SYNTHESIS OF A STATIC ANTI-WINDUP COMPENSATOR FOR SYSTEMS WITH MAGNITUDE AND RATE LIMITED ACTUATORS

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Abstract: In this paper, we propose a design method of a static anti-windup compensator for systems with input magnitude and rate saturations. First, we present an anti-windup controller for systems with input magnitude and rate saturations. Then, we show that the design problem of the anti-windup compensator that guarantees the local stability of the closed-loop system against saturation nonlinearities and optimizes the robust control performance during the saturation period can be reduced to a linear matrix inequality (LMI) problem.

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1. INTRODUCTION

Many practical control systems involve magnitude saturation on control input, which causes severe performance degradation called windup phenomena in the case where a controller has slow dynamics. An anti-windup scheme is one way to deal with the windup phenomena, and it is usually performed by adding an anti-windup compensator to the controller which gives acceptable performance in the unsaturated region. If the compensator is a constant gain matrix, it is referred to as a static anti-windup compensator in this paper. The general structure of the static anti-windup compensator for the system with input magnitude saturation is presented in Kothare, Campo, Morari and Nett (1994). Recently, in Wada and Saeki (1999), it is shown, for the first time, that the design problem of the static anti-windup compensator which guarantees robust stability against saturation functions can be represented as an equivalent LMI problem, and an optimal solution can be obtained efficiently. Since the design method of Wada et al. (1999) treats a nominal performance control problem, it is extended to the robust performance problem in Saeki and Wada (2000).

In many practical control systems, there exist not only magnitude saturations but also rate saturations. It is pointed out in Berg, Hammett, Schwartz and Banda (1996) that the phase-lag associated with rate saturation has a destabilizing effect. Several design methods of a controller for the system with both magnitude and rate saturations have been proposed. For example, a dynamic output feedback is used in Tyan and Bernstein (1997), Lin (1997), Kapila and Haddad (1998); the error governor is constructed in Kapasouris and Athans (1990). However, as far as we know, the static anti-windup scheme has not been applied to the problem of input magnitude and rate saturations.

In this paper, we extend the result of Wada et al. (1999) to the input magnitude and rate saturation problem. First, a new structure of the anti-windup controller is derived for systems with input magnitude and rate saturations. As a rate saturation model, a position-feedback-type system with saturation inside the closed-loop system is utilized, and the closed-loop system is represented as a linear system with two saturation functions. Secondly, a closed-loop stability condition is derived based on the multivariable circle theorem, and the design problem is reduced to an LMI problem. Finally, a numerical example is given.
where $H_m \in \mathbb{R}^{n_u \times n_p}$, $H_r \in \mathbb{R}^{n_r \times n_p}$ and $u_r = \dot{u}_m$. The controller shown in Fig.1(b) can be considered the observer based anti-windup controller (see, Åström and Wittenmark, 1984) with a compensator $f_r = H_r(u_r - \dot{u}_r)$ that works when the rate of $u_m$ exceeds its maximum value. We suppose that an original linear time-invariant (LTI) controller $K(s)$ has been already designed to meet some design specifications (e.g., a closed-loop stability, time domain criterion) when there is no input saturation. Therefore, the matrices $H_m$ and $H_r$ are only design parameters.

Next, let us consider a closed-loop system shown in Fig.2. $G(s)$ denotes a generalized plant, and its state equation is given by

$$\begin{cases}
\dot{x}_G = A_Gx_G + B_Gw_1 + B_G2u \\
z_1 = C_G1x_G + D_G1w_1 + D_G12u \\
y = C_G2x_G + D_G2w_1
\end{cases}$$

where $x_G \in \mathbb{R}^{n_x}$, $z_1 \in \mathbb{R}^{n_z}$, $w_1 \in \mathbb{R}^{n_w}$. This system can be represented as a system shown in Fig.3. It is described by

$$\begin{cases}
\dot{x}_M = A_Mx_M + B_M1w_1 + B_M2\xi + B_M3f \\
z_1 = C_M1x_M + D_M11w_1 + D_M12\xi + D_M13f \\
\xi = C_M2x_M + D_M21w_1 + D_M22\xi + D_M23f
\end{cases}$$

where

$$A_M = \begin{bmatrix} A_G & 0 & 0 \\ BCG_2 & A_0 & 0 \end{bmatrix}, B_{M1} = \begin{bmatrix} B_{G1} \\ BDG_21 \\ 0 \end{bmatrix}$$

$$B_{M2} = \begin{bmatrix} B_{G2} \\ 0 & 0 & 0 \end{bmatrix}, B_{M3} = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}$$

$$C_{M1} = \begin{bmatrix} C_{G1} & 0 & 0 \end{bmatrix}, C_{M2} = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$$

$$D_{M11} = D_{G11}, D_{M12} = [DG12, 0], D_{M13} = RDC_{G2} \quad R$$

$$D_{M21} = \begin{bmatrix} 0 \\ RDD_{G21} \end{bmatrix}, D_{M22} = 0, D_{M23} = 0$$

$$H = [H_m, H_r], \phi(\cdot) = \begin{bmatrix} \phi_m(\cdot) \\ \phi_r(\cdot) \end{bmatrix}$$

Fig. 1. Anti-windup controller for input magnitude and rate saturations

Fig. 2. Closed-loop system with an anti-windup controller
and $\xi = [u^T, u_m^T]^T$, $\tilde{\xi} = [\tilde{u}^T, \tilde{u}_m^T]^T$, $f = f_m + f_r$, $x_M = [x_G^T, x_c^T, u_m^T]^T$. The static matrix $H$ is the design parameter of the closed-loop system (6)-(8). We refer to $H$ as a static anti-windup compensator. In the next section, we consider a design method of $H$.

3. SYNTHESIS OF AN ANTI-WINDUP COMPENSATOR

3.1 Problem setting

In order to consider the robust control performance during the saturation period, we add an imaginary uncertainty:

$$w_1 = \Delta_p \delta z_1$$

(10)

to the system of Fig.3, where $\Delta_p \in \{\text{block-diag} (\Delta_{p1}, \cdots, \Delta_{pn}) | \Delta_{pi} \in [-1, 1], \forall i\}$ and $\delta$ is a positive scalar. We study a design method of $H$ that satisfies following design specifications.

Note 1. Kothare et al. (1994) proposed the following anti-windup controller.

$$\begin{align*}
\dot{x}_e &= Ax_e + By + f_1 \\
\dot{u} &= Cx_e + Dy + f_2 \\
f_1 &= \Lambda_1 (u - \tilde{u}) \\
f_2 &= \Lambda_2 (u - \tilde{u}) \\
u &= \phi(\tilde{u})
\end{align*}$$

(9)

where $\Lambda_1$ and $\Lambda_2$ are constant matrices. The closed-loop system consisted of (5) and (9) is included in the anti-windup structure shown in Fig.3 as a special case. Actually, the closed-loop system (5) and (9) can be written in the form of (6)-(8) with $\xi = u$, $\tilde{\xi} = \tilde{u}$, $f = [f_1, f_2]^T$, $x_M = [x_G^T, x_c^T]^T$ and

$$\begin{align*}
A_M &= \begin{bmatrix} A_G & 0 \\ BC_{G2} & A \end{bmatrix}, B_{M1} = \begin{bmatrix} B_{G1} \\ BD_{G21} \end{bmatrix} \\
B_{M2} &= \begin{bmatrix} B_{G2} \\ 0 \end{bmatrix}, B_{M3} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\
C_{M1} &= \begin{bmatrix} C_{G1} \\ 0 \end{bmatrix}, C_{M2} = \begin{bmatrix} 0 & C \end{bmatrix} \\
D_{M11} &= D_{G11}, D_{M12} = D_{G12}, D_{M13} = 0 \\
D_{M21} &= D_{G21}, D_{M22} = 0, D_{M23} = [0, I] \\
H &= \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}
\end{align*}$$

For simplicity, we assume $D_{M13} = 0$ and $D_{M23} = 0$ in the following discussion.

3.2 Robust stability condition

In this section, we derive a robust stability condition of the system of Fig.3 with (10), and reduce the design problem of $H$ which satisfies this condition to an LMI problem.

The saturation function $\phi_i(\cdot)$ can be represented as $\phi_i(\xi_i) = \tilde{\xi}_i - \phi_{di}(\xi_i)$ by using a dead zone function $\phi_{di}(\cdot)$. We model $\phi_{di}(\cdot)$ as the sector bounded nonlinearity that lies inside the sector $[0, \bar{\kappa}_i]$, i.e. $\phi_{di}(\cdot) \in [0, \bar{\kappa}_i]$, as shown in Fig.4. Namely, we model $\phi_{di}(\cdot)$ as $\phi_{di}(\xi_i) = \Delta_{di}\xi_i\bar{\kappa}_i\xi_i$, where $\Delta_{di} \in [0, 1]$. When $\bar{\kappa}_i = 1$, the sector entirely covers $\phi_{di}(\cdot)$, and when $\bar{\kappa}_i < 1$, the sector covers a finite region with the center $\xi_i = 0$. By using this model, we represent the saturation function $\phi(\cdot)$ as

$$\begin{align*}
\xi &= \bar{\xi} - \tilde{w}_2 \\
\bar{\xi}_2 &= \bar{\kappa}\tilde{\xi} \\
\tilde{w}_2 &= \Delta_{di}\bar{\kappa}_i\tilde{\xi}_2
\end{align*}$$

(11)

where $\tilde{w}_2 \in \mathbb{R}^{n_s}$, $\bar{\xi}_2 \in \mathbb{R}^{n_s}$, $\bar{\kappa} = \text{diag}(\bar{\kappa}_1, \cdots, \bar{\kappa}_n)$, $\Delta_{di} = \text{diag}(\Delta_{di1}, \cdots, \Delta_{din})$. Substitution of (11) into (7) gives

$$\begin{align*}
f &= H(\xi - \bar{\xi}) \\
&= H((\xi - \tilde{w}_2) - \bar{\xi}) \\
&= -H\tilde{w}_2
\end{align*}$$

(12)

From (12), we can obtain an interpretation that the static anti-windup compensator estimates the disturbance $\tilde{w}_2$, and makes use of it as a feedforward signal.

The transformation $\Delta_{pi} = (\Delta_{pi} + 1)/2$ bijectively maps $\Delta_{pi} \in [-1, 1]$ into $\Delta_{pi} \in [0, 1]$ (e.g. Safonov, Jonckheere, Verma and Limebeer, 1987). By using this relation, (10) can be represented as

$$\begin{align*}
w_1 &= 2\tilde{w}_1 - \bar{z}_1 \\
\bar{z}_1 &= \delta\tilde{z}_1 \\
\tilde{w}_1 &= \Delta_p\tilde{z}_1
\end{align*}$$

(13)
Next, by applying the multivariable circle theorem (see e.g., Boyd, Ghaoui, Feron and Balakrishnan, 1994) For a given transfer function $G(s) := C(sI - A)^{-1}B + D \in RH_{\infty}$, the following statements are equivalent:

\begin{enumerate}
  \item $G(j\omega) + G(j\omega)^* > 0, \forall \omega$
  \item There exists $Q = Q^T > 0$ such that
    \[
    \begin{bmatrix}
      QA^T + AQ & B - QC^T \\
      B^T - CQ & -(D + D^T)
    \end{bmatrix} < 0
    \]
\end{enumerate}

From Lemma 1, the condition (16) holds if and only if there exist $Q = Q^T > 0$ and $H$ such that

\[
F(Q, H) = \begin{bmatrix}
QA_N^T + ANQ & \tilde{B}_N + QC_N^T \\
\tilde{B}_N^T + C_NQ & D_N + D_N^T - 2I
\end{bmatrix} < 0
\]

Note 2. When the above LMI problem is solved with $\kappa = I$, the global stability is assured by using the obtained $H$. When the above LMI problem can not be solved with $\kappa = I$, the only local stability is assured. We do not discuss the domain of attraction (see e.g., Hindi and Boyd, 1998) directly in this paper. We simply make the sector larger in order to enlarge the maximum value of $\xi$ that satisfies the sector condition.

3.3 Introduction of a constant scaling matrix

The conservativeness of the design result can be reduced by using the following scaled passivity condition instead of (16).

\[(I - \tilde{N}(j\omega))W + W(I - \tilde{N}(j\omega))^* > 0, \forall \omega\]

where $W \in \{\text{block-diag}(w_1I_1, \cdots, w_mI_m, w_{m+1}, \cdots, w_{m+n_k-1}, 1)|w_i > 0, \forall i\}$. From Lemma 1, the condition (19) holds if and only if there exist $Q = Q^T > 0, W$ and $H$ such that

\[
F_W(Q, H, W) = \begin{bmatrix}
QA_N^T + ANQ & \tilde{B}_N W + QC_N^T \\
W\tilde{B}_N^T + C_NQ & D_NW + W\tilde{D}_N^T - 2W
\end{bmatrix} < 0
\]

Unfortunately, (20) is BMI with respect to $W$ and $H$, thus it is difficult to obtain an optimal solution in general. Nevertheless, the conservativeness of the design result can be reduced as compared with the case without a scaling matrix by using the following alternating minimization algorithm.
**step1:** Let $i = 0$ and fix $W_0 = I$.

**step2:** Fix $W = W_i$.

- minimize $\sigma$ subject to $F_W(Q, H, W) < \sigma I$ and $Q = Q^T > 0$. Let $H_i = H_i$.

**step3:** Fix $H = H_i$.

- minimize $\sigma$ subject to $F_W(Q, H, W) < \sigma I$ and $Q = Q^T > 0$. Let $W_{i+1} = W$ and $\sigma_{i+1} = \sigma$.

**step4:** If $\sigma_{i+1} < 0$ or $|\sigma_{i+1} - \sigma_i| < \epsilon$ for sufficiently small $\epsilon > 0$, then stop.

Otherwise, let $i \rightarrow i + 1$ and go to step2.

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**4. NUMERICAL EXAMPLE**

We consider a closed-loop system shown in Fig.6. $P(s) = C_p(sI - A_p)^{-1}B_p$ is a plant, and its coefficient matrices are given by

$$ A_p = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, B_p = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}, C_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} $$

$K(s) = C(sI - A)^{-1}B + D$ is a controller, and its coefficient matrices are given by

$$ A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 10 & 0 \\ 0 & -10 \end{bmatrix} $$

The above $P(s)$ and $K(s)$ are considered in Campo et al. (1990). Although this problem looks simple, the closed-loop system becomes unstable because of the change of the direction of control inputs even in the case where there exists only input magnitude saturation. Thus this problem is appropriate for evaluating the effectiveness of our design method.

We solve the matrix inequality (20) by using the alternating minimization algorithm with $R = 100I$, $\bar{R} = 0.9I$, $\delta = 0.03$. After three iterations, a feasible solution is obtained as

$$ H = \begin{bmatrix} 5.090 & 3.548 & 0.629 & -0.363 \\ -4.040 & -2.586 & -0.560 & 0.339 \end{bmatrix} \quad (21) $$

We conduct numerical simulations for a step disturbance $d(t) = [1, 2]^T, (t \geq 0)$ for three cases shown in Table1. Fig.7 and Fig.8 show the results of case1 and case2. The solid lines show the responses of the system with the proposed anti-windup compensator, and the dashed lines show the responses of the system without anti-windup compensation. The closed-loop stability is assured by using the anti-windup compensator. Fig.9 and Fig.10 show the results of case3. Although simulation conditions are severer than case2, the closed-loop stability is assured, and the step disturbance is attenuated in these figures.
Table 1. Simulation conditions

<table>
<thead>
<tr>
<th>Case</th>
<th>W/O AWC</th>
<th>$u_{m1,max} = 3$</th>
<th>$u_{r1,max} = 3$</th>
<th>$u_{m2,max} = 10$</th>
<th>$u_{r2,max} = 3$</th>
<th>$u_{m1,max} = 1.1$</th>
<th>$u_{r1,max} = 0.2$</th>
<th>$u_{m2,max} = 2.5$</th>
<th>$u_{r2,max} = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 2</td>
<td>W/O AWC</td>
<td>$u_{m1,max} = 3$</td>
<td>$u_{r1,max} = 3$</td>
<td>$u_{m2,max} = 10$</td>
<td>$u_{r2,max} = 3$</td>
<td>$u_{m1,max} = 3$</td>
<td>$u_{r1,max} = 3$</td>
<td>$u_{m2,max} = 3$</td>
<td>$u_{r2,max} = 3$</td>
</tr>
<tr>
<td>Case 3</td>
<td>W/O AWC</td>
<td>$u_{m1,max} = 1.1$</td>
<td>$u_{r1,max} = 0.2$</td>
<td>$u_{m2,max} = 2.5$</td>
<td>$u_{r2,max} = 0.2$</td>
<td>$u_{m1,max} = 3$</td>
<td>$u_{r1,max} = 3$</td>
<td>$u_{m2,max} = 3$</td>
<td>$u_{r2,max} = 3$</td>
</tr>
</tbody>
</table>

5. CONCLUSION

We propose a design method of an anti-windup compensator which guarantees the local stability of the closed-loop system with input rate and magnitude saturations. The proposed design method has the following useful properties.

- The design problem is reduced to an LMI problem. Thus, the solution can be obtained efficiently by a numerical optimization method.
- The proposed design method is applicable to the design of the anti-windup compensator for any class of LTI controllers (e.g., centralized controllers, decentralized controllers, PID controllers).

6. REFERENCES


