BORELSUMMABILITY OF SOME SEMILINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we are interested in the Borel summability of formal solutions with a parameter of first order semilinear system of partial differential equations with n independent variables. In [2], Balser and Kostov proved the Borel summability of formal solutions with respect to a singular perturbation parameter for a linear equation with one independent variable. We will extend their results to a semilinear system of equations with general independent variables.

1. Introduction

In this paper we shall study the Borel summability of formal solutions of partial differential equations with a parameter. Since the pioneering results obtained by Lutz-Miyake-Schäfke, Balser et al the Borel summability of formal solutions of heat operators has been studied extensively. (cf. [9], [3]. See also the recent papers by Michalik, [11] or/and Ichinobe, [6]). Another class of Borel summable operators which are perturbations of an ordinary differential equation were studied by Ouchi. (cf. [12]). We also refer to the extension by Tahara and Yamazawa. (cf. [13]). As for the first order single equation, we refer to Hibino, [5]. Concerning the summability of solutions of a partial differential equation with a singular perturbation parameter we cite the papers [2] and [4]. (cf. [10], [8], [2] and [7]).

In this paper we shall extend the results in [2] to a semilinear system of partial differential equations with general independent variables. We note that our system is not contained in the class of equations studied in the above, nor can be decomposed into first order single equations. We use the method of characteristics in order to prove our theorem which is different from that of [2]. We note that our method also yields the summability when the independent variable moves in a given bounded open set.

This paper is organized as follows. In Section 2, we state the main theorem and give some remarks on the theorem. In Section 3, we study formal solutions and Gevrey estimate. In Section 4, we prepare an elementary lemma needed for the proof of the main theorem. In Section 5, we give the proof. In Section 6, we give an extension of the main theorem when the independent variable moves in some open set not containing the origin.

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2. Statement of results

Let $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $n \geq 1$ be the variable in $\mathbb{C}^n$. For $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$ ($j = 1, 2, \dots, n$), define

$$\mathcal{L} := \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \tag{2.1}$$

Let $N \geq 1$ be an integer and let $f(x, u) = (f_1(x, u), \dots, f_N(x, u))$, $u = (u_1, \dots, u_N) \in \mathbb{C}^N$ be the holomorphic vector function in some neighborhood of the origin of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$. We consider Borel summability of formal solutions of the semilinear system of equations

$$\eta^{-1}\mathcal{L}u = f(x, u), \tag{2.2}$$

where $\eta \in \mathbb{C} \setminus \{0\}$ is a complex parameter. We assume

$$f(0, 0) = 0, \qquad \det(\nabla_u f(0, 0)) \neq 0 \tag{2.3}$$

where $\nabla_u f(0, 0)$ denotes the Jacobi matrix of $f(x, u)$ with respect to u at the point $x = 0, u = 0$.

We will construct a formal power series solution $v(x, \eta)$ of (2.2) in the form

$$v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots , \tag{2.4}$$

where the series is a formal power series in η^{-1} with coefficient $v_\nu(x)$ being a holomorphic vector function of x in some open set independent of ν. We denote by Ω_0 the open connected set containing the origin on which every coefficient $v_\nu(x)$ is defined.

The formal Borel transform of $v(x, \eta)$ is defined by

$$B(v)(x, \zeta) := \sum_{\nu=0}^{\infty} v_\nu(x) \frac{\zeta^\nu}{\Gamma(\nu + 1)}, \tag{2.5}$$

where $\Gamma(z)$ is the Gamma function. For an opening $\theta > 0$ and the bisecting direction ξ, define the sector $S_{\theta,\xi}$ by

$$S_{\theta,\xi} = \left\{ z \in \mathbb{C};\ |\arg z - \xi| < \frac{\theta}{2} \right\}. \tag{2.6}$$

We say that $v(x, \eta)$ is 1- Borel summable in the direction ξ with respect to η if $B(v)(x, \zeta)$ converges in some neighborhood of the origin of (x, ζ), and there exist a neighborhood U of the origin $x = 0$ and a $\theta > 0$ such that $B(v)(x, \zeta)$ can be analytically continued to $(x, \zeta) \in U \times S_{\theta,\xi}$ and of exponential growth of order 1 with respect to ζ in $S_{\theta,\xi}$. For the sake of simplicity we denote the analytic continuation with the same notation $B(v)(x, \zeta)$. The Borel sum $V(x, \eta)$ of $v(x, \eta)$ is, then, given by the Laplace transform

$$V(x, \eta) := \int_{L_\xi} \zeta^{-1} e^{-\zeta\eta} B(v)(x, \zeta) d\zeta \tag{2.7}$$

where the integral is taken on the ray starting from the origin to the infinity in the direction ξ. We assume that $\nabla f(0,0)$ is a diagonal matrix,

$$\nabla f(0,0) = \mathrm{diag}\,(\mu_1, \ldots, \mu_N). \tag{2.8}$$

Moreover, we assume

$$\mathrm{Re}\,\lambda_j > 0,\ \mathrm{Re}\,\mu_k > 0, \quad \mathrm{Re}\,\frac{\lambda_j}{\mu_k} > 0\ (j = 1, \ldots, n,\ k = 1, \ldots, N). \tag{2.9}$$

Let C_0 be the convex closed positive cone with vertex at the origin containing λ_j $(j = 1, 2, \ldots, n)$ and λ_j/μ_k $(j = 1, 2, \ldots, n; k = 1, \ldots, N)$. Write

$$C_0 = \{z \in \mathbb{C}; -\theta_1 \le \arg z \le \theta_2\} \tag{2.10}$$

for some $0 \le \theta_1 < \pi/2$ and $0 \le \theta_2 < \pi/2$. Note that $S_{\pi+\theta,\xi}$ is equal to $\mathbb{C} \setminus C_0$ with $\xi = \pi + \frac{\theta_2-\theta_1}{2}$ and $\theta = \pi - \theta_1 - \theta_2$. We have

Theorem 1. *Suppose (2.3), (2.8) and (2.9). Then $v(x, \eta)$ is 1- Borel summable in the direction $\eta \in S_{\theta,\xi}$ with $\xi = \pi + \frac{\theta_2-\theta_1}{2}$ and $\theta = \pi - \theta_1 - \theta_2$ when x is in some neighborhood of the origin. Moreover, there exists a neighborhood U of $x = 0$ such that $V(x, \eta)$ is holomorphic and satisfies* (2.2) *when $(x, \eta) \in U \times S_{\pi+\theta,\xi}$.*

Example. Let $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$. Consider a holomorphic singular vector field $\mathcal{X} = \sum_{j=1}^{n} a_j(y)\frac{\partial}{\partial y_j}$ with $a_j(0) = 0$ for $j = 1, \ldots, n$. Assume that the change of coordinates $y = u(x)$, $u(0) = 0$ transforms $\mathcal{X}$ to its linear part $x\Lambda$, where Λ is some constant matrix. Moreover, suppose that Λ is a diagonal matrix with diagonal entries λ_j $(j = 1, 2, \ldots, n)$. The linearizability is equivalent to $A(u(x))\left(\frac{\partial u}{\partial x}\right)^{-1} = x\Lambda$, where $\frac{\partial u}{\partial x}$ denotes the Jacobi matrix of u. We consider the approximate equation by introducing a parameter η^{-1} in front of the right-hand side, namely $A(u(x))\left(\frac{\partial u}{\partial x}\right)^{-1} = \eta^{-1}x\Lambda$. If we recall $\mathcal{L} = x\Lambda\frac{\partial u}{\partial x}$, then we obtain (2.2). Therefore, by applying Theorem 1 we have the summability of the formal solution (2.4).

Remark 1. *(a) In* [2] *the summability of the formal solution (2.4) was shown for (2.2) with $N = 1$ and $n = 1$ assuming that f is a polynomial of degree 1 with respect to u. In fact, in Theorem 2 of* [2] *the summability was proved under the condition equivalent to (2.9). It was also shown that (2.9) is necessary in general.*

(b) An interesting phenomenon shown in [2] *is that a certain Diophantine phenomenon appears in the summability, while it does not appear for an irregular singular equation. (cf.*[4]*). In the case of general independent variables one can easily see that a similar multi-dimensional Diophantine condition enters in the analysis. Because we do not know how to generalize the proof in* [2] *to a semilinear multi-dimensional case, we use the method of characteristics in order to prove the summability. More precisely, the stable behavior of the characteristics in our proof corresponds to the Diophantine type condition in* [2]. *We note that our method also shows the summability in the case when the independent variable is outside the origin without assuming (2.9). We will briefly mention the extension in the last section.*

3. Formal power series in the perturbation parameter

In this section we construct the formal solution of (2.2) and obtain some estimate of the formal series.

Construction of formal solution. By substituting the expansion (2.4) into (2.2), we obtain

$$\mathcal{L}v_j = \sum_{\nu=0}^{\infty} \mathcal{L}v_\nu^j(x)\eta^{-\nu}, \tag{3.1}$$

$$\begin{aligned} f(x,v) &= f(x, v_0 + v_1\eta^{-1} + v_2\eta^{-2} + \cdots) \\ &= f(x, v_0) + \eta^{-1}(\nabla_u f)(x, v_0)v_1 + O(\eta^{-2}). \end{aligned} \tag{3.2}$$

By comparing the coefficients of η, we obtain for $\eta^0 = 1$

$$f(x, v_0(x)) = 0 \tag{3.3}$$

and for η^{-1}

$$\mathcal{L}v_0 = (\nabla_u f)(x, v_0)v_1. \tag{3.4}$$

In order to determine $v_\nu(x)$ ($\nu \geq 2$) we compare the coefficients of $\eta^{-\nu}$ of (2.2). Differentiate (3.2) $(\nu-1)$-times with respect to $\varepsilon = \eta^{-1}$ and put $\varepsilon = 0$. Then we obtain

$$\mathcal{L}v_{\nu-1} = (\nabla_u f)(x, v_0)v_\nu + (\text{terms consisting of } v_k^j,\ j = 1, \dots, n,\ k < \nu). \tag{3.5}$$

First, note that there exists an analytic solution $v_0(x)$, $v_0(0) = 0$ of (3.3) in some domain containing the origin $x = 0$ by (2.3). We define

$$\Sigma_0 := \{x;\ \det\left((\nabla_u f)(x, v_0(x))\right) = 0,\ f(x, v_0(x)) = 0\}. \tag{3.6}$$

The next theorem gives the existence of a formal solution.

Proposition 1. *Assume (2.3). Then every coefficient of (2.4) is uniquely determined as a holomorphic function in some neighborhood of $x = 0$ independent of ν.*

Proof. By (2.3) and the implicit function theorem, $v_0(x)$ is uniquely determined as the holomorhic function at the origin such that $v_0(x) = O(|x|)$. Suppose that $v_k(x)$ is determined up to some $\ell - 1$ in some neighborhood of the origin. Then, by the implicit function theorem one can determine $v_\ell(x)$ uniquely in some neighborhood of the origin depending on ℓ. Because $v_k(x)$ are determined recursively by differentiations and algebraic calculations, the recurrence formula for $v_\ell(x)$ implies that $v_\ell(x)$ is holomorphic in some neighborhood of the origin independent of ν. □

Remark 2. *Let $\Omega_0 \subset \mathbb{C}^n$ be the domain containing the origin on which every coefficient of $v(x,\eta)$ is defined. Let $\widetilde{\Omega_0 \setminus \Sigma_0}$ be the universal covering space of $\Omega_0 \setminus \Sigma_0$. Then every coefficient of $v(x,\eta)$ is analytically continued from the origin to $\widetilde{\Omega_0 \setminus \Sigma_0}$, provided that $f(x,u)$ is an entire function of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$.*

Gevrey estimate of order 1. We will show the convergence of the formal Borel transform of (2.4).

Theorem 2. *Assume that $f(x,u)$ is an entire function of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$. Let v in (2.4) be analytically continued as in Remark 2. Let K be the compact set in $\widetilde{\Omega_0 \setminus \Sigma_0}$. Suppose that every $v_\nu(x)$ in v be analytic in some neighborhood of K independent of ν. Then there exist a neighborhood U of K and a neighborhood W of the origin $\zeta = 0$ in $\mathbb{C}$ such that the formal Borel transform $B(v)(x,\zeta)$ converges in $U \times W$.*

Remark 3. *By the same argument as in the proof of Theorem 2 we have the formal Borel summability when K is a neighborhood of the origin $x = 0$. In fact, we only need to assume that $f(x,u)$ is analytic in some neighborhood of the origin $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$. Note that every $v_\nu(x)$ in v is analytic in some neighborhood of the origin independent of ν.*

Proof of Theorem 2. The compact set K can be covered by a finite number open balls. Hence it is sufficient to show our theorem when K is a subset of an open small ball. One may also assume that the center of the ball is the origin. We use the majorant relation $u \ll v$ when v is the majorant function of u. Namely, for $u = \sum_\alpha x^\alpha u_\alpha$ and $v = \sum_\alpha x^\alpha v_\alpha$ the relation $u \ll v$ holds if $|u_\alpha| \le v_\alpha$ for every α. If u and v are vector functions, then $u \ll v$ means that for every j, the j-th component u_j of u and v_j of v satisfy $u_j \ll v_j$. If v is a scalar function, then $u \ll v$ means that $u_j \ll v_j$ for every j. For $\rho > 0$, define

$$\phi_\rho(x) := \left(1 - \frac{x_1 + \cdots + x_n}{\rho}\right)^{-1}. \tag{3.7}$$

The set of holomorphic functions at the origin such that $u \ll \phi_\rho C$ for some $C \ge 0$ forms a Banach space with the norm $\|u\|$ given by the infimum of C satisfying $u \ll \phi_\rho C$.

First we will estimate the differentiation. For any integers $1 \le j \le n$ and $k \ge 1$ we have

$$\frac{\partial}{\partial x_j}\phi_\rho(x)^k = \frac{k}{\rho}\phi_\rho(x)^{k+1}. \tag{3.8}$$

On the other hand, because $x_j(\nabla_u f)(x,v_0)^{-1}$ is analytic at the origin for $1 \le j \le n$ we have, for sufficiently small $\rho > 0$

$$x_j(\nabla_u f)(x,v_0)^{-1} \ll K\phi_\rho \tag{3.9}$$

for some $K > 0$. Similarly, we have $v_0 \ll \|v_0\|\phi_\rho$.

We next estimate v_1. By virtue of (3.4) we have $v_1 = (\nabla_u f)(x,v_0)^{-1}\mathcal{L}v_0$. Hence, by (3.8) and (3.9) we have $v_1 \ll \|v_0\|C_0\phi_\rho^3$ for some $C_0 > 0$. We will show that there exists $C \ge 1$ independent of $\nu \ge 1$ such that

$$v_m \ll C^{2m-1}m!\phi_\rho^{4m-1}, \quad m = 1, 2, \ldots \tag{3.10}$$

Suppose that (3.10) holds up to $m \le \nu - 1$ and consider v_ν. In view of (3.5) we first consider $(\nabla_u f)(x,v_0)^{-1}\mathcal{L}v_{\nu-1}$.

$$(\nabla_u f)(x,v_0)^{-1}\mathcal{L}v_{\nu-1} \ll C^{2\nu-3}(\nu-1)!(4\nu-5)\phi_\rho^{4\nu-3}C_1 \le 4C_1C^{2\nu-3}\nu!\phi_\rho^{4\nu-3} \tag{3.11}$$

for some $C_1 > 0$ depending only on K and $\mathcal{L}$. Hence if $4C_1 \le C$ and $C > 1$, then we have the estimate like (3.10) since $1 \ll \phi_\rho$.

Next we estimate the nonlinear term. Set $v = v_0 + u$, $u = \eta^{-1}v_1 + \eta^{-2}v_2 + \cdots$ and expand

$$f(x,v) = f(x,v_0) + \nabla_u f(x,v_0)\cdot u + \sum_{|\beta|\ge 2} r_\beta(x,v_0)u^\beta. \tag{3.12}$$

By inserting the expansion of u and by comparing the coefficients of $\eta^{-\nu}$ of the right-hand side of (3.12) we see that the nonlinear term in (3.5) is given by

$$\sum_{|\beta|\ge 2}\sum_{\ell=2}^{|\beta|}\sum_{\nu_1+\cdots+\nu_\ell=\nu,\nu_j\ge 1} r_\beta(x,v_0)v_{\nu_1}\cdots v_{\nu_\ell}. \tag{3.13}$$

By inductive assumption on v_m we have

$$\sum_{\ell=2}^{|\beta|}\sum_{\nu_1+\cdots+\nu_\ell=\nu,\nu_j\ge 1,\ell\ge 2} v_{\nu_1}\cdots v_{\nu_\ell} \ll \sum_{\ell=2}^{|\beta|}\sum \nu_1!\cdots\nu_\ell! C^{2\nu-\ell}\phi_\rho^{4\nu-\ell}. \tag{3.14}$$

We recall the inequality

$$\sum_{\nu_1+\cdots+\nu_\ell=\nu,\nu_j\ge 1,\ell\ge 2} \frac{\nu_1!\cdots\nu_\ell!}{\nu!} \le 1. \tag{3.15}$$

Then the right-hand side of (3.14) is bounded by

$$\ll C^{2\nu-2}\nu!\sum_{\ell=2}^{|\beta|} C^{2-\ell}\phi_\rho^{4\nu-2} \ll C^{2\nu-2}C_2\nu!\phi_\rho^{4\nu-2},$$

for some $C_2 > 0$ independent of ν because $\sum_{\ell=2}^{\infty} C^{2-\ell} < \infty$ by $C > 1$.

In order to estimate $(\nabla_u f)(x,v_0)^{-1}$ times (3.13) we consider

$$(\nabla_u f)(x,v_0)^{-1}\sum_{|\beta|\ge 2} r_\beta(x,v_0). \tag{3.16}$$

By virtue of (3.12) we have

$$\sum_{|\beta|\ge 2} r_\beta(x,v_0) = f(x,v_0+e) - f(x,v_0) - \nabla_u f(x,v_0)\cdot e, \tag{3.17}$$

where $e = (1,\dots,1)$. By using the scale change of variables $u \mapsto \varepsilon u$, $\varepsilon > 0$, one may assume that $f(x, v_0 + e)$ is analytic at $x = 0$, if necessary. Therefore one can estimate (3.16) like $\ll K\phi_\rho$ for some $K > 0$.

Therefore $(\nabla_u f)(x,v_0)^{-1}$ times (3.13) can be estimated by $C^{2\nu-2}C_2K\nu!\phi_\rho^{4\nu-1}$. By inserting this estimate and (3.11) into (3.5) we obtain (3.10) for $m = \nu$. By (3.10) and the definition of majorant functions we obtain the convergence of formal Borel transform in some neighborhood of $x = 0$ and $\zeta = 0$. This ends the proof.

4. Convolution estimate

We estimate the convolution. Let Ω be the smallest open set containing the sector $S_{\theta,\pi}$ in (2.6) and the disk $\{|z| < r_0\}$ for small $r_0 > 0$ such that $z \in \Omega$ implies $z+t \in \Omega$ for every real number $t \le 0$. For $c > 0$, define the space $\mathcal{H}(\Omega)$ by

$$\mathcal{H}(\Omega) := \left\{ f \in H(\Omega) \mid \exists K \text{ such that } |f(z)| \le Ke^{-c\mathrm{Re}\, z}(1+|z|)^{-2}, \forall z \in \Omega \right\}, \tag{4.1}$$

where $H(\Omega)$ is the set of holomorphic functions in Ω. Obviously, $\mathcal{H}(\Omega)$ is the Banach space with the norm

$$\|f\|_\Omega := \sup_{z\in\Omega} |f(z)|(1+|z|)^2 e^{c\mathrm{Re}\, z}. \tag{4.2}$$

The convolution $f * g$ ($f, g \in \mathcal{H}(\Omega)$) is defined by

$$(f * g)(z) := \frac{d}{dz}\int_0^z f(z-t)g(t)dt = \frac{d}{dz}\int_0^z f(t)g(z-t)dt. \tag{4.3}$$

Write $f'(z) = (df/dz)(z)$. Then we have

Proposition 2. *For every $f, g \in \mathcal{H}(\Omega)$ such that $f(0) = g(0) = 0$ and $f', g' \in \mathcal{H}(\Omega)$ we have $f * g \in \mathcal{H}(\Omega)$ with the estimate*

$$\|f * g\|_\Omega \le 8\|f'\|_\Omega \|g\|_\Omega, \quad \|f * g\|_\Omega \le 8\|f\|_\Omega \|g'\|_\Omega. \tag{4.4}$$

Proof. Because $f * g = g * f$ we will prove the first inequality of (4.4). We have

$$(f * g)(z) = \frac{d}{dz}\int_0^z f(z-t)g(t)dt = f(0)g(z) + \int_0^z f'(z-t)g(t)dt = \int_0^z f'(z-t)g(t)dt.$$

By (4.2) and by taking the path of integration from 0 to z we have

$$\begin{aligned} \left|\int_0^z f'(z-t)g(t)dt\right| &\le \|f'\|_\Omega \|g\|_\Omega e^{-c\mathrm{Re}\, z} \int_0^z (1+|z-t|)^{-2}(1+|t|)^{-2}|dt| \\ &\le \|f'\|_\Omega \|g\|_\Omega e^{-c\mathrm{Re}\, z} \int_0^{|z|} (1+|z|-s)^{-2}(1+s)^{-2}ds. \end{aligned} \tag{4.5}$$

We divide the integral in the right-hand side into two parts, $s \le \frac{|z|}{2}$ and $s > \frac{|z|}{2}$. If $s \le \frac{|z|}{2}$, then we have $(1+|z|-s)^{-2} \le 4(1+|z|)^{-2}$, while in case $s > \frac{|z|}{2}$ we have $(1+s)^{-2} \le 4(1+|z|)^{-2}$. Hence we have

$$\int_0^{|z|/2} \frac{1}{(1+|z|-s)^2(1+s)^2} ds \le \frac{4}{(1+|z|)^2}\int_0^{|z|/2}(1+s)^{-2}ds \le \frac{4}{(1+|z|)^2}. \tag{4.6}$$

One can similarly estimate the other part like $\int_{|z|/2}^{|z|}(1+|z|-s)^{-2}(1+s)^{-2}ds \le 4(1+|z|)^{-2}$. Therefore we see that the left-hand side term of (4.5) can be estimated by $8\|f'\|_\Omega \|g\|_\Omega e^{-c\mathrm{Re}\, z}(1+|z|)^{-2}$. This ends the proof.

5. Proof of Theorem 1

Proof of Theorem 1. We first show the summability of $v(x,\eta)$ in the direction $\arg\eta = \pi$ when x is in some neighborhood of the origin. One may assume $\lambda_n = 1$ without loss of generality by dividing the equation with $\lambda_n \neq 0$. In terms of (2.2) with u replaced by $v_0 + u$, (3.12) and $f(x, v_0) = 0$ we obtain

$$\mathcal{L}u = -\mathcal{L}v_0 + \eta \nabla_u f(x, v_0)u + \eta \sum_{|\beta|\geq 2} r_\beta(x, v_0)u^\beta. \tag{5.1}$$

Let $\hat{u}(y) := \mathcal{B}(u)$ be the Borel transform of u with respect to η, where y is the dual variable of η. By the Borel transform of (5.1) we obtain

$$\mathcal{L}\hat{u} = -\mathcal{L}v_0 + \nabla_u f(x, v_0)\frac{\partial \hat{u}}{\partial y} + \frac{\partial}{\partial y}\sum_{|\beta|\geq 2} r_\beta(x, v_0)(\hat{u})^\beta_*, \tag{5.2}$$

where $(\hat{u})^\beta_* = (\hat{u}_1)^{\beta_1}_* \cdots (\hat{u}_N)^{\beta_N}_*$, $\beta = (\beta_1, \dots, \beta_N)$, and $(\hat{u}_j)^{\beta_j}_*$ is the β_j-convolution product, $(\hat{u}_j)^{\beta_j}_* = \hat{u}_j * \cdots * \hat{u}_j$.

Let v be the formal solution given by Proposition 1 and consider the formal Borel transform $B(v)$. Define $\hat{u}(x,y) := B(v) - v_0$. Then $\hat{u}(x,y)$ is analytic in some neighborhood of the origin, $x = 0$, $y = 0$, and $\hat{u}$ is the solution of (5.2) in some neighborhood of $y = 0$ such that $\hat{u}(x, 0) \equiv 0$ in x. We will show that every solution of (5.2) analytic at $y = 0$ and satisfying $\hat{u}(x, 0) \equiv 0$ is uniquely determined. Indeed, by definition the convolution product of $y^i/i!$ and $y^j/j!$ is equal to $y^{i+j}/(i+j)!$. Hence, if we expand $\hat{u}$ in the power series of y and insert (5.2), then every coefficient of the expansion can be uniquely determined from the recurrence relation because $\nabla_u f(x, v_0)$ is invertible. Therefore, if we can show the existence of the solution of (5.2) being analytic in (x,y) with x in some open set and $y \in \Omega$ which is of exponential growth with respect to y in Ω, then we have the analytic continuation of the formal Borel transform of v with exponential growth in $y \in \Omega$. Hence we have the summability of v.

We will prove the solvability of (5.2) when x is in some open set and $y \in \Omega$. We introduce the function space similar to $\mathcal{H}(\Omega)$ in (4.1). Let D be the open connected set in some neighborhood of the origin of $\mathbb{C}^n$. Then we define

(5.3)
$$\mathcal{H}(D, \Omega) := \left\{ f \in H(D, \Omega) \mid \exists K,\ \sup_{x\in D} |f(x,y)| \leq K e^{-c\mathrm{Re}\, y}(1 + |y|)^{-2}, \forall y \in \Omega \right\},$$

where $H(D, \Omega)$ is the holomorphic function in $(x,y) \in D \times \Omega$. The space $\mathcal{H}(D, \Omega)$ is a Banach space with the norm $\|f\| = \inf K$ where K is given in (5.3).

We consider the system of equations

$$\mathcal{L}w - (\nabla_u f)(x, 0)\frac{\partial w}{\partial y} = g, \tag{5.4}$$

where $g \in \mathcal{H}(D, \Omega)$ is a given vector function. First, assume that $(\nabla_u f)(x, 0)$ is a diagonal matrix and let $(\nabla_u f)_j(x, 0)$ be the j-th diagonal component of $(\nabla_u f)(x, 0)$.

We use the method of characteristics in order to solve (5.4). Namely, we consider

$$\frac{d\zeta}{\zeta} = \frac{dx_k}{\lambda_k x_k} = -\frac{dy}{(\nabla_u f)_j(x,0)}, \quad k = 1, 2, \dots, n-1. \tag{5.5}$$

By integration we have

$$x_k = c_k \zeta^{\lambda_k} \ (k = 1, 2, \dots, n-1), \ y = y_0 - \Phi(\zeta, b), \tag{5.6}$$

where c_k's and y_0 are some constants, and $\Phi(\zeta, b) = (\Phi_1(\zeta, b), \dots, \Phi_n(\zeta, b))$ with

$$\Phi_j(\zeta, b) = \int_b^\zeta \frac{(\nabla_u f)_j(x,0)}{s} ds, \quad j = 1, \dots, n, \tag{5.7}$$

where $x = (x_1, \dots, x_n)$, $x_k = c_k s^{\lambda_k}$ $(k = 1, 2, \dots, n-1)$ and $b \in \mathbb{C}$. Note that the relations (5.6) give the (multi-valued) change of variables between (x_k, ζ, y) and (c_k, ζ, y_0).

Let $v_0(x)$ and Σ_0 be given by (3.3) and (3.6), respectively. We fix j, $1 \le j \le n$ and write $\Phi(\zeta, b) \equiv \Phi_j(\zeta, b)$. The line starting from some point in Σ_0 such that $\operatorname{Im}\Phi(\zeta, b) = \operatorname{Im}\Phi(\zeta_0, b)$ on the curve is denoted by γ_{ζ,ζ_0}. Let b be in some neighborhood of the origin of $\mathbb{C}$. Then the solution of (5.4) such that $w(\zeta) \to 0$ as $\zeta \to 0$ is given by

$$w \equiv P_0 g = \int_{\gamma_{\zeta,\zeta_0}} g(s^{\lambda_1} c_1, \cdots, s^{\lambda_{n-1}} c_{n-1}, s; y_0 - \Phi(s, b)) ds, \tag{5.8}$$

where the integral for the j-th component $g = g_j$ is taken along the curve γ_{ζ,ζ_0} defined in the above for $\Phi(\cdot, b) \equiv \Phi_j(\cdot, b)$ which emanates from the origin and passes ζ and ζ_0 in this order. Here we change the variables in (5.8) via (5.6) after integration. In order to verify that the integrand is well defined we first show that

$$\Phi(s, b) = \mu_j \log\left(\frac{s}{b}\right) + o(s, b) \quad \text{when } s, b \to 0. \tag{5.9}$$

Indeed we know that $(\nabla_u f)_j(x, 0) = \mu_j + O(|x|)$ as $x \to 0$. Because $\operatorname{Re}\lambda_k > 0$, the integral $\int_b^s t^{-1}(\nabla_u f)_j(x, 0)dt$ with $x_k = c_k t^{\lambda_k}$ has the limit when $s \to 0$ in some sector. Hence we have (5.9).

We will show that the integrand in (5.8) is well defined. By (5.6) and (5.7) we have $y_0 - \Phi(s, b) = y - \Phi(s, b) + \Phi(\zeta, b) = y + \Phi(\zeta, s)$. By the definition of γ_{ζ,ζ_0} we have that $\mathrm{Im}\Phi(\zeta, s) = 0$ if $s \in \gamma_{\zeta,\zeta_0}$. On the other hand one can easily show that $\operatorname{Re}\Phi(\zeta, s)$ is a monotone function of ζ on γ_{ζ,ζ_0}. In view of (5.9) $\operatorname{Re}\Phi(\zeta, s)$ tends to $-\infty$ as $\zeta \to 0$. Hence $\operatorname{Re}\Phi(\zeta, s)$ is a monotone increasing function on the curve as $|\zeta|$ increases. We have $\operatorname{Re}\Phi(\zeta, s) \le 0$ if $s \in \gamma_{\zeta,\zeta_0}$. In view of the assumption on Ω we have $y_0 - \operatorname{Re}\Phi(s, b) \in \Omega$ for every $y \in \Omega$ and $s \in \gamma_{\zeta,\zeta_0}$.

Next we take a neighborhood U_0 of the origin such that the formal solution is holomorphic in U_0. Let γ_{ζ,ζ_0} be as in the above. We want to substitute $x_k = s^{\lambda_k} c_k$ into the integrand of (5.8) for $s \in \gamma_{\zeta,\zeta_0}$. In order to show that this is possible uniformly when ζ tends to zero along the curve γ_{0,ζ_0} emanating from the origin we will consider

$$\log x_k = \log c_k + \lambda_k \log s = \log(c_k b^{\lambda_k}) + \frac{\lambda_k}{\mu_j}\mu_j \log\left(\frac{s}{b}\right). \tag{5.10}$$

By virtue of (5.9), $\mu_j \log(s/b)$ is close to $\Phi(s,b)$ and hence $\mathrm{Im}\,(\mu_j \log(s/b))$ is close to $\mathrm{Im}\,\Phi(s,b)$. Because $\mathrm{Im}\,\Phi(s,b)$ is a constant function of s on every curve γ_{ζ,ζ_0}, we may consider $\mathrm{Im}\,(\mu_j \log(\zeta_0/b))$ instead of $\mathrm{Im}\,(\mu_j \log(s/b))$. It follows that there exists $K_0 > 0$ depending only on ζ_0/b such that

$$-K_0 < \mathrm{Im}(\mu_j \log(s/b)) < K_0.$$

Because $\mathrm{Re}\,\Phi(s,b)$ is monotone increasing on s along the curve, it is bounded by $\mathrm{Re}\,\Phi(\zeta_0,b)$. By taking the maximum on $|\zeta_0| = const$ there exists K_1 independent of $|\zeta_0| = const$ such that $\mathrm{Re}\,(\lambda_j \log(s/b)) \le K_1$ for all $s \in \gamma_{\zeta,\zeta_0}$. On the other hand, by (2.9) we see that there exists $\varepsilon_0 > 0$ such that $(\lambda_k/\mu_j)\mu_j \log(s/b)$ is contained in the set $\{z; |arg\, z - \pi| < \pi/2 - \varepsilon\}$ for all $s \in \gamma_{\zeta,\zeta_0}$ except for a bounded set.

Because $\mathrm{Re}\,(\log(c_k b^{\lambda_k}))$ tends to $-\infty$ when b tends to zero, we choose b sufficiently small, then choose ζ_0 so that $|\zeta_0|/|b|$ so small. We see that the right-hand side of (5.10) stays in the left-half plane such that the real part is arbitrarily small. Therefore we see that x_k lies in a sufficiently small neighborhood of the origin for $s \in \gamma_{\zeta,\zeta_0}$ uniformly when ζ moves to 0 along γ_{0,ζ_0}. This proves that the substitution $x_k = s^{\lambda_k} c_k$ for $s \in \gamma_{\zeta,\zeta_0}$ into the integrand of (5.8) is well defined uniformly when $\zeta \to 0$ and ζ_0. The integrability in (5.8) is clear for every given b because the integrand is continuous and the integral is taken on a compact smooth curve.

We estimate w and its derivative w_y of (5.8) for $g \in \mathcal{H}(D,\Omega)$. In the following we assume that there exists an $\varepsilon_0 > 0$ such that $|\zeta|/|\zeta_0| \ge \varepsilon_0$. We now estimate w in (5.8). We recall that $\Phi(s,b)$ is asymptotically equals to $\mu_j \log(s/b)$ as $s \to 0$. Therefore one may assume that the integral is taken along the curve $\mathrm{Im}\,\mu_j \log(s/b) = c$ for some c. Set $\mu_j = \alpha + i\beta$ $(\alpha > 0)$ and $\log(s/b) = x + iy$. Then one can see that the curve $\mathrm{Im}\,\mu_j \log(s/b) = c$ can be written in $\beta x + \alpha y = c$, and the integration is taken for some $x_1 \ge x \ge x_0$, where x_0 corresponds to ζ. In view of the relation $s = be^{x+iy}$, we have $ds = be^{x+iy}(dx + idy) = be^{x+iy}(1 - \beta i/\alpha)dx$. Because there appears a positive power of s in the integrand of (5.8) in view of the above argument, a positive power of e^x appears from the integrand.

We next estimate the growth of $y_0 - \Phi(s,b)$. In terms of (5.6) we have

$$(5.11) \qquad \begin{aligned} \exp\left(-c\mathrm{Re}\,(y_0 - \Phi(s,b))\right) &= \exp\left(-c\mathrm{Re}\,(y + \Phi(\zeta,b) - \Phi(s,b))\right) \\ &= \exp\left(-c\mathrm{Re}\,(y + \Phi(\zeta,s))\right). \end{aligned}$$

Because $\mathrm{Re}\,\Phi(\zeta,s)$ is decreasing in ζ as ζ tends to zero along γ_{0,ζ_0}, we have $\mathrm{Re}\,\Phi(\zeta,s) \le 0$. Hence we need to estimate $e^{-c\mathrm{Re}\,\Phi(\zeta,s)}$. We have that $\Phi(\zeta,s)$ is asymptotically equal to $\mu_j \log(\zeta/s)$. Set $\log(\zeta/s) = x + iy$ and $\mu_j = \alpha + i\beta$ with $\alpha > 0$. Then we have $\mathrm{Re}\,(\mu_j \log(\zeta/s)) = \alpha x - \beta y$. On the other hand, by definition we have $\beta x + \alpha y = c$ for some c. Hence $\alpha x - \beta y = (\alpha + \beta^2\alpha^{-1})x - c\beta\alpha^{-1}$. Noting that $x = \log(|\zeta|/|s|) > \log(|\zeta|/|\zeta_0|) > \log \varepsilon_0$, we have

$$\begin{aligned} \exp(-c(\alpha x - \beta y)) &= \exp(-(\alpha + \beta^2\alpha^{-1})cx - c^2\beta\alpha^{-1}) \\ &\le \exp\left((\alpha + \beta^2\alpha^{-1})c \log \varepsilon_0^{-1} - c^2\beta\alpha^{-1}\right) =: K_0. \end{aligned}$$

This proves

$$\exp\left(-c\mathrm{Re}\left(y_0 - \Phi(s,b)\right)\right) \le K_0 \exp\left(-c\mathrm{Re}\, y\right). \tag{5.12}$$

We estimate $|y_0 - \Phi(s,b)| = |y + \Phi(\zeta, s)|$ from the below. Because $\mathrm{Im}\,\Phi(\zeta,s) = 0$ and $\mathrm{Re}\,\Phi(\zeta,s) \le 0$ on the curve, there exists $C_1 > 0$ independent of ζ and s such that

$$(1 + |y_0 - \Phi(s,b)|)^{-2} \le C_1(1+|y|)^{-2} \quad \text{for all } y \in \Omega. \tag{5.13}$$

Therefore we get, from (5.12) and (5.13) that

$$\begin{aligned} \|w\| &\le \sup\left((1+|y|)^2 \exp\left(c\mathrm{Re}\, y\right) \int \|g\| \frac{\exp\left(-c\mathrm{Re}\left(y_0 - \Phi(s,b)\right)\right)}{(1+|y_0 - \Phi(s,b)|)^2} |ds|\right) \\ &\le C_2 \|g\| \int |ds| \le C_3 \|g\|, \end{aligned} \tag{5.14}$$

for some $C_2 > 0$ and $C_3 > 0$.

We shall show

$$\|w_y\| \le C_4 \|g\| \tag{5.15}$$

for some $C_4 > 0$ independent of g. Noting that $y_0 - \Phi(s,b) = y + \Phi(\zeta, s)$ we make the change of variable $\sigma = y + \Phi(\zeta, s)$ in (5.8) from s to σ. We have $d\sigma = -\frac{(\nabla_u f)_j}{s} ds$. Note that the right-hand side is independent of y. We have $\sigma = y$ for $s = \zeta$ and $\sigma = y + \tilde{\zeta_0}$ for $s = \zeta_0$, where $\tilde{\zeta_0} = \Phi(\zeta, \zeta_0)$. Clearly, $s \in \gamma_{\zeta_0,\zeta}$ is expressed as $\sigma \in y + \gamma_{\tilde{\zeta_0},\zeta}$, where $\gamma_{\tilde{\zeta_0},\zeta}$ is the straight line connecting 0 and $\tilde{\zeta_0}$. Then (5.8) is written in

$$w = -\int_{\gamma_{\tilde{\zeta_0},\zeta}} g(s^{\lambda_1} c_1, \cdots, s^{\lambda_{n-1}} c_{n-1}, s; \sigma) \frac{d\sigma}{\partial_s \Phi(s,\zeta)}, \tag{5.16}$$

where $\sigma - y = \Phi(\zeta, s) \sim \mu_j \log(\zeta/s)$ and s is independent of y. Hence we have

$$\begin{aligned} w_y &= -g(\zeta_0^{\lambda_1} c_1, \cdots, \zeta_0^{\lambda_{n-1}} c_{n-1}, \zeta_0; y + \tilde{\zeta_0}) \frac{1}{\partial_s \Phi(\zeta_0, \zeta)} \\ &+ g(\zeta^{\lambda_1} c_1, \cdots, \zeta^{\lambda_{n-1}} c_{n-1}, \zeta; y) \frac{1}{\partial_s \Phi(\zeta,\zeta)}. \end{aligned} \tag{5.17}$$

Using (5.17) we have (5.15) by the same argument as $\|w\|$ since $\partial_s \Phi(\zeta_0,\zeta)^{-1}$ and $\partial_s \Phi(\zeta,\zeta)^{-1}$ are bounded.

We consider the case where $(\nabla_u f)(x,0)$ is not a diagonal matrix. We write $(\nabla f)(x,0) = A + (\nabla f)(x,0) - A$ with $A = (\nabla f)(0,0)$. Choose Q_0 such that $Q_0 A Q_0$ is a Jordan canonical form. By considering new unknown quantity $Q_0 w$ one may assume that A is the Jordan canonical form. If $(\nabla f)(x,0) - A$ is upper (resp. lower) triangular matrix, then one can solve (5.4) inductively in case there is a nilpotent part. In fact, we have the same estimate for w and w_y. Hence one can extend the definition of P_0 in the Jordan case. On the other hand, if $(\nabla f)(x,0) - A$ is not a triangular matrix, then we take the lower triangular matrix $R(x)$ so that $(\nabla f)(x,0) - A - R(x)$ is the upper triangular matrix. Because $R(x) = O(|x|)$ as $x \to 0$, one can estimate $\|R(x) w_y\| \le K_3 \varepsilon \|w_y\|$ for some $K_3 > 0$ independent of $\varepsilon > 0$, where ε can be taken arbitrarily small if we take the neighborhood of the origin sufficiently small. In this case we subtract the term corresponding to $R(x)$ in the iteration step.

We will solve (5.2) in $\mathcal{H}(D,\Omega)$. First we note

$$\nabla f(x,v_0)\frac{\partial \hat{u}}{\partial y} = \nabla f(x,0)\frac{\partial \hat{u}}{\partial y} + (\nabla f(x,v_0) - \nabla f(x,0))\frac{\partial \hat{u}}{\partial y}. \tag{5.18}$$

We note $\|\nabla f(x,v_0) - \nabla f(x,0)\| = O(\|v_0\|)$ when $\|v_0\| \to 0$. We also note that small perturbation terms appear from $R(x)\frac{\partial \hat{u}}{\partial y}$ when we define P_0. Note that these terms are also estimated by $K_4\varepsilon\|w_y\|$, where ε is small and K_4 is some constant.

We define the approximate sequence $\hat{u}_k$ $(k = 0,1,2,\dots)$ by $\hat{u}_0 = 0$ and

$$\hat{u}_1 = -P_0\mathcal{L}v_0 \tag{5.19}$$

$$\begin{aligned} \hat{u}_2 &= P_0\sum_{|\beta|\ge 2} r_\beta(x,v_0)\frac{\partial}{\partial y}(\hat{u}_1)_*^\beta - P_0\mathcal{L}v_0 + P_0R(x)\frac{\partial}{\partial y}\hat{u}_1 \\ &+ P_0(\nabla f(x,v_0) - \nabla f(x,0))\frac{\partial \hat{u}_1}{\partial y}, \end{aligned} \tag{5.20}$$

$$\vdots$$

$$\begin{aligned} \hat{u}_{k+1} &= P_0\sum_{|\beta|\ge 2} r_\beta(x,v_0)\frac{\partial}{\partial y}(\hat{u}_k)_*^\beta - P_0\mathcal{L}v_0 + P_0R(x)\frac{\partial}{\partial y}\hat{u}_k \\ &+ P_0(\nabla f(x,v_0) - \nabla f(x,0))\frac{\partial \hat{u}_k}{\partial y}, \end{aligned} \tag{5.21}$$

where $k = 1,2,\dots$

In order to show that the sequence is well defined we make an apriori estimate. Given $\varepsilon > 0$. We can take a sufficiently small domain D such that $\|\mathcal{L}v_0\| \le \varepsilon$. By (5.14) we have

$$\|\hat{u}_1\| \le \|P_0\mathcal{L}v_0\| \le C\|\mathcal{L}v_0\| \le C\varepsilon. \tag{5.22}$$

Similarly by using (5.15) we have $\|(\hat{u}_1)_y\| \le C\varepsilon$.

Next we will estimate $\|\hat{u}_2\|$ and $\|(\hat{u}_2)_y\|$. Because the argument is similar we consider $\|\hat{u}_2\|$. Because $v_0(x) = O(|x|)$, there exist $K_5 > 0$ and $K_6 > 0$ such that for every $\varepsilon > 0$ we have $|r_\beta|_\infty := \sup_{x\in D}|r_\beta(x,v_0(x))| \le \varepsilon K_5 K_6^{|\beta|}$ for all $|\beta| \ge 2$ if D is sufficiently small. By (5.20), (5.22), (4.4) and the elementary property of convolution we have

$$\begin{aligned} \|\hat{u}_2\| &\le C\|\mathcal{L}v_0\| + C\sum_{|\beta|\ge 2}\left\| r_\beta\frac{\partial}{\partial y}(\hat{u}_1)^\beta\right\| + 2C^2\varepsilon^2K_4 \\ &\le C\varepsilon + C\sum_\beta |r_\beta|_\infty (C\varepsilon)^{|\beta|} + 2C^2\varepsilon^2K_4 \\ &\le C\varepsilon\left(1 + C\varepsilon K_5\sum_{|\beta|\ge 2} K_6^{|\beta|}n^{|\beta|}(C\varepsilon)^{|\beta|-1}\right) + 2C^2\varepsilon^2K_4. \end{aligned} \tag{5.23}$$

If we take $C\varepsilon K_6 n < 1$, then there exists $K_7 > 0$ such that the right-hand side of (5.23) can be estimated by $C\varepsilon(1 + 2C\varepsilon K_4 + C^2 nK_5 K_6^2 K_7 \varepsilon^2)$. Hence, if we take ε so that $C^2 nK_5 K_6^2 K_7 \varepsilon \le 1$, then we have $\|\hat{u}_2\| \le C\varepsilon K_9$ for some $K_9 > 0$ independent of ε. Similarly we have $\|(\hat{u}_2)_y\| \le C\varepsilon K_9$.

We continue to estimate $\|\hat{u}_3\|$ and $\|(\hat{u}_3)_y\|$. Clearly, we see that the same argument works if we replace K_6 with some K_8. By induction we have the apriori estimate

$$\|\hat{u}_k\| \le C\varepsilon K_9, \quad \|(\hat{u}_k)_y\| \le C\varepsilon K_9, \; k = 0, 1, 2, \ldots \tag{5.24}$$

We will show the convergence of $\hat{u}_k$. Let $l > m$ and write $\hat{u}_l - \hat{u}_m = \sum_{j=m}^{l-1} (\hat{u}_{j+1} - \hat{u}_j)$. Hence we estimate

$$\begin{aligned}
\hat{u}_{j+1} - \hat{u}_j &= P_0 \sum_{|\beta| \ge 2} r_\beta \frac{\partial}{\partial y} \left((\hat{u}_j)_*^\beta - (\hat{u}_{j-1})_*^\beta \right) \\
&- \quad P_0 R(x) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}) + P_0(\nabla f(x, v_0) - \nabla f(x, 0)) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}), \\
&= \quad P_0 \sum_\beta r_\beta \frac{\partial}{\partial y} \left(\sum_{\nu=1}^n (\hat{u}_{j,\nu} - \hat{u}_{j-1,\nu}) * R_\nu(\hat{u}_j, \hat{u}_{j-1}) \right) \\
&- \quad P_0 R(x) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}) + P_0(\nabla f(x, v_0) - \nabla f(x, 0)) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}),
\end{aligned} \tag{5.25}$$

where $R_\nu(\hat{u}_j, \hat{u}_{j-1})$ is some polynomial of $\hat{u}_j, \hat{u}_{j-1}$ of degree greater than or equal to $|\beta| - 1 \ge 1$ with respect to the convolution product. By (5.24) and Proposition 2 one can easily show that $\|\hat{u}_{j+1} - \hat{u}_j\| \le 2^{-1} \|(\hat{u}_j - \hat{u}_{j-1})_y\|$ and $\|(\hat{u}_{j+1} - \hat{u}_j)_y\| \le 2^{-1} \|(\hat{u}_j - \hat{u}_{j-1})_y\|$ if ε is sufficiently small. Indeed, by (5.15) one can show the latter one. The former one can be proved directly. These estimates show that $\hat{u}_k$ is a Cauchy sequence and converges to some $\hat{u}$. Hence we have the solution $\hat{u}$.

Let u be the Laplace transform of $\hat{u}$. Then $v := v_0 + u$ is the Borel sum of the formal solution with respect to η when $x \in D$. Note that u and $\hat{u}$ are analytic with respect to x in D. We denote u by u_D and define $v_D := v_0 + u_D$. Similarly, writing $\hat{u}$ by $\hat{u}_D$ we define $\hat{v}_D := v_0 + \hat{u}_D$.

Let D' be a domain such that $D \cap D' \ne \emptyset$ and let v_D and $v_{D'}$ be the corresponding Borel sum in D and D', respectively. Because the Borel sum with respect to η is unique for every x, we have that $v_D = v_{D'}$ on $D \cap D'$, from which we have an analytic continuation of v_D to $D \cup D'$. By choosing the sequence of open sets D we make an analytic continuation of v_D to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, where B_0 is some open ball centered at the origin. By the uniqueness of the Borel sum the analytic continuation of $\hat{v}_D(x, y)$ with respect to x to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, $y \in \Omega$ is single-valued. We also note that in view of the construction of $\hat{v}_D$ the growth estimate with respect to y of $\hat{v}_D(x, y)$ is uniform for $x \in (\mathbb{C} \setminus 0)^n \cap B_0$. Therefore we can define $\hat{v}(x, y) := \hat{v}_D(x, y)$ on $x \in (\mathbb{C} \setminus 0)^n \cap B_0$ and $y \in \Omega$ by taking $x \in D$.

The function $\hat{v}(x, y)$ may have singularity on $x \in (\mathbb{C}^n \setminus (\mathbb{C} \setminus 0)^n) \cap B_0$, $y \in \Omega$. We will prove that the singularity is removable. First consider the singularity with codimension 1. For simplicity take $y_0 \in \Omega$, $x'_0 = (x_2^0, \ldots, x_n^0)$ with $x_j^0 \ne 0$ and consider the expansion

$$\hat{v}(x,y) = \sum_{\nu\geq 0, j\geq 0} \hat{v}_{\nu,j}(x_1)(x'-x'_0)^\nu (y-y_0)^j. \tag{5.26}$$

By what we have proved in the above, the right-hand side is convergent if $x'-x'_0$ and $y-y_0$ are sufficiently small and $x_1 \neq 0$. Moreover, by the boundedness of $\hat{v}(x,y)$ when $x_1 \to 0$ and the Cauchy's integral formula we have that $\hat{v}_{\nu,j}(x_1)$ is holomorphic and single-valued and bounded in some neighborhood of the origin except for $x_1 = 0$. Hence its singularity is removable. In the same way one can show that the singularity of codimension 1 is removable.

Next we consider the singularity of codimension 2. For the sake of simplicity, consider the one $x_1 = x_2 = 0$, $x''_0 = (x^0_3, \dots, x^0_n)$ with $x^0_j \neq 0$. By considering in the same way as in the codimension one case we have the expansion similar to (5.26) where $x'-x'_0$ and $\hat{v}_{\nu,j}(x_1)$ are replaced by $x''-x''_0$ and $\hat{v}_{\nu,j}(x_1,x_2)$, respectively. Because $\hat{v}_{\nu,j}(x_1,x_2)$ is holomorphic and single-valued except for $x_1 = x_2 = 0$, we see that the singularity is removable by Hartogs theorem. As for the singularity of higher codimension ≥ 3 we can argue in the same way by using Hartogs theorem. We see that $\hat{v}(x,y)$ is holomorphic and single-valued on $x \in \mathbb{C}^n \cap B_0$, $y \in \Omega$.

The exponential growth of $\hat{v}(x,y)$ when $y \to \infty$ in $y \in \Omega$ for $x \in \mathbb{C}^n \cap B_0$ can be proved by putting some c_k to be equal to zero when constructing $\hat{v}_D(x,y)$. Indeed, we have already proved the fact in the above argument. Hence we have proved the solvability of (5.2), and the summability of the our solution as desired. Next we make the same argument for every j-th equation of the system, $1 \leq j \leq n$. If we choose a neighborhood of $x = 0$ sufficiently small, then we have the summability of every component of the formal solution.

We will prove the summability in the direction $\eta \in S_{\theta,\xi}$. By multiplying the equation (2.2) with $e^{-i\theta}$ we see that η^{-1}, λ_k, μ_j are replaced by $\eta^{-1}e^{-i\theta} = (\eta e^{i\theta})^{-1}$, λ_k and $\mu_j e^{-i\theta}$, respectively. Noting that the conditions (2.9) are satisfied for $0 \leq \theta < \pi/2 - \theta_2$, the summability holds for $\eta = e^{i(\pi-\theta)}$ with $0 \leq \theta < \pi/2 - \theta_2$. Hence the summability holds for $\pi/2 + \theta_2 < \arg \eta \leq \pi$. Next, by replacing η and λ_k by $\eta e^{-i\theta}$ and $\lambda_k e^{-i\theta}$ we see that (2.9) are satisfied for $0 \leq \theta < \pi/2 - \theta_1$. It follows that the summability holds for $\pi < \arg \eta \leq 3\pi/2 - \theta_1$. Therefore, the summability holds for $\pi/2 + \theta_2 < \arg \eta < 3\pi/2 - \theta_1$. This proves the latter half in view of the definition of Borel sum. This completes the proof.

6. Some remarks

In Theorem 1 we proved Borel summability of $v(x,\eta)$ in some neighborhood of the origin $x = 0$. We want to extend Theorem 1 to the case $x \neq 0$. Instead of (2.3) we assume that there exist $a \in \mathbb{C}^n$ and $b \in \mathbb{C}^N$ such that

$$f(a,b) = 0, \qquad \det(\nabla_u f(a,b)) \neq 0. \tag{6.1}$$

By the implicit function theorem one can construct $v_0(x)$ analytic at $x = a$ such that $v_0(a) = b$ and $f(x, v_0(x)) \equiv 0$ in some neighborhood of a. Let Σ_0 be given by (3.6). Note that $a \notin \Sigma_0$. Let $\Omega_1 \subset \mathbb{C}^n \setminus \Sigma_0$ be the maximal domain containing

a and not containing the origin on which v_0 is holomorphic. One can construct formal solution $v(x,\eta)$ in (2.4). By virtue of Theorem 2 the formal Borel transform of $v(x,\eta)$ converges for x in some domain $\Omega' \subset \Omega_1$ with compact closure. For the sake of simplicity we assume $\Omega' = \Omega_1$ in the following. We will study Borel summability of $v(x,\eta)$ for $x \in \Omega_1$.

Theorem 3. *Assume that $f(x,u)$ is an entire function of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$ such that $\nabla_u f(x, v_0(x))$ is a diagonal matrix for every $x \in \Omega_1$. Then $v(x,\eta)$ is 1- Borel summable in the direction ξ, $\frac{\pi}{2} < \arg\xi < \frac{3\pi}{2}$ with respect to η for any $x \in \Omega_1$.*

Before starting the proof we remark that the condition (2.9) is not necessary in the above theorem.

Proof of Theorem 3. Suppose that we have proved Borel summability of $v(x,\eta)$ in some neighborhood of every $a \in \Omega_1$. Denote its Borel sum at a by $\hat{v}_a(x,\eta)$. Let $a, a' \in \Omega_1$. If there is an open set D_0 for which $\hat{v}_a(x,\eta)$ and $\hat{v}_{a'}(x,\eta)$ are defined for $x \in D_0$, then we have $\hat{v}_a(x,\eta) = \hat{v}_{a'}(x,\eta)$ for $x \in D_0$ by the uniqueness of the Borel sum. Hence we can make the analytic continuation of $\hat{v}_a(x,\eta)$. This proves that the Borel sum $\hat{v}_a(x,\eta)$ is independent of the choice of $a \in \Omega_1$. Hence we write $\hat{v}(x,\eta)$ instead of $\hat{v}_a(x,\eta)$. We will prove the Borel summability at every point $a \in \Omega_1$.

We follow the argument in the proof of Theorem 1. Because we do not assume (2.9) it is necessary that P_0 in (5.8) is well defined. Consider (5.6) in the formula (5.8). By assumption there exists k such that $a_k \neq 0$. Without loss of generality we may assume $k = n$. Hence, in the transformation (5.6) ζ is close to $a_n \neq 0$. If we fix the branch, then the relations (5.6) give one to one correspondence between the neighborhood D of a_n in the ζ space and some open set in the x_k space. Hence we see that if $\zeta \in D$ varies and c_k's in (5.6) are chosen appropriately, then x moves in some open set in a neighborhood of a. Hence the substitution in the x variable in (5.8) is well defined.

Next we consider the substitution of y variable, (5.6) in (5.8). We note that the integrand in $\Phi(s,\cdot)$ is a regular function because $a_n \neq 0$. We will show that either $\operatorname{Re}\Phi(\zeta,\zeta_0) \le 0$ or $\operatorname{Re}\Phi(\zeta_0,\zeta) \le 0$ holds along the curve γ_{ζ,ζ_0}. Indeed, set $\int^s (u + iv)dt = \Phi(s,\cdot)$ and $dt = dx + idy$. By the definition of γ_{ζ,ζ_0} we have $\operatorname{Im}((u+iv)(dx + idy)) = 0$. It follows that $udx + vdy = 0$. On the other hand we have

$$\operatorname{Re}\,((u+iv)(dx+idy)) = udx - vdy = 2udx. \tag{6.2}$$

Because $a \notin \Sigma_0$, one sees that the curves $\{u = 0\}$ and γ_{ζ,ζ_0} are transversal because $f_j = u + iv$ does not vanish. Hence, by taking ζ_0 in $\{u < 0\}$ and ζ in $\{u > 0\}$ sufficiently close to $\{u = 0\}$, we have $\operatorname{Re}\Phi(\zeta,\zeta_0) \le 0$. Because $\nabla_u f(x, v_0(x))$ is a diagonal matrix for every $x \in \Omega_1$ one can define P_0.

If we define P_0, then we can solve (5.2) by the same argument as in the case of the origin $x = 0$ if we assume that $\|\mathcal{L}(v_0 - b)\|$ and/or $\|v_0 - b\|$ is sufficiently small. The condition is clearly satisfied if x is in some neighborhood of a. Note that the smallness of the coefficients does not hold in the present case, while the apriori estimate and the convergence can be proved by the smallness. Finally, in case some $a_k = 0$, then we need to remove the singularity at $x_k = 0$ in order to show the analyticity of the

Borel sum at $x_k = 0$. This can be done in the same way as in the proof of Theorem 1. This ends the proof of Theorem 3.

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