AN ALTERNATIVE PROOF OF A THEOREM OF MORIMOTO ON COMPOSITE TUNNEL NUMBER ONE LINKS

MASAKAZU TERAGAITO

Abstract. Morimoto characterized composite tunnel number one links and their unknotting tunnels. In this note, we will give its alternative proof which is relatively straightforward.

1. Introduction

In [3] Morimoto proved the following.

Theorem 1.1. *A composite link L in S^3 has tunnel number one if and only if it is the connected sum of a 2-bridge knot and the Hopf link. Moreover, any unknotting tunnel for L is obtained from the upper or lower tunnel for the 2-bridge knot as in Figure 1.*

His methods are difficult however, involving isotopy of type A argument [1] and the characterization of essential annuli in a genus two handlebody [2]. We give here an alternative proof of the theorem, which is relatively straightforward.

2. Proof

Let L be a link in S^3. Suppose that L is composite and the tunnel number of L is one. Then L consists of two components, K_1 and K_2, since n-component links have tunnel number at least $n-1$. Let γ be an unknotting tunnel for L. Put $V_1 = N(K_1 \cup K_2 \cup \gamma)$ and $V_2 = \mathrm{cl}(S^3 - V_1)$. We can regard V_1 as the union of two solid tori $N(K_1)$, $N(K_2)$ and the 1-handle $N(\gamma)$. Let S be a decomposing sphere for L. We can assume that $S \cap K_1 =$ two points and $S_2 \cap K_2 = \emptyset$. We can also assume that S meets V_1 in a disjoint union of disks as in Figure 2.

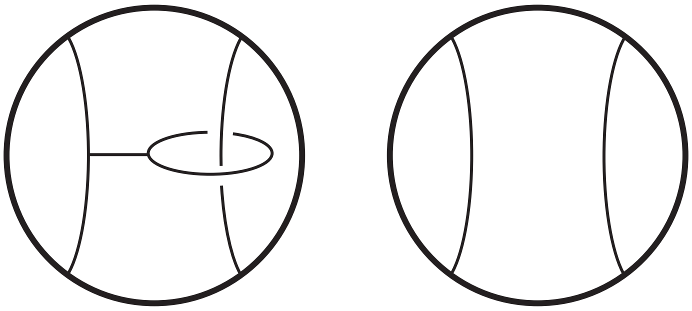

Figure 1

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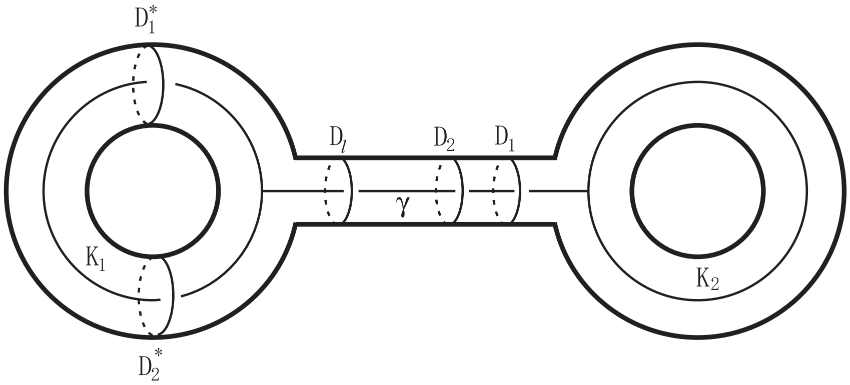

FIGURE 2

Here $l = |\gamma \cap S| \geq 0$. We assume that l is minimal among all decomposing spheres for L, meeting V_1 in such disks. Let $S_1 = S \cap V_1 (= D_1^* \cup D_2^* \cup D_1 \cup D_2 \cup \cdots \cup D_l)$ and $S_2 = S \cap V_2$. Then S_2 is incompressible in V_2 by the minimality of l. (Note that L is unsplittable and so the exterior of L is irreducible by the hypothesis on L.)

Lemma 2.1. $l > 0$.

Proof. If $l = 0$, then S_2 is an annulus cutting off a nontrivial knot exterior W from V_2. Because S_2 is incompressible, it follows from Van Kampen's theorem that the inclusion $\pi_1(W) \to \pi_1(V_2)$ is injective. This implies that $\pi_1(W)$ is free, which is a contradiction. □

Let $\{E, E'\}$ be a pair of disjoint disks in V_2 which cuts V_2 into a 3-ball. By Lemma 2.1, $\pi_1(S_2)$ is a free group of rank at least two. Therefore $S_2 \cap E \neq \emptyset$ (and $S_2 \cap E' \neq \emptyset$). We can assume that each component of $S_2 \cap E$ is an essential arc in S_2. We can also assume that ∂E meets each component of $\mathrm{cl}(\partial V_1 - \partial D_1^* \cup \partial D_2^* \cup \partial D_1 \cup \ldots \partial D_l)$ in essential arcs. Consider $S_2 \cap E$ in E. Each end of each arc is a point in ∂D_i^* or ∂D_i. Assign the label i to the end if it lies in ∂D_i and the label it i^* if it lies in ∂D_i^*. Thus, each arc of $S_2 \cap E$ has a pair of labels at its ends. Since (V_1, V_2) gives a Heegaard splitting of S^3, we may assume that ∂E runs over the 1-handle of V_1 separated by $D_1^* \cup D_2^*$ at least once. If ∂E does not meet ∂D_l, E would be a Seifert surface for K_1. This would imply that L is splittable. Thus ∂E has the label sequence $l, \ldots, 2, 1, 1, 2, \ldots, l$ at least once.

Lemma 2.2. *Any outermost arc of $S_2 \cap E$ in E is labeled* $\{1, 1\}$.

Proof. The alternatives are

(a) $\{1, 1\}$,
(b) $\{i, i+1\}$, $1 \leq i \leq l-1$,
(c) $\{l, l\}$, $l > 1$,
(d) $\{l, i^*\}$, $i = 1, 2$,
(e) $\{1^*, 2^*\}$,
(f) $\{i^*, i^*\}$, $i = 1, 2$.

Let α be an outermost arc of $S_2 \cap E$ in E. Let Δ be the outermost 2-gon cut off by α and let $\partial\Delta = \alpha \cup \beta$, where $\beta \subset \partial E$. In cases (b) and (d), we can use Δ to reduce $|\gamma \cap S|$.

In case (c), α separates ∂D_1^* and ∂D_2^* in S_2, otherwise a meridian disk of K_1 would be null-homotopic in $S^3 - K_1$. Take a disk δ bounded by α and a subarc α' of ∂D_l in S, such

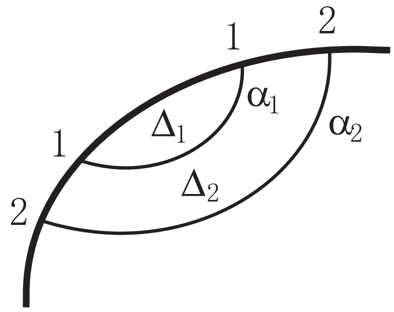

FIGURE 3

that $D_l \subset \delta$. Let $Q = (S - \mathrm{Int}\delta) \cup \Delta$. Then $\partial Q = \beta \cup \alpha'$, and it is isotopic to a meridian of $N(K_1)$ in ∂V_1. Capping off Q by a suitable meridian disk of $N(K_1)$ and isotoping it slightly from $N(\gamma)$, we obtain a new decomposing sphere S' with $|\gamma \cap S'| < |\gamma \cap S|$.

In case (e), consider the 3-ball B bounded by S such that $\Delta \subset B$. Since S is a decomposing sphere, $K_2 \subset \mathrm{Int}B$. Then this contradicts that L is unsplittable.

In case (f), let A be the annulus in ∂V_1 bounded by $\partial D_1^* \cup \partial D_2^*$, and let P be the 2-sphere with 3-holes in ∂V_1 bounded by $\partial D_1^* \cup \partial D_2^* \cup \partial D_l$. Any arc of $A \cap \partial E$ runs from ∂D_1^* to ∂D_2^*, and any arc of $P \cap \partial E$ runs from ∂D_1^* to ∂D_2^* or runs parallel to β. Thus ∂E would be non-compact. □

S separates S^3 into two 3-balls $\mathcal{B}$ and $\mathcal{W}$, called the black and white regions, respectively. We always assume that $K_2 \subset \mathcal{B}$.

Lemma 2.3. *Any outermost 2-gon in E lies in the black region $\mathcal{B}$.*

Proof. Consider an outermost 2-gon Δ in E. If Δ lies in the white region, then $l = 1$. The same argument as in case (c) of the proof of Lemma 2.2 works. □

Lemma 2.4. *K_2 is isotopic to a meridian of K_1 in $S^3 - K_1$.*

Proof. Let Δ be an outermost 2-gon in E. Let $\partial\Delta = \alpha \cup \beta$, where $\alpha \subset S_2 \cap E$, $\beta \subset \partial E$. If β does not run over $N(K_2) \subset V_1$, a meridian of K_2 would be null-homotopic in $S^3 - K_2$. Thus β runs over $N(K_2)$ and then $\partial N(K_2)$ is compressible in $S^3 - K_2$. Hence K_2 is unknotted, and β runs over $N(K_2)$ exactly once. Since L is unsplittable, α separates ∂D_1^* and ∂D_2^* in S_2. Thus there is a disk δ in $S - \mathrm{Int}D_1$ bounded by α and a subarc of ∂D_1 such that δ meets K_1 in a single point. From $\Delta \cup \delta$ we can easily find disk bounded by K_2, meeting K_1 in a single point. □

Thus L has the Hopf link summand. Since K_1 has tunnel number one, K_1 is prime [4, 5]. Hence S is the only decomposing sphere for L.

Lemma 2.5. $l = 1$.

Proof. Suppose that $l > 1$. Then there are an outermost arc α_1 and an arc α_2 of $S_2 \cap E$ in E, such that α_1 and α_2 are parallel and adjacent. By Lemma 2.2, α_1 is labeled $\{1, 1\}$, and hence α_2 is labeled $\{2, 2\}$. See Figure 3.

Let Δ_1 be the outermost 2-gon cut off by α_1 in E and Δ_2 the rectangle between α_1 and α_2 in E. Since Δ_1 lies in the black region by Lemma 2.3, Δ_2 lies in the white region. Then we can use $\Delta_1 \cup \Delta_2$ to reduce $|\gamma \cap S|$ by two. □

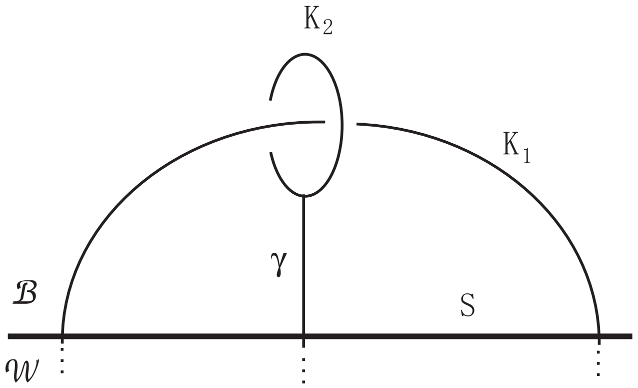

FIGURE 4

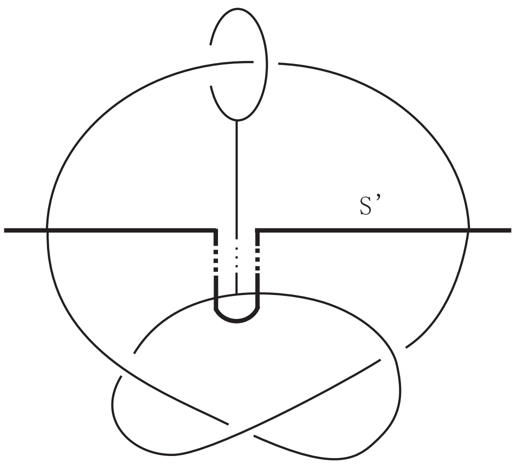

FIGURE 5

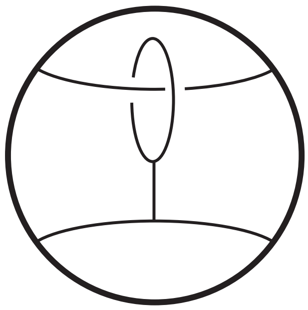

FIGURE 6

By Lemma 2.5, we can assume that the black region is as in Figure 4.

We change S to S' by an isotopy as drawn in Figure 5, that is, we push S in the direction of the white region along γ. Then S' meets K_1 in 4 points.

Let B_1 and B_2 be the 3-balls bounded by S' in S^3, with $K_2 \subset \mathrm{Int} B_2$. We may assume that $B_1 \cap V_2 = \mathcal{W} \cap V_2$. Then $(B_1, B_1 \cap L)$ is a 2-string tangle and $(B_2, B_2 \cap (L \cap \gamma))$ is equivalent to the "tangle" as shown in Figure 6.

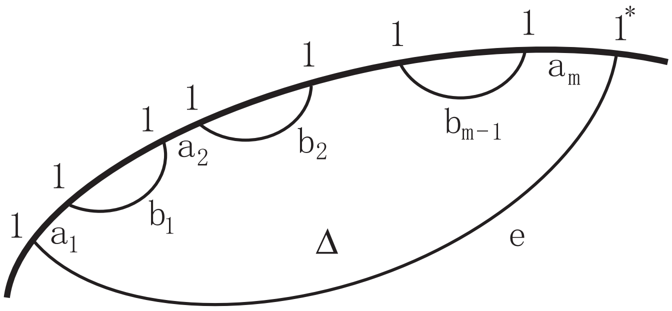

FIGURE 7

Lemma 2.6. $(B_1, B_1 \cap L)$ *is equivalent to a trivial tangle.*

Proof. Any arc of $S_2 \cap E$ in E is labeled $\{1,1\}$, $\{1,1^*\}$ or $\{1,2^*\}$, since S_2 is a 2-sphere with 3-holes and any arc of $S_2 \cap E$ is essential in S_2. Consider all the arcs of $S_2 \cap E$ in E labeled $\{1,1^*\}$ or $\{1,2^*\}$. Let e be an outermost one among them. We may assume that e is labeled $\{1,1^*\}$. Then there is a 2-cell Δ in E such that $\partial\Delta = e \cup a_1 \cup b_1 \cup a_2 \cup b_2 \cup \cdots \cup a_{m-1} \cup b_{m-1} \cup a_m$, where a_i are arcs in ∂E, b_i are arcs of $S_2 \cap E$, and $m \geq 2$ (see Figure 7).

Because a_m runs from ∂D_1^* to ∂D_1, Δ lies in the white region. By using Δ, one component of $B_1 \cap L$ can be pushed into ∂B_1. It is easy to see that $\pi_1(B_1 - B_1 \cap L) \cong \pi_1(\mathcal{W} \cap V_2)$ is a free group of rank two. Therefore, the other component of $B_1 \cap L$ must be an unknotted arc in B_1, and $(B_1, B_1 \cap L)$ is equivalent to a trivial tangle. □

Thus S' divides $L \cup \gamma$ into two tangles as in Figure 1. This completes the proof of Theorem 1.1.

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DEPARTMENT OF MATHEMATICS AND MATHEMATICS EDUCATION, HIROSHIMA UNIVERSITY, 1-1-1, KAGAMIYAMA, HIGASHI-HIROSHIMA 739-8524, JAPAN
E-mail address: teragai@hiroshima-u.ac.jp