j}(x, \xi) \in S_\sigma^\nu(\mathbb{T}^n \times \mathbb{R}^n)$ then $c(x, D) := a(x, D) \circ b(x, D)$ is in $OS_\sigma^{\mu+\nu}(\mathbb{T}^n)$ with symbol $c(x, \xi) = a(x, \xi) \circ b(x, \xi) \sim \sum_{j=0}^{\infty} c_{\mu+\nu-j}(x, \xi)$ given by (see [15])

$$c_{\mu+\nu-j}(x, \xi) = \sum_{p+q+|\beta|=j} \frac{1}{\beta!} D_\xi^\beta a_{\mu-p}(x, \xi) \partial_x^\beta b_{\nu-q}(x, \xi), \quad j \in \mathbb{Z}_+.$$

Furthermore, if $a(x, \xi)$ is elliptic, i.e. $a_\mu(x, \xi) \neq 0$ for $(x, \xi) \in \mathbb{T}^n \times (\mathbb{R}^n \setminus 0)$, then $a(x, D)$ admits a (formal) inverse $a^{-1}(x, D) \in OS^{-\mu}$ satisfying $a \circ a^{-1} - id \in OS_\sigma^{-\infty}$ and $a^{-1} \circ a - id \in OS_\sigma^{-\infty}$. Recall that for a given $P \in SL(n; \mathbb{Z})$, the change of variables $y = Px$ induces the linear (symplectic) change $(x, \xi) \rightarrow (y, \eta)$, $x = P^{-1}y$ and $\xi = {}^{tr}P\eta$. We have $(P^*a)(y, \eta) = a(P^{-1}y, {}^{tr}P\eta)$ for $y \in \mathbb{T}^n$, $\eta \in \mathbb{R}^n$. Thus the composition is invariant, i.e. for two p.d.o.-s $a = a(x, D)$ and $b = b(x, D)$ we have

$$(3.2) \quad P^*(a \circ b) = P^*a \circ P^*b.$$

Moreover, if a is a p.d.o. having a symbol depending in x only on resonant variables, the same is true for P^*a .

We denote by $C^\infty(\mathbb{R}^n \setminus \{0\}; g^\sigma(\mathbb{T}_x^n))$ as the set of functions $a(x, \xi)$ smooth in $\xi \in \mathbb{R}^n \setminus \{0\}$ and valued in $G^\sigma(\mathbb{T}_x^n)$. Now we state the first main result on simultaneous normal forms.

Theorem 3.1. *Let $1 \leq \sigma \leq \infty$ and suppose that $(Si)_\sigma$ holds for ω . If $r = \dim_{\mathbb{Z}} \Gamma_{\mathbb{Z}}^\omega \geq 1$ fix a resonant (canonical) basis $\{\kappa^j\}_1^r$. Assume that if $\sigma < \infty$ then $a_{-j}^k(x, \xi)$ is $C^\infty(\mathbb{R}^n \setminus \{0\}; G^\sigma(\mathbb{T}_x^n))$ for $k = 1, \dots, d$, $j \in \mathbb{Z}_+$. Then there exists an elliptic symbol $q(x, \xi) \sim \sum_{j=0}^{\infty} q_{-j}(x, \xi) \in FS^0(\mathbb{T}^n \times \mathbb{R}^n)$ and*

symbols $\alpha^k(y, \xi) \sim \sum_{j=0}^{\infty} \alpha_{-j}^k(y, \xi) \in FS^0(\mathbb{T}^r \times \mathbb{R}^n)$ such that for $1 \leq k \leq d$

(3.3)

$$(-iX_k + a^k(x; D)) \circ q(x; D) = q(x; D) \circ (-iX_k + \alpha^k(\kappa x; D) + R_k(x, D)),$$

where $R_k(x, \xi) \in S^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$ and $\kappa x := (\langle \kappa^1, x \rangle, \dots, \langle \kappa^r, x \rangle)$.

Proof. Let $P \in SL(n; \mathbb{Z})$ be as in Theorem 2.1, with the first r rows being $\kappa^1, \dots, \kappa^r$, and define $y = Px$. In view of (3.2), we will consider the new variables (y, η) , with $\kappa^j = \bar{e}_j$. The resonant set is the lattice $\mathbb{Z}_{y'}^r$ in $\mathbb{R}_{y'}^r$, where $y = (y', y'')$ and $y' = (y_1, \dots, y_r)$. For brevity $a(y, \eta)$ is used instead of $P^*a(y, \eta) = a(P^{-1}y, {}^{tr}P\eta)$ for the symbol $a(x, \xi)$. We write terms of the same order of homogeneity on both sides of (3.3) in the new variables y . The first system of d equations involves both $q_0(y, \eta)$ and $\alpha_0^k(y', \eta)$ ($1 \leq k \leq d$)

$$(3.4) \quad \langle \theta^k, D_{y''} \rangle q_0 + a_0^k(y, \eta)q_0 = \alpha_0^k(y', \eta)q_0, \quad \eta \in \mathbb{R}^n \setminus 0.$$

The advantage of the expression (3.4) is that we can regard the resonant variables y' as parameters if $r \geq 1$. A nonzero solution of (3.4) can be found if for every $k \in \{1, \dots, d\}$, α_0^k is chosen to satisfy

$$(3.5) \quad \int_{\mathbb{T}^{n-r}} a_0^k(y', y'', \eta) dy'' = \alpha_0^k(y', \eta), \quad y' \in \mathbb{T}^r, \eta \in \mathbb{R}^n \setminus \{0\}$$

i.e. the Fourier coefficients $\mathcal{F}_{y \rightarrow \zeta}(a^k(y, \eta) - \alpha_0^k(y', \eta))$ vanish on the resonant set, which in the canonical variables is defined by $\zeta_{r+1} = \dots = \zeta_n = 0$. By the commutativity we have $\langle \theta^j, D_{y''} \rangle a_0^k(y, \eta) = \langle \theta^k, D_{y''} \rangle a_0^j$ for $1 \leq j, k \leq d$. As $\langle \theta^j, D_{y''} \rangle a_0^k \equiv 0$, it follows from Frobenius theorem and some simple calculations that the general solution of (3.4) is given by

$$q_0(y, \eta) = \ell(y', \eta)e^{i\psi(y, \eta)}, \quad \ell(y', \eta) \in G^\sigma(\mathbb{T}^r \times (\mathbb{R}^n \setminus 0)), \text{ord}_\eta \ell = 0,$$

where $\psi(y, \eta)$ is the unique solution of the following system of equations with Fourier coefficients vanishing on the resonant set $\Gamma_{\mathbb{Z}}^{P\omega}$,

$$(3.6) \quad \langle \theta^k, \partial_{y''} \rangle \psi = f_0^k(y, \eta) := -a_0^k(y, \eta) + \alpha_0^k(y', \eta), \quad 1 \leq k \leq d.$$

In fact, ψ is defined by

$$\mathcal{F}_{y'' \rightarrow \zeta''}(\psi(y', y'', \eta)) = \frac{\mathcal{F}_{y'' \rightarrow \zeta''}(f_0^k(y', y'', \eta))}{i \langle \theta^k, \zeta'' \rangle}$$

if $\langle \theta^k, \zeta'' \rangle \neq 0$ for some $1 \leq k \leq d$, and is 0 otherwise. In view of $(Si)_\sigma$ and the results in [22], $\psi(y, \eta)$ (and therefore $q_0(y, \eta)$) is G^σ in x . The composition rule for p.d.o.s yields

$$(q_0^{-1} \circ b^k \circ q_0)(y, \eta) = \langle \theta^k, \eta'' \rangle + \alpha_0^k(y', \eta) + (O(|\eta|^{-1})), |\eta| \rightarrow \infty$$

for all $1 \leq k \leq d$, where b^k stands for $P^*(iX_k + a^k)$. We proceed by induction: suppose that there exist $\alpha_{-j}^k(y', \eta) \in C^\infty(\mathbb{R}^n \setminus \{0\}; G^\sigma(\mathbb{T}_{y'}^r))$,

$\text{ord}_\eta \alpha_{-j}^k(y', \eta) = -j$, $0 \leq j \leq N-1$, $1 \leq k \leq d$, such that

$$a^k(y, \eta) \sim \sum_{j=0}^{N-1} \alpha_{-j}^k(y', \eta) + \sum_{j=N}^{\infty} a_{-j}^k(y, \eta), \quad 1 \leq k \leq d.$$

Now we look for an elliptic symbol $q(y, \eta) \sim 1 + q_{-N}(y, \eta)$, $\text{ord}_\eta q_{-N} = -N$, and $\alpha_{-N}^k(y', \eta) \in C^\infty(\mathbb{R}^n \setminus \{0\}; G^\sigma(\mathbb{T}^r))$ such that

$$(q^{-1} \circ b^k \circ q)(y, \eta) = \langle \theta^k, \eta'' \rangle + \sum_{j=0}^N \alpha_{-j}^k(y', \eta) + O(|\eta|^{-N-1}),$$

when $|\eta| \rightarrow \infty$ for all $1 \leq k \leq d$. For $k = 1, \dots, d$ this gives the system of equations,

$$(3.7) \quad -i \langle \theta^k, \partial_{y''} \rangle q_{-N}(y, \eta) = f_{-N}^k(y, \eta) := \alpha_{-N}^k(y', \eta) - a_{-N}^k(y, \eta).$$

for $q_{-N}(y, \eta)$ and $\alpha_{-N}^k(y', \eta)$. In view of (3.5) and (3.6) define α_{-N}^k by

$$(3.8) \quad \mathcal{F}_{y' \rightarrow \zeta'}(\alpha_{-N}^k(y', \eta)) = \mathcal{F}_{y \rightarrow \zeta}(a_{-N}^k(y', y'', \eta))|_{\zeta=(\zeta', 0)}.$$

By the Frobenius theorem, (3.8) and $(Si)_\sigma$, (3.7) admits a unique solution $q_{-N}(y, \xi)$, if $\mathcal{F}_{y' \rightarrow \zeta'}(q_{-N}(y', y'', \eta)) = 0$ for $\zeta' \in \mathbb{Z}^r$. Moreover, in view of $(Si)_\sigma$, $q_{-N}(y, \xi)$ is $C^\infty(\mathbb{R}^n \setminus \{0\}; G^\sigma(\mathbb{T}^n))$.

Finally, construct an elliptic p.d.o. q by

$$(3.9) \quad q(y, \eta) = \dots \circ (1 + q_{-N}(y, \eta)) \circ \dots \circ (1 + q_{-1}(y, \eta)) \circ q_0(y, \eta).$$

The right-hand side in (3.9) defines a formal symbol in $FS^0(\mathbb{T}^n \times (\mathbb{R}^n \setminus 0))$. Taking any realization of this formal symbol, the above calculations yield (3.3) and the estimates on the remainders. \square

Remark 3.2. *Let $\omega \in \mathbb{R}^n$ be nonresonant and assume that ω does not satisfy $(Si)_\sigma$ for some $1 \leq \sigma \leq \infty$. Then we can find $f(x) \in G^\sigma(\mathbb{T}^n)$ such that the pseudo-differential operator with symbol $\langle \omega, \xi \rangle + f(x)|\xi|^{-1}$ cannot be transformed via conjugation (3.3) into a constant p.d.o.*

Indeed, in view of the assumption on ω there exists a positive constant $\delta > 0$ and a sequence $\rho^k \in \mathbb{Z}^n \setminus 0$ ($k = 1, 2, \dots$) such that

$$\lim_{k \rightarrow \infty} e^{-2\delta|\rho^k|^{1/\sigma}} |\langle \omega, \rho^k \rangle|^{-1} = \infty.$$

We choose a function $f(x) = \sum_\eta \hat{f}(\eta)e^{i\eta x} \in G^\sigma(\mathbb{T}^n)$ such that $\hat{f}(\eta) = e^{-\delta|\rho^k|^{1/\sigma}}$ if $\eta = \rho^k$, $\hat{f}(\eta) = 0$ for $\eta \neq \rho^k$. The solution $q_0(x, \xi)$ of the equation $\langle \omega, D_x \rangle q_0 = f(x)\xi/|\xi|$ satisfies $|\mathcal{F}_{x \rightarrow \eta}(q_0(x, \xi))| \geq Ce^{\delta|\eta|^{1/\sigma}}$ for $\eta = \rho^k$, $k \in \mathbb{N}$, which contradicts the fact that $q_0 \in G^\sigma$ in x . Thus, our Diophantine conditions are optimal.

Remark 3.3. *In Theorem 3.1 perturbations of constant vector fields X_k are considered. If X_k is a variable coefficient vector field, the problem is essentially a global nonlinear one. There follows two examples of a system of overdetermined vector fields being simultaneously transformed into constant*

vector fields. First, any nonsingular hypoelliptic real vector field on \mathbb{T}^2 is diffeomorphic to a nonzero multiple of $\partial_t + \rho\partial_x$ with ρ being an irrational non Liouville number (cf. [29]). The second example is an overdetermined system of vector fields, $L = dt + \omega(t) \wedge \partial_x$, $x \in \mathbb{T}^1$, $t = (t_1, \dots, t_n) \in \mathbb{T}^n$, $\omega(t) = \sum_{j=1}^n \omega_j(t) dt_j$ being a real-valued smooth closed one form on \mathbb{T}^n . The corresponding family of n commuting vector fields associated with L (cf. [5]) is given by $L_j = \partial_{t_j} + \omega_j(t)\partial_x$, $1 \leq j \leq n$. Straightforward calculations show the existence of a real-valued function $h(t)$ on \mathbb{T}^n such that $\partial_{t_j} h(t) = \omega_j(t) - c_j$, with c_j being the mean value $[\omega_j(\cdot)]_n$ of ω_j , $1 \leq j \leq n$. Then the family $\{L_j\}_1^n$ is transformed into $\{\partial_{s_j} + c_j\partial_y\}_1^n$ via the diffeomorphism of \mathbb{T}^{n+1} : $y = x - h(t)$, $s = t$.

We now study the simultaneous reduction of a family of commuting vector fields with variable coefficients on \mathbb{T}^2

$$X_j = \partial_t + h_j(t, x)\partial_x, \quad j = 1, \dots, d.$$

The commutativity relations are given by

$$(3.10) \quad \partial_t(h_k - h_j) = h_k\partial_x h_j - h_j\partial_x h_k, \quad j, k = 1, \dots, d.$$

Let $\psi_j^t : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ be the time map of X_j , defined by $\psi_j^t(\xi) = x_j(t; \xi)$, where $x_j(t; \xi)$ is a solution of $\dot{x}_j = h_j(t, x_j)$, $x_j(0) = \xi$. Clearly (3.10) is equivalent to $\psi_j^t \circ \psi_k^s = \psi_k^s \circ \psi_j^t$, $s, t \in \mathbb{R}$, $j, k = 1, \dots, d$.

Recall that the Poincaré map of X_j , given by $P_j := \psi_j^{2\pi} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$, is a diffeomorphism of \mathbb{T}^1 and assume the orientation preserving property of P_j . The rotation number of P_j is denoted by $\rho_j \in \mathbb{R}$ (ρ_j is also called the rotation number of the vector field X_j). Moreover, suppose that there exists a smooth diffeomorphism u simultaneously conjugating the maps P_1, \dots, P_d to the rotations $R_{\rho_1}, \dots, R_{\rho_d}$ where $R_{\rho_j}(z) = z + \rho_j$, $z \in \mathbb{T}$ ($j = 1, \dots, d$), so that

$$(3.11) \quad u^{-1} \circ P_j \circ u = R_{\rho_j}, \quad j = 1, \dots, d.$$

Theorem 3.4. *Let $1 \leq \sigma \leq +\infty$. If $d \geq 2$ assume that the Poincaré maps P_1, \dots, P_d are orientation preserving and that there exists an index $j \in \{1, \dots, d\}$ such that $(2\pi)^{-1}\rho_j$ is irrational. Then if a G^σ diffeomorphism u on \mathbb{T}^1 satisfying (3.11) can be found, the map $\Phi: t = s, x = \phi(s, y)$, where $\phi(s, y) = x_1(s, u(y - s(2\pi)^{-1}\rho_1))$ if $d = 1$ and $\phi(s, y) = x_j(s, u(y - s(2\pi)^{-1}\rho_j))$ if $d \geq 2$, is a G^σ diffeomorphism of \mathbb{T}^2 transforming $\partial_t + h_k(t, x)\partial_x$ to $\partial_s + (2\pi)^{-1}\rho_k\partial_y$ for all $1 \leq k \leq d$.*

Proof. We want to reduce the vector fields X_j to $\partial_t + \rho_j\partial_x$ by the change of variables on \mathbb{T}^2 , $(t, x) = \Phi(s, y)$ with $t = s$ and $x = \phi(s, y)$ simultaneously. By simple computations it can be seen that $\partial_s + (2\pi)^{-1}\rho_j\partial_y = \partial_t + (\phi_s + \rho_j\phi_y)\partial_x$. Hence it is necessary to solve the equation

$$\phi_s + \rho_j\phi_y = h_j(s, \phi(s, y)), \quad j = 1, \dots, d.$$

By the definition of $x_j(t; \xi)$, we obtain for $\xi = u(y - s(2\pi)^{-1}\rho_j)$, $t = s$

$$\begin{aligned} \partial_s \phi + (2\pi)^{-1} \rho_j \partial_y \phi &= \dot{x}_j(t, \xi) - (2\pi)^{-1} \rho_j \partial_\xi x_j(t; \xi) u'(y - s(2\pi)^{-1} \rho_j) \\ &+ (2\pi)^{-1} \rho_j \partial_\xi x_j(t; \xi) u'(y - s(2\pi)^{-1} \rho_j) = \dot{x}_j(t; \xi) \\ &= h_j(t, x_j(t; \xi)) = h_j(s, \phi(s, y)). \end{aligned}$$

Hence ϕ transforms $\partial_t + h_j(t, x) \partial_x$ into $\partial_s + (2\pi)^{-1} \rho_j \partial_y$. If $d \geq 2$ for $i \neq j$, $\partial_t + h_i(t, x) \partial_x$ is transformed to $\partial_s + g_i(s, y) \partial_y$ for some $g_i \in G^\sigma(\mathbb{T}^2; \mathbb{R})$. The commutativity implies that

$$(\partial_s + (2\pi)^{-1} \rho_j \partial_y) g_i(s, y) = 0, \quad i = 1, \dots, d, i \neq j.$$

Since $(2\pi)^{-1} \rho_j$ is irrational it is readily verified from the Fourier expansion that the g_i are constants. Because the rotation number is invariantly defined for vector fields on \mathbb{T}^2 , we have $g_i(s, y) \equiv (2\pi)^{-1} \rho_i$.

In order to prove that the map Φ is a G^σ diffeomorphism on \mathbb{T}^2 note that the Jacobian of Φ does not vanish as it is equal to $\partial_y \phi$. For $\xi = u(y - s(2\pi)^{-1} \rho_j)$ we have $\partial_y \phi(s, y) = \partial_\xi x_j(t; \xi) u'(y - s(2\pi)^{-1} \rho_j)$, and since u is a G^σ diffeomorphism $u'(y) \neq 0$ for every $y \in \mathbb{R}$. On the other hand, $\theta(t; z) := \partial_z x_j(t; z)$ satisfies

$$\dot{\theta} = \partial_x h_j(t, x_j(t; z)) \theta, \quad \theta|_{t=0} = 1$$

which implies that $\theta(t; \xi) = \partial_\xi x_j(t; \xi) \neq 0$ for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}$. Thus $\partial_y \phi(s, y) \neq 0$ for all $s \in \mathbb{R}$, $y \in \mathbb{R}$ and the map Φ is a G^σ diffeomorphism. In order to complete the proof, we will show that $\phi(s + 2\pi, y) = \phi(s, y)$ and $\phi(s, y + 2\pi) = \phi(s, y)$. The latter relation is easy; for the former one recall that $x_j(2\pi, u(y)) = u(y + \rho_j)$ by the definition of u . The function $x_j(t + 2\pi, u(y - \rho_j - s\rho_j/2\pi))$ solves $\dot{x}_j = h_j(t, x_j)$ with initial value $\xi = x_j(2\pi, u(y - \rho_j - s\rho_j/2\pi)) = u(y - s\rho_j/2\pi)$. By uniqueness it is therefore equal to $x_j(t, u(y - s\rho_j/2\pi))$. \square

Remark 3.5. a) Let $d = 1$ and suppose either that $\sigma = 1$ and $\rho/(2\pi) = \rho_1/(2\pi)$ satisfies the Bruno condition [9] or that $\sigma = \infty$ and $\rho/(2\pi)$ satisfies the usual Siegel condition. Then by the global reduction theorems due to M. Herman and A. Bruno, the circle map $P = P_1$ is conjugated to a rotation by a G^1 or C^∞ diffeomorphism.

b) Let $d = 1$ and $1 < \sigma < +\infty$. We are not aware of any result on global reduction in G^σ Gevrey classes.

c) Let $d \geq 2$. If $\sigma = +\infty$ or $\sigma = 1$, there is a local simultaneous reduction theorem due to J. Moser [36], under a simultaneous Siegel condition. A global C^∞ simultaneous reduction theorem is shown by Y. Katznelson, B. Kra and D. Ornstein [32].

d) If $d \geq 1$ and $1 \leq \sigma < +\infty$, T. Gramchev and M. Yoshino [24] proved local simultaneous reductions of commuting G^σ circle maps under weak exponential arithmetic conditions.

4. NEKHOROSHEV TYPE ESTIMATES FOR G^σ P.D.O.

The next result shows that perturbations with G^σ p.d.o.s are conjugated with G^θ elliptic p.d.o.s for some $\theta > \sigma$ if simultaneous small divisor conditions in the nonresonant case are satisfied. This shows a sharp contrast with the case of commuting diffeomorphisms, where even weaker arithmetic conditions are sufficient.

Theorem 4.1. *Let $\Gamma_{\mathbb{Z}}^\omega = \{0\}$ and let $(Si)_\infty$ in §2 hold for some $\tau \leq -1$. Let $\theta = \max\{1, 1 + \tau\}$. Assume that $a^k(x, D)$ ($1 \leq k \leq d$) is a G^σ p.d.o. Then there exist symbols $q_{-j}(x, \xi)$ of the conjugating p.d.o. with the following properties: there exists a neighbourhood $\tilde{S} \subset \mathbb{C}^n$ of the unit sphere S^{n-1} such that for every $s > n/2$, $\varepsilon > 0$ we can find $M > 0$, $N > 0$ satisfying*

$$(4.1) \quad \sup_{\xi \in \tilde{S}} \|\partial_x^\gamma q_{-j}(\cdot, \xi)\|_{H^s} \leq \varepsilon M^j N^{|\gamma|} ((\theta j + |\gamma|)!)^\sigma, \quad \gamma \in \mathbb{Z}_+^n, j \in \mathbb{N},$$

$$(4.2) \quad \sup_{\xi \in \tilde{S}} |\alpha_{-j}^k(\xi)| \leq \varepsilon M^j ((\theta j)!)^\sigma, \quad k = 1, \dots, d, j \in \mathbb{N},$$

where $\|\cdot\|_{H^s}$ denotes the usual Sobolev norm. It follows that $\sum_{j=0}^\infty q_{-j}(x, \xi) \in FS_{\theta\sigma}^0$ and $\sum_{j=0}^\infty \alpha_{-j}(\xi) \in FS^0(\mathbb{R}^n \setminus \{0\})$. Set $\theta' = \theta$ if $\sigma > 1$ or $\tau > 0$, and $\theta' = 1 + \rho$, $0 < \rho \ll 1$ if $\theta = 1$ (i.e. $\sigma = 1$ and $\tau \leq 0$). Then there exist $G^{\theta\sigma}$ realizations $q(x, \xi)$, $\alpha^k(\xi)$, $1 \leq k \leq d$ and positive constants C , δ such that

$$\sup_{x \in \mathbb{T}^n} |\partial_x^\beta \partial_\xi^\gamma R_k(x, \xi)| \leq C_2^{|\beta|+|\gamma|} (\gamma!)^\sigma (\beta!)^{\sigma\theta'} |\xi|^{-|\gamma|} \exp(-\delta|\xi|^{1/(\sigma\theta)}),$$

for all $\beta, \gamma \in \mathbb{Z}_+^n$, $\xi \in \mathbb{R}^n$ and $1 \leq k \leq d$. Thus $R_k(x, D) \in OS_{\sigma\theta}^{-\infty}(\mathbb{T}^n)$, $1 \leq k \leq d$. Finally, if the symbols $a_j(x, \xi)$ depend on one base variable, the system can be transformed into a simultaneous normal form without any restrictions on τ in the Diophantine condition for $\omega^1, \dots, \omega^d$. In this case there is no loss of regularity in the remainders; more precisely if $a_k(x, D)$, $1 \leq k \leq d$ are G^σ p.d.o. then $R_k(x, D)$ are G^σ smoothing p.d.o.s.

First we need a technical assertion, where $C_0^\infty(\mathbb{T}^n)$ denotes the set of smooth functions with mean value zero.

Lemma 4.2. *Suppose that $(Si)_\infty$ holds. Then for every $f_k(x) \in C_0^\infty(\mathbb{T}^n)$ such that $X_k f_r = X_r f_k$, $1 \leq r, k \leq d$ there exists a unique solution $q(x) \in C_0^\infty(\mathbb{T}^n)$ of the system $\langle \omega^k, \partial_x \rangle q(x) = f_k(x)$, $1 \leq k \leq d$, satisfying $\hat{q}(\xi) = 0$, $\xi \in \Gamma_{\mathbb{Z}}^\omega$. Furthermore, for some $C_\tau > 0$ this unique solution satisfies the estimate*

$$\|q\|_{H^s} \leq C_\tau \max_{1 \leq k \leq d} \|f_k\|_{H^{s+\tau}}, \quad s \geq 0.$$

Proof. By Fourier transform we obtain $\langle \omega^k, \eta \rangle \hat{q}(\eta) = \hat{f}_k(\eta)$. It follows that $\hat{f}_k(\eta) = 0$ when $\langle \omega^k, \eta \rangle = 0$. The compatibility condition and the definition of $\Gamma_{\mathbb{Z}}^\omega$ implies that we can decompose $\mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^\omega = \bigcup_{m=1}^d Z'(m)$, $Z'(m) \cap Z'(j) = \emptyset$, $1 \leq j < m \leq d$ with the property $\langle \omega^k, \eta \rangle \neq 0$ for

$\eta \in Z'(k)$, $1 \leq k \leq d$. Hence for $1 \leq k \leq d$ the inverse Fourier transform gives,

$$\begin{aligned} \|q\|_{H^s} &\leq (2\pi)^{-n/2} \sum_{k=1}^d \left(\sum_{\eta \in Z'(k)} \frac{|\eta|^{2s}}{|\langle \omega^k, \eta \rangle|^2} |\widehat{f}_k(\eta)|^2 \right)^{1/2} \\ &\leq C_\tau \max_{1 \leq k \leq d} \left(\sum_{\eta \in \mathbb{Z}^n \setminus 0} \langle \eta \rangle^{2(s+\tau)} |\widehat{f}_k(\eta)|^2 \right)^{1/2} = C_\tau \max_{1 \leq k \leq d} \|f_k\|_{H^{s+\tau}}, \end{aligned}$$

where $\langle \eta \rangle = (1 + |\eta|^2)^{1/2}$. \square

Proof of Theorem 4.1. Suppose that τ is an integer (the proof is the same for $\tau \notin \mathbb{Z}$, but there are some technical complications). Clearly, after conjugation with the symbol $e^{i\psi(x,\xi)}$, we may suppose that $a_0^k(x, \xi) = \alpha_0(\xi)$ for $1 \leq k \leq d$ and hence that $\psi = 0$, $q_0 = 1$ and the conjugating p.d.o. is of the form $q(x, \xi) \sim 1 + \sum_{j=1}^{\infty} q_{-j}(x, \xi)$. Then, by the composition rule, q_{-j} satisfies the following equations

$$(4.3) \quad \langle \omega^k, D_x \rangle q_{-j} = F_{-j}^k(x, \xi) + G_{-j}^k(x, \xi) - a_{-j}^k(x, \xi) + \alpha_{-j}^k(\xi)$$

for $1 \leq k \leq d$, where

$$\begin{aligned} F_{-j}^k(x, \xi) &= - \sum_{r=1}^{j-1} \sum_{|\beta| \leq j-r} \frac{1}{\beta!} D_\xi^\beta a_{-(j-r-|\beta|)}^k(x, \xi) \partial_x^\beta q_{-r}(x, \xi); \\ G_{-j}^k(x, \xi) &= \sum_{r=1}^{j-1} q_{-r}(x, \xi) \alpha_{-(j-r)}^k(\xi). \end{aligned}$$

Clearly in order to solve (4.3) we must choose

$$(4.4) \quad \alpha_{-j}^k(\xi) = -[F_{-j}^k(\cdot, \xi)]_n - [G_{-j}^k(\cdot, \xi)]_n + [a_{-j}^k(\cdot, \xi)]_n, \quad 1 \leq k \leq d.$$

Because $a^k(x, \xi) \in S_\sigma^0(\mathbb{T}^n)$, $1 \leq k \leq d$ there exists a complex neighbourhood \tilde{S} in \mathbb{C}^n of the unit sphere S^{n-1} in \mathbb{R}^n so that for fixed $s > n/2$ and τ there exists a positive constant A such that

$$(4.5) \quad \sup_{\xi \in \tilde{S}} \|\partial_x^\beta a_{-j}^k(\cdot, \xi)\|_{H^{s+\tau}} \leq A^{j+|\beta|+1} (j! \beta!)^\sigma, \quad j \in \mathbb{Z}_+, \beta \in \mathbb{Z}_+^n.$$

By the Schauder lemma for multiplication in Sobolev spaces, if $s + \tau > n/2$ there exists $\omega > 0$ such that

$$(4.6) \quad \|fg\|_{H^{s+\tau}} \leq \omega \|f\|_{H^{s+\tau}} \|g\|_{H^{s+\tau}}, \quad f, g \in H_0^{s+\tau}(\mathbb{T}^n).$$

Estimates (4.1) and (4.2) will be proved by induction. They both hold for $j = 0$. Suppose that they are true for $0 \leq j \leq i - 1$. By (4.6) we obtain for

$\xi \in \tilde{S}$ that

$$\begin{aligned}
(4.7) \quad & \|\partial_x^\gamma F_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} \leq \omega \sum_{r=1}^{i-1} \sum_{|\beta| \leq i-r} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} \frac{1}{\beta!} \\
& \times \|\partial_x^{\gamma-\gamma'} D_\xi^\beta a_{-(i-r-|\beta|)}^k(\cdot, \xi)\|_{H^{s+\tau}} \|\partial_x^{\beta+\gamma'} q_{-r}(\cdot, \xi)\|_{H^{s+\tau}} \\
& \leq \varepsilon \omega A \sum_{r=1}^{i-1} \sum_{|\beta| \leq i-r} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} A^{i-r+|\gamma|-|\gamma'|} ((i-r-|\beta|)! (\gamma-\gamma')!)^\sigma M^r \\
& \times N^{|\beta|+\tau+|\gamma'|} ((\theta r + |\beta| + |\gamma'| + \tau)!)^\sigma \leq \varepsilon M^i N^{|\gamma|} ((\theta i + |\gamma|)!)^\sigma \omega A N^\tau K
\end{aligned}$$

with

$$\begin{aligned}
(4.8) \quad & K = \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta| \leq i-r} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} \\
& \times (A/N)^{|\gamma|-|\gamma'|} \left(\frac{(i-r-|\beta|)! (\gamma-\gamma')! (\theta r + |\beta| + \tau + |\gamma'|)!}{(\theta i + |\gamma|)!} \right)^\sigma \\
& \leq \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta| \leq i-r} \sum_{\gamma' \leq \gamma} \frac{(|\gamma'|+1) \dots (|\gamma'|+|\gamma|-|\gamma'|)}{(\gamma-\gamma')!} \\
& \times (A/N)^{|\gamma|-|\gamma'|} \left(\frac{(\gamma-\gamma')!}{(|\gamma'|+1) \dots (|\gamma'|+|\gamma|-|\gamma'|)} \right)^\sigma \\
& \times \left(\frac{(i-r-|\beta|)! (|\gamma'|+1) \dots (|\gamma'|+|\gamma|-|\gamma'|) (\theta r + |\beta| + |\gamma'| + \tau)!}{(\theta i + |\gamma|)!} \right)^\sigma \\
& \leq \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta| \leq i-r} \sum_{\gamma' \leq \gamma} (A/N)^{|\gamma|-|\gamma'|} C_1^{\sigma-1} C_2^\sigma
\end{aligned}$$

where $C_1 = \frac{(\gamma-\gamma')!}{(|\gamma'|+1) \dots (|\gamma'|+|\gamma|-|\gamma'|)} \leq 1$, while

$$\begin{aligned}
C_2 &= \frac{(i-r-|\beta|)! (|\gamma'|+1) \dots (|\gamma'|+|\gamma|-|\gamma'|) (\theta r + |\beta| + |\gamma'| + \tau)!}{(\theta i + |\gamma|)!} \\
&\leq \frac{(i-r-|\beta|)! (\theta r + |\beta| + |\gamma| + \tau)!}{(\theta i + |\gamma|)!} \leq \frac{(\theta i + |\gamma| - (\theta - 1)(i-r) + \tau)!}{(\theta i + |\gamma|)!} \leq 1
\end{aligned}$$

for all $\gamma, \gamma', \beta \in \mathbb{Z}_+^n$, $\gamma' \leq \gamma$, $r, i \in \mathbb{N}$, $i-r \geq 1$. We used the inequality $(\theta - 1)(i-r) \geq \theta - 1 \geq \tau$ if $i-r \geq 1$. Choosing first $N > A$ and then $M > 2AN$ large enough so that $AN/M(1 - A/N)^{-n} \sum_{\nu=0}^{\infty} 2^{-\nu} (\nu + 2)^n \leq (6C_\tau \omega AN^\tau)^{-1}$, and using the estimates on C_1 and C_2 , we get from (4.8)

that

$$\begin{aligned}
K &\leq \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta| \leq i-r} \sum_{\gamma' \leq \gamma} (A/N)^{|\gamma| - |\gamma'|} \\
&\leq \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta| \leq i-r} (1 - A/N)^{-n} \\
&\leq \frac{AN}{M} (1 - A/N)^{-n} \sum_{r=1}^{i-1} (AN/M)^{i-r-1} (i-r+1)^n \\
&\leq AN/M (1 - A/N)^{-n} \sum_{\nu=0}^{\infty} 2^{-\nu} (\nu+2)^n \leq (6C_\tau \omega AN^\tau)^{-1}.
\end{aligned}$$

Hence, by the choice of M and N ,

$$(4.9) \quad \sup_{\xi \in \tilde{S}} \|\partial_x^\gamma F_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} \leq (3C_\tau)^{-1} \varepsilon M^i N^{|\gamma|} ((\theta i + |\gamma|)!)^\sigma, \quad \gamma \in \mathbb{Z}_+^n.$$

Next, for any $\xi \in \tilde{S}$

$$\begin{aligned}
\|\partial_x^\gamma G_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} &\leq \sum_{r=1}^{i-1} \|\partial^\gamma q_{-r}(\cdot, \xi)\|_{H^{s+\tau}} |\alpha_{-(i-r)}^k(\xi)| \\
&\leq \varepsilon^2 M^i N^{|\gamma| + \tau} ((\theta i + |\gamma|)!)^\sigma L,
\end{aligned}$$

$$\begin{aligned}
(4.10) \quad L &= \sum_{r=1}^{i-1} \left(\frac{(\theta r + |\gamma| + \tau)! (\theta(i-r))!}{(\theta i + |\gamma|)!} \right)^\sigma \\
&= \left(\frac{(\tau+1)!}{\theta i + |\gamma|} \right)^\sigma \sum_{r=1}^{i-1} \left(\frac{(\tau+2) \dots (\tau+1 + \theta r + |\gamma| - 1)}{(\theta(i-r) + 1) \dots (\theta(i-r) + \theta r + |\gamma| - 1)} \right)^\sigma.
\end{aligned}$$

Since $i-r \geq 1$, $r \geq 1$ and $\theta \geq (1+\tau)$

$$\begin{aligned}
\tilde{c} &:= \frac{(\tau+2) \dots (\tau+1 + \theta r + |\gamma| - 1)}{(\theta(i-r) + 1) \dots (\theta(i-r) + \theta r + |\gamma| - 1)} \\
(4.11) \quad &\leq \frac{\theta \dots (\theta + \theta r + |\gamma| - 1)}{(\theta(i-r) + 1) \dots (\theta(i-r) + \theta r + |\gamma| - 1)} \leq 1.
\end{aligned}$$

It is readily verified from (4.10) that, for all $\gamma \in \mathbb{Z}_+^n$

$$\begin{aligned}
L &\leq \left(\frac{(\tau+1)!}{\theta i + |\gamma|} \right)^\sigma \sum_{r=1}^{i-1} 1 \\
&= \left(\frac{(\tau+1)!}{\theta i + |\gamma|} \right)^\sigma (i-1) \leq \frac{((\tau+1)!)^\sigma}{(\theta i + |\gamma|)^{\sigma-1}} \leq ((\tau+1)!)^\sigma.
\end{aligned}$$

Hence, by choosing $\varepsilon > 0$ and $N > 0$ so that $\varepsilon N^\tau ((\tau+1)!)^\sigma \leq 1/3$ we obtain, from (4.1) and (4.2) with $1 \leq j \leq i-1$

$$(4.12) \quad \sup_{\xi \in \tilde{S}} \|\partial_x^\gamma G_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} \leq 3^{-1} \varepsilon M^i N^{|\gamma|} ((\theta i + |\gamma|)!)^\sigma, \quad \gamma \in \mathbb{Z}_+^n.$$

Finally, observe that (4.5) leads to

$$(4.13) \quad \sup_{\xi \in \tilde{S}} \|\partial_x^\gamma a_{-j}^k(\cdot, \xi)\|_{H^{s+\tau}} \leq 3^{-1} \varepsilon M^j N^{|\gamma|} ((\theta j + |\gamma|)!)^\sigma, \quad j \in \mathbb{N}, \gamma \in \mathbb{Z}_+^n$$

provided that $M \geq 3A/\varepsilon$ and $N \geq A$. By Lemma 4.2, (4.4), (4.12) and (4.13) we have, for any $\xi \in \tilde{S}$

$$\begin{aligned} |\alpha_{-i}^k(\xi)| &\leq |[F_{-i}^k(\cdot, \xi)]_n| + |[G_{-i}^k(\cdot, \xi)]_n| + |[a_{-i}^k(\cdot, \xi)]_n| \\ &\leq \|F_{-i}^k(\cdot, \xi)\|_{L^2} + \|G_{-i}^k(\cdot, \xi)\|_{L^2} + \|a_{-i}^k(\cdot, \xi)\|_{L^2} \\ &\leq \|F_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} + \|G_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} + \|a_{-i}^k(\cdot, \xi)\|_{H^{s+\tau}} \\ &\leq \varepsilon (C_\tau)^{-1} M^i ((\theta i)!)^\sigma \leq \varepsilon M^i ((\theta i)!)^\sigma. \end{aligned}$$

Thus (4.1) and (4.2) for $1 \leq j \leq i-1$ imply (4.2) for $j = i$, while (4.1) for $j = i$ follows from Lemma 4.2 applied to $f_k = F_{-i}^k(x, \xi) + G_{-i}^k(x, \xi) - a_{-i}^k(x, \xi) + \alpha_{-i}^k(\xi)$, $1 \leq k \leq d$, and the fact that the $H^{s+\tau}(\mathbb{T}^n)$ norm of f_k does not depend on α_{-i}^k . \square

Remark 4.3. *As the proof of Theorem 4.1 shows, our approach resembles methods used for obtaining effective stability (Nekhoroshev type) estimates (cf. [2], [21], [25], [23], [31], [34]). However, we stress that the study of the Gevrey regularity of the conjugating p.d.o. $q(x, D)$ presents new features and difficulties in comparison to the aforementioned results in dynamical systems. This is due to the presence of compositions of p.d.o.s and the global regularity in $\xi \in \mathbb{R}^n$. In fact, the composition rule of p.d.o.s is a major technical obstacle for getting Gevrey estimates in the presence of resonances. Finally, we point out that our iterative approach shows that the dominating term in $q_j(x, \xi)$, $|\xi| = 1$, is given by $C^{j+1} |D^{j\theta} q_{-1}(x, \xi)| \sim (j!)^\theta$ which suggests that the Gevrey index σ is sharp.*

5. GLOBAL PROPERTIES OF PERTURBATIONS OF RESONANT VECTOR FIELDS

In this section we study the global hypoellipticity of first order overdetermined systems by use of global normal forms.

Let \mathcal{L}_j be defined by (1.2) with $a_j \in G^\sigma(\mathbb{T}^n)$. We say that \mathcal{L}_j is G^σ globally hypoelliptic ($1 \leq \sigma \leq \infty$) if every distribution u such that $\mathcal{L}_j u \in G^\sigma$ ($j = 1, \dots, d$) is in G^σ . Set $r = \dim_{\mathbb{Z}} \Gamma_{\mathbb{Z}}^\omega$ and for $r \geq 1$, take a canonical basis η^1, \dots, η^r of $\Gamma_{\mathbb{Z}}^\omega$. Suppose that it is possible to find $\alpha_j(y) \in G^\sigma(\mathbb{T}^r)$ ($j = 1, \dots, d$) for which the system

$$(5.1) \quad \langle \omega^j, \partial_x \rangle \psi = a_j(x) - \alpha_j(\langle \eta^1, x \rangle, \dots, \langle \eta^r, x \rangle), \quad 1 \leq j \leq d$$

admits a unique solution ψ with $\hat{\psi}(\xi) = 0$ for $\xi \in \Gamma_{\mathbb{Z}}^{\omega}$. Note that (5.1) is weaker than the simultaneous σ -Siegel type condition on $\mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^{\omega}$ since if that holds we can solve (5.1) by Fourier series expansion.

If the multiplication operator q is defined by $qv(x) = e^{i\psi(x)}v(x)$, the proof of Theorem 3.1 and (5.1) imply that for every $j \in \{1, \dots, d\}$ we have

$$q^{-1} \circ \mathcal{L}_j \circ q = L_j := \langle \omega^j, D_x \rangle + \alpha_j(\langle \eta^1, x \rangle, \dots, \langle \eta^r, x \rangle)$$

Indeed, in (3.7) $a_{-N}^k(x, \xi) = 0$ for $N \geq 1$. Therefore the system $\mathcal{L}_j u = f_j$, ($1 \leq j \leq d$) is equivalent to the system

$$(5.2) \quad L_j v = g_j, \quad 1 \leq j \leq d$$

where $g_j = e^{i\psi(x)} f_j(x)$, $1 \leq j \leq d$.

We define the change of variables $x = T\tilde{x}$, $\tilde{x} = (y, z)$ with $y = (y_1, \dots, y_r)$ and $z = (z_1, \dots, z_{n-r})$ by $y_j = \langle \eta^j, x \rangle$, ($1 \leq j \leq r$), and $z_j = \langle \kappa^j, x \rangle$, ($1 \leq j \leq n-r$). Then the system (5.2) becomes

$$(5.3) \quad \langle \theta^j, D_z \rangle w + \alpha_j(y)w = h_j(y, z), \quad j = 1, \dots, d,$$

where the $\theta^j \in \mathbb{R}^{n-r}$ are given by $(0, \theta^j) = {}^{tr}T\omega^j$, $1 \leq j \leq d$. The partial Fourier transform $z \rightarrow \zeta$ reduces (5.3) to

$$(5.4) \quad (\langle \theta^j, \zeta \rangle + \alpha_j(y))\hat{w}(y, \zeta) = \hat{h}_j(y, \zeta), \quad \zeta \in \mathbb{Z}^{n-r}, j = 1, \dots, d.$$

Define $\Phi(y, \zeta) = \sum_{j=1}^d |\langle \theta^j, \zeta \rangle + \alpha_j(y)|$ and let $d_0 \leq n-r$ be the rank of $\theta^1, \dots, \theta^d$. Then we have

Theorem 5.1. *Suppose $\Gamma_{\mathbb{Z}}^{\omega} \neq \{0\}$ and let η^1, \dots, η^r be a canonical basis of $\Gamma_{\mathbb{Z}}^{\omega}$. Assume that (5.1) has a unique solution and that $a_{-j}^k(x, \xi) \in C^\infty(\mathbb{R}_x^n \setminus \{0\}; G^\sigma(\mathbb{T}_x^n))$ if $\sigma < \infty$. Then*

(i) *The system (5.3) is G^σ globally hypoelliptic if there exists $c > 0$ such that $\Phi(y, \zeta) \geq c$ for all $\zeta \in \mathbb{Z}^{n-r}$ and $y \in \mathbb{T}^r$. If $d_0 = n-r$ the preceding condition is replaced by a discrete condition $\Phi(y, \zeta) \neq 0$ for all $\zeta \in \mathbb{Z}^{n-r}$ and all $y \in \mathbb{T}^n$. Note that the completely resonant case $r = n-1$ satisfies $d_0 = n-r$.*

(ii) *Suppose $d_0 = n-r$. Then the system (5.3) is G^σ globally hypoelliptic for $1 < \sigma \leq \infty$ if and only if the discrete condition is satisfied.*

Proof. By the representation (5.4), w satisfies $\hat{w}(y; \zeta) = \hat{h}_j(y; \zeta)(\langle \theta^j, \zeta \rangle + \alpha_j(y))^{-1}$ for some $1 \leq j \leq d$. Hence $w \in G^\sigma$. If $d_0 = n-r$, then $\sum_{j=1}^d |\langle \theta^j, \zeta \rangle| \rightarrow \infty$ as $|\zeta| \rightarrow \infty$, $\zeta \in \mathbb{Z}^{n-r}$. Hence the condition $\Phi \geq c$ can be replaced by the discrete condition proving (i).

As to (ii) we define the distribution $w(y, z)$ by $\hat{w}(y, \zeta) \equiv 0$ if $\zeta \neq \zeta^0$, and $\hat{w}(y, \zeta^0) = \delta(y - y^0)$, with $y^0 \in \mathbb{T}^r$ and $\zeta^0 \in \mathbb{Z}^{n-r}$ satisfying $\zeta^0 + \alpha_k(y^0) = 0$ for $1 \leq k \leq d$. Here $\delta(y - y^0)$ stands for the Dirac measure massed at $y = y^0$. The distribution $w(y, z)$ gives a nonsmooth zero solution of (5.3). \square

Remark 5.2. *If $d_0 < n-r$ the condition $\Phi(y, \zeta) \geq c$ is not a discrete condition because the vectors $(\langle \theta^1, \zeta \rangle, \dots, \langle \theta^d, \zeta \rangle)$ ($\zeta \in \mathbb{Z}^{n-r}$) fill a*

dense subset of a d_0 -dimensional subspace of \mathbb{R}^d and if $d_0 < d$ this actually occurs. Next, if the simultaneous G^σ Siegel condition holds, the solvability of (5.1) is superfluous; that is instead of the Diophantine conditions on ω^j , the solvability of (5.1) for special $a_j(x)$ is enough. This allows us to characterize larger classes of first order operators on \mathbb{T}^2 with nonresonant $\omega = \omega^1$ than those in Proposition 3.2 of [3] ($\sigma = \infty$) and to generalize some of the results for a class of overdetermined systems of vector fields on \mathbb{T}^n in Theorem 3.3 of [4] ($\sigma = 1$). Note that the Gevrey spaces $G^\sigma(\mathbb{T}^n)$ are not metrisable if $1 \leq \sigma < \infty$ and therefore Baire's theorem is not applicable.

Now we study the global properties of $\{b^k(x, D)\}$ in the nonresonant case $\Gamma_{\mathbb{Z}}^\omega = \{0\}$. Let a corresponding normal form $\{\omega^k \cdot \xi + \alpha^k(\xi)\}_1^d$ be as in Theorem 3.1 and $1 \leq \theta \leq \infty$. We say that $\{\omega^k \xi + \alpha^k(\xi)\}_1^d$ satisfies the simultaneous inhomogeneous θ -Siegel condition if

$$\liminf_{|\xi| \rightarrow \infty, \xi \in \mathbb{Z}_+^n \setminus 0} \exp(\varepsilon |\xi|^{1/\theta}) \max_{1 \leq k \leq d} |\omega^k \cdot \xi + \alpha^k(\xi)| > 0, \quad \text{for every } \varepsilon > 0 \quad (TSi)_\theta$$

for $1 \leq \theta < \infty$, and

$$\liminf_{|\xi| \rightarrow \infty, \xi \in \mathbb{Z}_+^n \setminus 0} |\xi|^N \max_{1 \leq k \leq d} |\omega^k \cdot \xi + \alpha^k(\xi)| > 0, \quad (TSi)_\infty$$

for some $N \in \mathbb{R}$ when $\theta = \infty$.

If $\theta = \infty$, $(TSi)_\infty$ is well defined in the sense that it does not depend on realizations of $\sum_{r=0}^\infty \alpha_{-r}(\xi)$. However, if $1 \leq \theta < \infty$, the condition $(TSi)_\theta$ is independent of realizations if q is a G^θ p.d.o. In this case, $\alpha^j(D)$ and the remainders $R^k(x, D)$ are G^θ p.d.o. as well.

Theorem 5.3. *Assume that ω^j , $j = 1, \dots, d$, are simultaneously nonresonant and satisfy $(Si)_\sigma$ for some $1 \leq \sigma \leq \infty$. Let the hypotheses of Theorem 3.1 be true. Then $(TSi)_\infty$ holds for one simultaneous canonical form of $\{b_j\}_1^d$ if and only if it holds for every simultaneous canonical form of $\{b^j\}_1^d$. Moreover, $\{b^j\}_1^d$ is globally hypoelliptic on \mathbb{T}^n if and only if a normal form of $\{b^j\}_1^d$ satisfies $(TSi)_\infty$. If $(Si)_\infty$ holds and $a^j(x, D)$, $j = 1, \dots, d$, are G^σ p.d.o. then the operator $\{b^j\}_1^d$ is G^θ globally hypoelliptic if and only if a normal form of $\{b^j\}_1^d$ satisfies $(TSi)_\theta$ for $\theta \geq \rho\sigma$, where $\rho = \max\{1, 1 + \tau\}$.*

Proof. The proof follows from Theorem 3.1 (respectively, the Nekhoroshev type estimates) if $\theta = \infty$ (respectively, $\theta \geq \rho\sigma$), and the following properties: Since $q(x, D)$ is elliptic in $G^{\rho\sigma}$ either both or neither systems $\{\omega^j D_x + a^j(x, D)\}$ and $\{\omega^j D_x + \alpha^j(D) + R^j(x, D)\}$ are globally hypoelliptic. The estimates for $q(x, D)$ and the remainders $R^j(x, \xi)$ imply the global G^θ hypoellipticity for $\theta \geq \rho\sigma$ if $(TSi)_\theta$ is true. \square

6. LOGARITHMIC HAUSDORFF DIMENSION

For $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ and $1 \leq \sigma \leq \infty$ we define

$$\begin{aligned} NG^\sigma(\beta; d) &= \{\omega = (\omega^1, \dots, \omega^d) \in \mathbb{R}^{nd} : \{\langle \omega^j, \xi \rangle + \beta_j\}_1^d \\ &\quad \text{does not satisfy } (TSi)_\sigma\}, \\ NG^\sigma(d) &= NG^\sigma(0; d). \end{aligned}$$

Clearly $NG^\sigma(d)$ with $\Gamma_{\mathbb{Z}}^\omega = \{0\}$ is the exceptional set for which $(Si)_\sigma$ (fails, and therefore no reduction to normal forms is possible. Similarly, if $\beta \neq 0$, $NG^\sigma(\beta; d)$ is related to the exceptional sets of $\{\omega^j\}_1^d$ for which the system $\{L_k + \beta_k\}_1^d$ is not globally G^σ hypoelliptic. The sizes of $NG^\sigma(d)$ and $NG^\sigma(\beta; 1)$ will be estimated in terms of the Hausdorff dimension and logarithmic Hausdorff dimension associated with the Gevrey index σ .

For $n > d$ let $E_\beta(\sigma) = E_\beta(\sigma; n, d)$ be the set

$$E_\beta(\sigma) = \{X \in \mathbb{R}^{nd} : |\mathbf{q}X - \beta| < e^{-|\mathbf{q}|^{1/\sigma}} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}.$$

Note that if X is identified with $\omega = (\omega^1, \dots, \omega^d)$, we have

$$E_0(\sigma) \subset NG^\sigma(d) \quad \text{and} \quad E_\beta(\sigma) \subset NG^\sigma(\beta; d)$$

A δ cover of a set A is a union of sets C with diameter less than or equal to δ such that $A \subseteq \cup C$.

Definition 6.1. Let $J \subseteq \mathbb{R}^{nd}$ and define the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $f(r) = r^{(n-1)d}(\log 1/r)^{-s}$. The logarithmic Hausdorff measure of J is

$$\mathfrak{L}^s(J) := \liminf_{\delta \rightarrow 0} \sum_{C \in \mathfrak{C}_\delta : \cup C \supset J} f(L(C)),$$

where the infimum is over all δ covers, \mathfrak{C}_δ of J and $L(C) \leq \delta$ is the diameter of C . The logarithmic Hausdorff dimension is

$$\text{Ldim}(J) := \sup\{s : \mathfrak{L}^s(J) = \infty\} = \inf\{s : \mathfrak{L}^s(J) = 0\}.$$

Hence $\text{Ldim}(J)$ is the unique value for which $\mathfrak{L}^s(J)$ changes from being ∞ to 0. The definition of Hausdorff dimension is the same as above with $f(L(C)) = L(C)^s$; its properties can be found in [20]. To denote the information given by the two dimensions, ordinary and logarithmic the dimension of J will be written as the ordered pair $\vec{\dim} J = (\dim J, \text{Ldim } J)$.

For the homogeneous case the following theorem will be proved.

Theorem 6.2. Let $\sigma \in \mathbb{R}^+$. Then $\vec{\dim} E_0(\sigma; n, d) = ((n-1)d, n\sigma)$.

By modifying the proof of Theorem 6.2 an inhomogeneous version will also be proved for the case $d = 1$.

Theorem 6.3. Let $\sigma \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$. Then $\vec{\dim} E_\beta(\sigma; n, 1) = (n-1, n\sigma)$.

Remark 6.4. Identify \mathbb{R}^{nd} with $(\mathbb{R}^n)^d$ and note that the sets $NG^\sigma(\beta; d)$ are invariant under the action of $SL(n; \mathbb{Z})$ ($P(NG^\sigma(\beta; d)) = NG^\sigma(\beta; d)$ for all $P \in SL(n; \mathbb{Z})$). Next, let \sim be an equivalence relation in $NG^\sigma(\beta; d)$ such

that $\omega \sim \omega'$ if and only if $\omega = P\omega'$ for some $P \in SL(n; \mathbb{Z})$ (i.e., they lie in the same orbit), then the space of $SL(n; \mathbb{Z})$ orbits $NG^\sigma(\beta; d)/\sim$ has the same $\vec{\dim}$.

Without loss of generality, as the two sets are invariant under translations of integer vectors, we can restrict ourselves to the sets $E_\beta(\sigma; n, d) \cap I^{nd}$ where $I = (-\frac{1}{2}, \frac{1}{2}]$. The proofs are done in three parts: firstly the Hausdorff dimensions are obtained; then an upper bound for the logarithmic Hausdorff dimension of $E_\beta(\sigma; n, d)$ is determined for all β , n and d and finally lower bounds for the logarithmic Hausdorff dimensions of the two sets are found.

In [14] the Hausdorff dimension of the set

$$W_0(\tau) = W_0(n, d; \tau) = \{X \in \mathbb{R}^{nd} : |\mathbf{q}X| < |\mathbf{q}|^{-\tau} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}$$

was shown to be $(n-1)d + n/(\tau+1)$. Plainly, the $(n-1)d$ -dimensional hyperplanes $R_{\mathbf{q}} = \{X \in I^{nd} : \mathbf{q}X = 0\}$ are contained in $E_0(\sigma)$ for all $\mathbf{q} \in \mathbb{Z}^n$ so that $\dim E_0(\sigma) \geq (n-1)d$. Also $E_0(\sigma) \subset W_0(\tau)$ which implies that $\dim E_0(\sigma) \leq \dim W_0(\tau)$ for all τ and all σ . Thus $\dim E_0(\sigma) = (n-1)d$. Similarly, in the inhomogeneous case the $(n-1)$ -dimensional hyperplanes $R_{\beta, \mathbf{q}} = \{\mathbf{x} \in I^n : \mathbf{q}\mathbf{x} + \beta = 0\}$ are contained in $E_\beta(\sigma; n, 1)$ which implies that $\dim E_\beta(\sigma; n, 1) \geq n-1$. Using the upper bound argument in the next section with $f(L(C))$ replaced by $L(C)^s$ it is readily verified that for any $s > n-1$ the Hausdorff s -measure of $E_\beta(\sigma; n, 1)$ is zero which implies that $\dim E_\beta(\sigma; n, 1) \leq n-1$ completing the result.

Now we deal with the logarithmic component of $\vec{\dim}$. In what follows $a \ll b$ (respectively $a \gg b$) means that there exists a constant $c > 0$ such that $a \leq cb$ ($a \geq cb$). If $a \ll b$ and $a \gg b$ then $a \asymp b$.

First the upper bounds are obtained for both theorems.

Lemma 6.5. *Let $n, d \in \mathbb{N}$, $n \geq 2$, $\sigma \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$. Then*

$$\text{Ldim } E_\beta(\sigma) \leq n\sigma.$$

Proof. For future reference note that

$$(6.1) \quad f(e^{-r^{1/\sigma}}/r) = \frac{(e^{-r^{1/\sigma}})^{(n-1)d}}{r^{(n-1)d}(r^{s/\sigma} + \log r)} \asymp \frac{(e^{-r^{1/\sigma}})^{(n-1)d}}{r^{(n-1)d+s/\sigma}}.$$

Define $R_{\beta, \mathbf{q}}$ as $R_{\beta, \mathbf{q}} = \{X \in I^{nd} : \mathbf{q}X + \beta = 0\}$ and let $C(\mathbf{q})$ denote a collection of hypercubes C of sidelength $L(C) = 2n^{1/2}e^{-|\mathbf{q}|^{1/\sigma}}/|\mathbf{q}|$ centred on $R_{\beta, \mathbf{q}}$ with centres at a distance of $e^{-|\mathbf{q}|^{1/\sigma}}/|\mathbf{q}|$ apart. There are $\ll (|\mathbf{q}|e^{|\mathbf{q}|^{1/\sigma}})^{(n-1)d}$ such hypercubes. Let C_N denote the set $C_N = \{C(\mathbf{q}) :$

$|\mathbf{q}| > N\}$. For each N this set is a cover of $E_\beta(\sigma)$. Hence, using (6.1)

$$\begin{aligned} \mathfrak{L}^s(E_\beta(\sigma)) &\leq \sum_{r>N} \sum_{|\mathbf{q}|=r} \sum_{C \in \mathcal{C}(\mathbf{q})} f(L(C)) \ll \sum_{r>N} \sum_{|\mathbf{q}|=r} \left(|\mathbf{q}|e^{|\mathbf{q}|^{1/\sigma}}\right)^{(n-1)d} \\ &\times \frac{\left(e^{-|\mathbf{q}|^{1/\sigma}}\right)^{(n-1)d}}{|\mathbf{q}|^{(n-1)d+s/\sigma}} \ll \sum_{r>N} r^{(n-1)-s/\sigma} < \varepsilon \end{aligned}$$

for $s > n\sigma$ and N sufficiently large. Therefore $\mathfrak{L}^s(E_\beta(\sigma)) = 0$ for $s > n\sigma$ and $\text{Ldim } E_\beta(\sigma) \leq n\sigma$. \square

Proof of Theorem 6.2. We now obtain a lower bound for $\text{Ldim } E_0(\sigma; n, d)$. First, a Cantor subset \mathbb{K} of $E_0(\sigma; n, d)$ and a probability measure supported on \mathbb{K} are constructed. Then the Mass Distribution Principle is used to show that the logarithmic Hausdorff measure of \mathbb{K} is infinity for $s < n\sigma$. This implies that the logarithmic Hausdorff dimension of \mathbb{K} and hence of $E_0(\sigma; n, d)$ is greater than or equal to $n\sigma$.

For the rest of the proof it is assumed that $s < n\sigma$. Also note that the following inequality will continually be used:

$$(6.2) \quad \frac{N^{n/d}}{16n \log N} \leq \left[\frac{N^{n/d}}{8n \log N} \right] \leq \frac{N^{n/d}}{8n \log N},$$

where $[\cdot]$ denotes integer part.

Similar Cantor constructions to the following can be found in [7, p 347] and [16] (for general dimension functions). Choose N_1 large enough so that

$$(6.3) \quad e^{N_1^{1/\sigma}} > \frac{5N_1^{n/d-1}}{2 \log N_1},$$

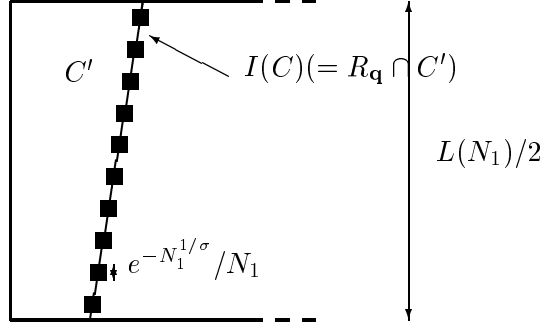
$$(6.4) \quad \frac{15N_1^{n-s/\sigma}}{5n^d 2^{4nd+4} \log^d N_1} > 1.$$

Starting in one corner, divide I^{nd} into $T(N_1) = [N_1^{n/d}/8n \log N_1]^{nd}$ hypercubes C with sidelength

$$L(C) = L_{N_1} = [N_1^{n/d}/8n \log(N_1)]^{-1}.$$

Define C' to be a hypercube with the same centre as C and $L(C') = \frac{1}{2}L(C)$. Let $R_N = \{R_{\mathbf{q}} : N \leq |\mathbf{q}| \leq 2N\}$ and let $G(N_1)$ denote the set of cubes C for which there exists an $R_{\mathbf{q}} \in R_{N_1}$ such that $C' \cap R_{\mathbf{q}} \geq (\frac{1}{10}L(C))^{(n-1)d}$; i.e., the intersection $C \cap R_{\mathbf{q}}$ is relatively large and $R_{\mathbf{q}}$ does not just "clip" one corner. From Lemma 2 in [14] it can be verified that for any $\varepsilon > 0$ and N_1 sufficiently large

$$(6.5) \quad \#G(N_1) \geq \left[\frac{N_1^{n/d}}{8n \log N_1} \right]^{nd} (1 - \varepsilon).$$



For each $C \in G(N_1)$ pick one such intersection $C' \cap R_{\mathbf{q}}$ and call it $I(C)$. Now draw smaller hypercubes centred on $I(C)$ and contained in C' of side-length $e^{-N_1^{1/\sigma}}/N_1$ with centres at a distance of $2e^{-N_1^{1/\sigma}}/N_1$ apart, so that they are disjoint, (see figure for $n = 2$). Let $\#I(C)$ denote the number of these small cubes lying on the segment $I(C)$. Then, using (6.3) it can be shown that

$$\begin{aligned} \left(\frac{1}{40}N_1L(C)e^{N_1^{1/\sigma}}\right)^{(n-1)d} &\leq \left(\frac{1}{20}L(C)N_1e^{N_1^{1/\sigma}} - 1\right)^{(n-1)d} \\ &\leq \#I(C) \leq \left(N_1e^{N_1^{1/\sigma}}L(C)\right)^{(n-1)d}. \end{aligned}$$

The union of these cubes over all $C \in G(N_1)$ forms the first level K_1 of the Cantor set and $\#K_1$ denotes the number of them. From (6.2), (6.5) (with $\varepsilon = 1/16$) and above

$$\begin{aligned} \#K_1 &\geq \frac{15}{16} \left[\frac{N_1^{n/d}}{8n \log N_1} \right]^{nd} \left(\frac{1}{40}N_1L(C)e^{N_1^{1/\sigma}} \right)^{(n-1)d} \\ &\geq \frac{15N_1^{nd-d+n} \left(e^{N_1^{1/\sigma}} \right)^{(n-1)d}}{2^{nd+4}(8n)^d 40^{(n-1)d} (\log N_1)^d}. \end{aligned}$$

Let E_1 denote a hypercube in K_1 , so that $L(E_1) = e^{-N_1^{1/\sigma}}/N_1$, then (6.4) yields

$$\#K_1 f(L(E_1)) \geq \frac{15N_1^{n-s/\sigma}}{5n^d 2^{4nd+4} \log^d N_1} > 1.$$

This will be needed later to use the Mass Distribution Principle.

To obtain the second level K_2 precisely the same construction is made but instead of I^{nd} the cubes from the previous level K_1 are used. Choose

N_2 large enough so that

$$(6.6) \quad \frac{N_2^{n/d}}{16n \log N_2} > N_1 e^{N_1^{1/\sigma}},$$

$$(6.7) \quad \frac{3N_2^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2} \log^d N_2} > \frac{e^{ndN_1^{1/\sigma}} N_1^{nd}}{\#K_1}$$

and $\varepsilon > 0$ such that

$$(6.8) \quad \varepsilon \leq 2^{-1} e^{-2N_1^{1/\sigma}} N_1^{-nd}.$$

Divide I^{nd} into $T(N_2)$ hypercubes of sidelength $L_{N_2} = [N_2^{n/d}/8n \log N_2]^{-1}$ as before. Since $N_2 > N_1$ from (6.2)

$$\#G(N_2) \geq \left[\frac{N_2^{n/d}}{8n \log N_2} \right]^{nd} (1 - \varepsilon) > \frac{N_2^{n^2}}{(16n)^{nd} \log^{nd} N_2} (1 - \varepsilon).$$

Let $T_{E_1}(N_2)$ denote the number of hypercubes contained in E_1 and $\#G_{E_1}(N_2)$ be the number of those which have large intersection with a resonant set $R_{\mathbf{q}} \in R_{N_2}$. It is readily verified that

$$\left(\frac{e^{-N_1^{1/\sigma}}}{N_1 L_{N_2}} - 1 \right)^{nd} \leq T_{E_1}(N_2) \leq \left(\frac{e^{-N_1^{1/\sigma}}}{N_1 L_{N_2}} + 2 \right)^{nd},$$

and (6.6) implies that

$$\left(\frac{e^{-N_1^{1/\sigma}}}{2N_1 L_{N_2}} \right)^{nd} \leq T_{E_1}(N_2) \leq \left(\frac{2e^{-N_1^{1/\sigma}}}{N_1 L_{N_2}} \right)^{nd}.$$

Equations (6.8) and (6.5) can be used to show that $\#G(N_2) \geq T(N_2) - T_{E_1}(N_2)/4$ whence

$$\#G_{E_1}(N_2) > \frac{3T_{E_1}(N_2)}{4} > \frac{3e^{-ndN_1^{1/\sigma}}}{4 \cdot 2^{nd} (16n)^{nd} N_1^{nd} N_2^{-d^2} \log^{nd} N_2}.$$

Do exactly the same as before with the segments $I(C)$ to construct the second level of the Cantor set K_2 . Let $\#H_2$ be the number of hypercubes of K_2 in one E_1 so that $\#K_2 = \#K_1 \#H_2$. Then

$$\#H_2 \geq \#G_{E_1}(N_2) (L_{N_2} N_2 e^{N_2^{1/\sigma}} / 40)^{(n-1)d}$$

and

$$\#H_2 \leq \#G_{E_1}(N_2) \left(L_{N_2} N_2 e^{N_2^{1/\sigma}} \right)^{(n-1)d}.$$

Thus

$$\#K_2 \geq \#K_1 \frac{3N_2^{nd-d+n} e^{(n-1)dN_2^{1/\sigma}}}{4 \cdot 2^{2nd} (8n)^n \cdot 40^{(n-1)d} \log^d N_2} \frac{e^{-ndN_1^{1/\sigma}}}{N_1^{nd}}$$

and, from (6.7)

$$\#K_2 f(L(E_2)) = \frac{3N_2^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2} \log^d N_2} \frac{e^{-ndN_1^{1/\sigma}}}{N_1^{nd}} \#K_1 > 1.$$

Similarly for the r 'th level. Assume that K_{r-1} has been constructed with hypercubes E_{r-1} of sidelength $e^{-N_{r-1}^{1/\sigma}}/N_{r-1}$. Choose N_r sufficiently large so that

$$(6.9) \quad \frac{N_r^{n/d}}{16n \log N_r} > N_r e^{N_{r-1}^{1/\sigma}},$$

$$(6.10) \quad \frac{3N_r^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2} \log^d N_r} > \frac{e^{ndN_{r-1}^{1/\sigma}} N_{r-1}^{nd}}{\#K_{r-1}}$$

and $\varepsilon > 0$ so that

$$(6.11) \quad \varepsilon \leq 2^{-1} e^{-2N_{r-1}^{1/\sigma}} N_{r-1}^{-nd}.$$

Let $T_{E_{r-1}}(N_r)$ denote the number of hypercubes of the r 'th level contained in E_{r-1} (a cube in K_{r-1}) and $\#G_{E_{r-1}}(N_r)$ the number which have large intersection with a resonant set $R_{\mathbf{q}} \in R_{N_{r-1}}$. Following the argument exactly as for the second level but using equations (6.9) and (6.11) rather than equations (6.6) and (6.8) we get

$$(6.12) \quad \#K_r \geq \#K_{r-1} \frac{3N_r^{nd-d+n} e^{(n-1)dN_r^{1/\sigma}}}{4 \cdot 2^{2nd} (8n)^d 40^{(n-1)d} \log^d N_r} \frac{e^{-ndN_{r-1}^{1/\sigma}}}{N_{r-1}^{nd}},$$

and

$$(6.13) \quad \#K_r f(L(E_r)) = \frac{3N_r^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2} \log^d N_r} \frac{e^{-ndN_{r-1}^{1/\sigma}}}{N_{r-1}^{nd}} \#K_{r-1} > 1,$$

from (6.10).

Let $\mathbb{K} = \bigcap_{r=1}^{\infty} K_r$. It can be readily verified that $\mathbb{K} \subset E_0(\sigma; n, d)$. Now we proceed to construct a probability measure supported on \mathbb{K} . In order to do this the following lemma from [20, page 55] is needed.

Lemma 6.6 (The Mass distribution principle). *Let μ be a probability measure supported on a subset F of \mathbb{R}^k . Suppose there are positive constants λ and δ such that $\mu(C) \leq \lambda f(L(C))$, for any cube C with sidelength $L(C) \leq \delta$. Then $\mathcal{L}^s(F) \geq \lambda^{-1}$.*

Recall that E_i represents a cube in K_i , that is in the i 'th level of \mathbb{K} and define a probability measure on \mathbb{K} in the following way: $\mu(E_i) = (\#K_i)^{-1} < f(L(E_i))$ from (6.13), for $i \in \mathbb{N}$. Thus the sum of the measures of all hypercubes in the r 'th level in one fixed cube on the $r-1$ 'th level is $\mu(E_{r-1})$, i.e. $\mu(E_r) = \mu(E_{r-1})/(\#H_r)$. To use Lemma 6.6 we need to show that the measure of an arbitrary cube B is also $\ll f(L(B))$ where $L(B)$ is the sidelength of the cube. Without loss of generality we can choose an arbitrary cube B contained in some hypercube E_{r-1} and containing at least two cubes of the

r 'th level so that $L(B) \leq e^{-N_{r-1}^{1/\sigma}}/N_{r-1}$. (If B never contains two hypercubes of any level then its measure is zero.) There are two cases to consider. The first is when B intersects hypercubes lying on at least two different segments $I(C)$. In this case $L(B) \geq L_{N_r} = [N_r^{n/d}/8n \log N_r]^{-1}$. Let S be the number of segments $I(C)$ that B intersects. Then $S \leq (L(B)/L_{N_r})^{nd}$. The maximum number of hypercubes E_r of K_r that B can intersect on each segment is $(L_{N_r} N_r e^{(N_r^{1/\sigma})})^{(n-1)d}$. Evidently

$$\mu(B) \leq \frac{S \left(L_{N_r} N_r e^{N_r^{1/\sigma}} \right)^{(n-1)d}}{\#K_r} \leq \frac{L(B)^{nd} N_r^{nd-d+n} e^{(n-1)d N_r^{1/\sigma}}}{\log^d N_r \#K_r}.$$

From (6.12)

$$\mu(B) \ll \frac{L(B)^{nd} e^{nd N_r^{1/\sigma}} N_r^{nd}}{\#K_{r-1}}.$$

However, $\#K_{r-1} \geq 1/f(L(E_{r-1}))$ which implies that

$$\mu(B) \ll L(B)^{nd} e^{nd N_r^{1/\sigma}} N_r^{nd} f(L(E_{r-1})).$$

As the function $f(r)/r^{nd}$ decreases for $r < 1$ and since $L(B) \leq e^{-N_{r-1}^{1/\sigma}}/N_{r-1} < 1$ we obtain $\mu(B) \leq f(L(B))$.

The second case is when B intersects cubes in K_r from only one segment $I(C)$. As before $L(B) \leq e^{-N_{r-1}^{1/\sigma}}/N_{r-1}$, and $L(B) \geq L(E_r) = e^{-N_r^{1/\sigma}}/N_r$. The number of cubes that B can intersect is $\ll (L(B)/L(E_r))^{(n-1)d}$. Therefore

$$\mu(B) \ll \frac{L(B)^{(n-1)d}}{L(E_r)^{(n-1)d} \#K_r} \leq \frac{L(B)^{(n-1)d} f(L(E_r))}{L(E_r)^{(n-1)d}}.$$

As $f(r)/r^{(n-1)d} = 1/\log^s(1/r)$ is an increasing function and $L(B) \geq L(E_r)$ it is easy to verify that $\mu(B) \leq f(L(B))$.

Thus from Lemma 6.6 (the Mass Distribution Principle) for $s < n\sigma$ we have $\mathfrak{L}^s(\mathbb{K}) \geq 1$ which implies that $\text{Ldim}(E_0(\sigma; n, d)) \geq n\sigma$ and completes the proof of Theorem 6.2. \square

Proof of Theorem 6.3. Only the lower bound is left to prove. Let $I_\delta^m = \{\mathbf{x} \in I^n : |\mathbf{x}| > \delta\}$ and let $\|\cdot\|$ denote the distance from the nearest integer, i.e. $\|\mathbf{x}\| = \min_{\mathbf{k} \in \mathbb{Z}^2} |\mathbf{x} - \mathbf{k}|$. The following lemma is needed.

Lemma 6.7. *For every $N > 2$ and almost all $\mathbf{x} \in I_\delta^n$ there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $N/\log^4 N \leq |\mathbf{q}| \leq N$ such that $\text{dist}(\mathbf{x}, R_{\beta, \mathbf{q}}) < N^{-2} \log^7 N$.*

Proof. From [33], pp. 74-75 it can be verified that for every $N > 2$ and almost all $\mathbf{x} \in I^n$ there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $N/\log^4 N \leq |\mathbf{q}| \leq N$ such that

$$(6.14) \quad \|\mathbf{q} \cdot \mathbf{x} - \beta\| \leq N^{-n} \log^3 N$$

The following argument is done in detail for $n = 2$ and any decreasing function $\phi(N)$ and is not difficult to extend to $n \geq 3$. Consider the line $x_2 = 1$. Then $|\mathbf{q} \cdot \mathbf{x} + \beta| < \phi(N)$ implies that $|q_1 x_1 + q_2 + \beta| < \phi(N)$. Thus we can consider the inequality $||q_1 x + \beta|| < \phi(N)$. Similarly consider the line $x_2 = \alpha$ for any $1 \geq \alpha > 0$. Then the inequality $|q_1 x_1 + q_2 \alpha + \beta| < \phi(N)$ is the same as the inequality $||qx + \beta'|| < \alpha^{-1} \phi(N)$ where $x = x_1/\alpha$ and $\beta' = \beta/\alpha$. Thus, if there exists a function ϕ for which almost all points on the lines $x_2 = \alpha$ are covered by intervals centered at points $(p + \beta)/q$ of length $\alpha^{-1} \phi(N)/q$ with $N/\log^4 N < q < N$ then for some fixed $\delta > 0$ with $\alpha > \delta$ almost all points in I_δ^2 will be covered by the set

$$\bigcup_{\mathbf{q}: N > |\mathbf{q}| > N/\log^4 N} \{\mathbf{x} : |\mathbf{q} \cdot \mathbf{x} + \beta| < \phi(|\mathbf{q}|)\}$$

for N sufficiently large. On letting $\phi(N) = N^{-1} \log^3 N$ and using (6.14) and the fact that $N > |\mathbf{q}| > N/\log^4 N$ the lemma is proved. Exactly the same argument can be used for $n \geq 3$. \square

Using the ubiquity argument from [33] it can be shown that as in the homogeneous case with $L_N = [N^2/8n \log^7 N]^{-1}$ we have

$$\#G(N) \geq \left[\frac{N^2}{8m \log^7 N} \right]^n (1 - \varepsilon).$$

Theorem 6.3 can now be proved by using the above and following the proof of Theorem 6.2 exactly. \square

Theorems 6.2 and 6.3 show that the sets $E_0(\sigma; m, n)$ and $E_\beta(\sigma; m, 1)$ consist of more than just the resonant hyperplanes $R_{\mathbf{q}, \beta}$.

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REFERENCES

- [1] V.I. Arnol'd, *Geometrical methods in the theory of ordinary differential equations*, Springer, New York - Heidelberg - Berlin, 1983. MR **84d**:58023
- [2] D. Bambusi and N.N. Nekhoroshev, *A property of exponential stability in nonlinear wave equations near the fundamental linear mode*, *Physica D*, **122** (1998), 73–104. MR **99h**:35137
- [3] A. Bergamasco, *Perturbations of globally hypoelliptic operators*, *J. Differential equations*, **141**:2 (1994), 513–526. MR **95j**:35046

- [4] A. Bergamasco, *Remarks about global analytic hypoellipticity*, Trans. Amer. Math. Soc., **351** (1999), 4113–4126. MR **99m**:35032
- [5] A. Bergamasco, P. Cordaro and P. Malagutti, *Globally hypoelliptic systems of vector fields*, J. Funct. Analysis, **114**:2 (1993), 263–285. MR **94e**:35048
- [6] F. Bonetto, G. Gallavotti, G. Gentile, and V. Mastropietro, *Lindstedt series, ultraviolet divergences and Moser’s theorem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 545–593. MR **99i**:58132
- [7] J.D. Bovey and M.M. Dodson, *The Hausdorff dimension of systems of linear forms*, Acta Arith. **XLV** (1986), 337–358. MR **87h**:11074
- [8] H.W. Broer, G.B. Huitema, and M.B. Sevryuk, *Quasi-periodic motions in families of dynamical systems. Order amidst chaos*, Lect. Notes in Mathematics, **1645** Springer-Verlag, Berlin, 1996. xii+196 pp. MR **99d**:58142
- [9] A.D. Bruno, *The analytic form of differential equations*, Tr. Mosk. Mat. O-va, **25** (1971), 119–262 and **26** (1972), 199–239 (in Russian). MR **51**#13365
- [10] A.D. Bruno and S. Walcher, *Symmetries and convergence of normalizing transformations*, J. Math. Anal. Appl. **183**, (1994), 571–576. MR **95i**:58160
- [11] Chen Wenyi and M.Y. Chi, *Hypoelliptic vector fields and almost periodic motions on the torus T^n* , Comm. Partial Differential Equations, **25** (2000), 337–354. MR **2000m**:35042
- [12] R. De La Llave, *A tutorial on KAM theory*, to appear in Proc. of 1999 Summer Research Institute on Smooth Ergodic Theory and Applications, July 26–August 13, 1999, Univ. of Washington, Seattle.
- [13] D. DeLatté, *Diophantine conditions for the linearization of commuting holomorphic functions*, Discr. & Cont. Dynam. Sys. **3**, (1997), 317–332. MR **98c**:30035
- [14] H. Dickinson, *The Hausdorff dimension of systems of simultaneously small linear forms*, Mathematika, **40** (1993), 367–374. MR **94m**:11093
- [15] D. Dickinson, T. Gramchev and M. Yoshino, *First order pseudodifferential operators on the torus: normal forms, Diophantine phenomena and global hypoellipticity*, Proc. of the Conf. “Differential Equations”, Ferrara, June 1996, Ann. Univ. Ferrara Sez. VII (N.S.) **41**, suppl. (1997), 51–64. MR **98k**:58221
- [16] D. Dickinson and S. L. Velani, *Hausdorff measure and linear forms*, J. reine angew. Math., **490** (1997), 1–36. MR **98e**:11094
- [17] M.M. Dodson, *Hausdorff dimension, lower order and Khintchine’s theorem in metric Diophantine approximation*, J. reine angew. Math., **432** (1992), 69–76. MR **94a**:11125
- [18] M.M. Dodson and J.A.G. Vickers, *Exceptional sets in Kolmogorov–Arnol’d–Moser theory*, J. Phys. A, **19** (1986), 349–374. MR **87m**:58054
- [19] L. H. Eliasson, *Absolutely convergent series expansions for quasi periodic motions*, Math. Phys. Electron. J. **2** (1996), Paper 4, 33 pp. (electronic). MR **97f**:58114
- [20] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & sons, 1990. MR **92j**:28008
- [21] F. Fass, M. Guzzo, and G. Benettin, *Nekhoroshev-stability of elliptic equilibria of Hamiltonian systems*, Comm. Math. Phys., **197** (1998), 347–360. MR **99j**:58117
- [22] T. Gramchev, P. Popivanov and M. Yoshino, *Global properties in spaces of generalized functions on the torus for second order differential operators with variable coefficients*, Rend. Sem. Mat. Univ. Pol. Torino, **51**:2 (1993), 144–174. MR **95k**:35047
- [23] T. Gramchev and G. Popov, *Nekhoroshev type estimates for billiard ball maps*, Ann. Inst. Fourier, **45**:3 (1995), 859–895. MR **97a**:58145
- [24] T. Gramchev and M. Yoshino, *Rapidly convergent iterative method for simultaneous normal forms of commuting maps*, Math. Z., **231** (1999), 745–770. MR **2001b**:37072
- [25] A. Giorgilli, A. Delshams, E. Fontich, L. Galgani, and C. Simó, *Effective stability for a Hamiltonian system near an elliptic equilibrium point, with an application to the restricted three body problem*, J. Differential Equations, **77** (1989), 167–198. MR **90c**:70026

- [26] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. IHES, **49** (1979), 5–233. MR **81h**:58039
- [27] M. Herman, *Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of \mathbb{C}^n near a fixed point*, Proc. of the VIIIth International Congress on Mathematical Physics, Marseille, 1986, eds. M. Mebkhout and R. S en eor, World Scientific Publ., Singapore, 1987, 138–184. MR **89k**:32013
- [28] A. Himonas, and G. Petronilho, *Global hypoellipticity and simultaneous approximability*, J. Funct. Anal., **170** (2000), 356–365. MR **2000m**:35043
- [29] J. Hounie, *Globally hypoelliptic vector fields on compact surfaces*, Comm. Partial Differential Equations, **7** (1982), 343–370. MR **83k**:35025
- [30] A. Jorba, R. Ramirez-Ros, and J. Villanueva, *Effective reducibility of quasi-periodic linear equations close to constant coefficients*, SIAM J. Math. Anal. **28** (1997), 178–188. MR **97j**:34008
- [31] A. Jorba and J. Villanueva, *On the normal behaviour of partially elliptic lower-dimensional tori of Hamiltonian systems*, Nonlinearity **10** (1997), 783–822. MR **98h**:58169
- [32] Y. Katznelson, B. Kra and D. Ornstein, *A global version of a theorem of J. Moser*, to appear in *Israel J. Math.*
- [33] J. Levesley, *A general inhomogeneous Jarn ik–Besicovitch theorem*, J. Number Theory, **71** (1998), 65–80. MR **99g**:11091
- [34] P. Loshak, *Canonical perturbation theory: an approach based on joint approximations*, Russian Math. Surveys **47**:6 (1992), 57–133. MR **94f**:58110
- [35] S. MacLane and G. Birkhoff, *Algebra*, MacMillan, New York 1979. MR **80d**:00002
- [36] J. Moser, *On commuting circle mappings and simultaneous Diophantine approximations*, Math. Z., **205** (1990), 105–121. MR **92e**:58120
- [37] J. P oschel, *Integrability of Hamiltonian systems on Cantor sets*, Comm. Pure Appl. Math., **35** (1982), 653–696. MR **84d**:58039
- [38] L. Rodino, *Linear partial differential operators in Gevrey spaces*, World Scientific, Singapore-New Jersey-London-Hong Kong 1993. MR **95c**:35001
- [39] H. R ussmann, *On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus*, Lect. Notes in Physics, **38**, Springer Verlag, Berlin - Heidelberg - New York (1975), 598–624. MR **57**#7675
- [40] W. Schmidt, *Diophantine approximation*, Lect. Notes in Mathematics, **785** Springer Verlag, Berlin - Heidelberg - New York 1980. MR **81j**:10038
- [41] L. Stolovitch, *Singular complete integrability*, Laboratoire de Math ematiques E. Picard, Universit e P. Sabatier, Toulouse III, Pr e-publication n o 142, Janvier 1999, 67 pp. (to appear in Publ. Math. I.H.E.S.)
- [42] J.-C. Yoccoz, *Recent developments in dynamics*, Proc. of the International Congress of Mathematicians, Z urich, Switzerland, August 1994, Birkh user, (1995), 246–265. MR **98e**:58146
- [43] M. Yoshino, *A class of globally hypoelliptic operators on the torus*, Math. Z. , **201** (1989), 1–11. MR **90m**:35046

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