Triple crossing numbers of graphs

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Information

This is a joint work with Horoyuki Tanaka.

A preprint “Triple crossing numbers of graphs” is available as arXiv:1002.4231.

Drawing of a graph

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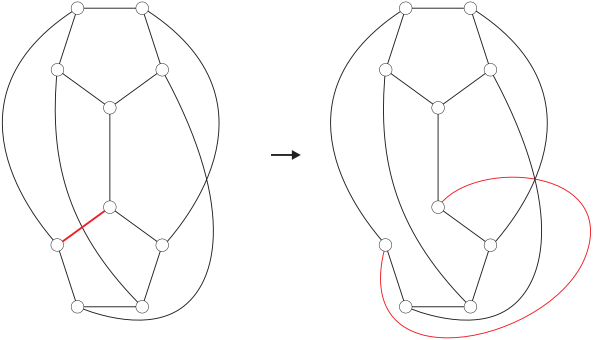
Clearly,

- $\operatorname{tcr}(G) = 0 \iff G$ is planar
- $\operatorname{cr}(G) \leq 3\operatorname{tcr}(G)$

An example: Petersen graph P

$\mathrm{cr}(P) = 2$, but $\mathrm{tcr}(P) = 1$.



Result

We determine the triple crossing numbers for all complete multipartite graphs as well as complete graphs.

Easy case

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Proof.

Assume that G has a drawing D with only triple crossings. If a new vertex is added to each triple crossing point, then we have a plane (simple) graph. The original vertices have degree at least $t - 1 \geq 6$, and the new vertices have degree 6, a contradiction. □

Algebraic criterion

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Assume G has p vertices and q edges. If G admits a drawing with only triple crossings, then $q \leq 3p - 6$.

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Proof.

Let D be such a drawing. Let k be the number of triple crossings of D. As before, we obtain a plane graph G' by adding a new vertex at each triple crossing. Then G' has $p + k$ vertices and $q + 3k$ edges. Hence,

$$q + 3k \leq 3(p + k) - 6.$$

□

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Proof.

Let G be a complete 5-partite graph K_{n_1,n_2,n_3,n_4,n_5} with $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq 1$. G has $p = \sum n_i$ vertices and $q = \sum_{i<j} n_i n_j$ edges. Then,

$$\begin{aligned} q - 3p + 6 &= (n_1 + n_4 - 3)(n_2 + n_3 - 3) + n_1 n_4 + n_2 n_3 \\ &\quad + n_5(n_1 + n_2 + n_3 + n_4 - 3) - 3 \\ &\geq (2n_4 - 3)^2 + 2n_4^2 + n_5 - 3 \geq 1. \end{aligned}$$

□

Complete graphs

Corollary

$$\mathrm{tcr}(K_n) = \begin{cases} 0 & \text{if } n \leq 4, \\ \infty & \text{otherwise.} \end{cases}$$

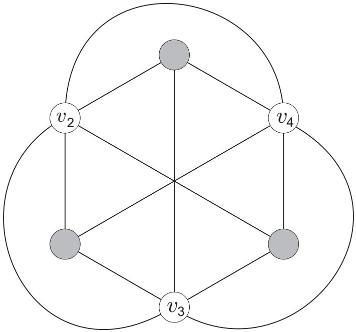
Complete 4-partite graphs

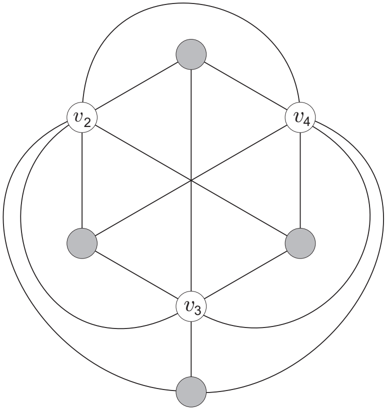
Theorem

Let G be a complete 4-partite graph K_{n_1,n_2,n_3,n_4} with $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$. Then $\mathrm{tcr}(G) = \infty$, except $K_{n_1,1,1,1}$ with $n_1 = 1, 2, 3, 4, 6$. For these exceptions,

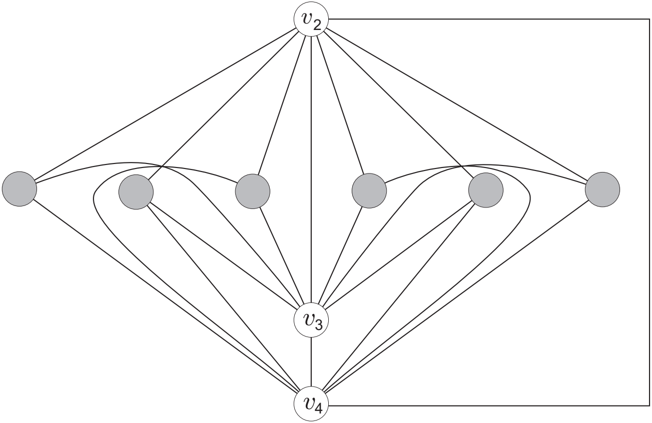
n_1	1, 2	3, 4	6
tcr	0	1	2

$K_{3,1,1,1}$ and $K_{4,1,1,1}$





$K_{6,1,1,1}$



Complete tripartite graphs

Theorem

Let G be a complete tripartite graph K_{n_1,n_2,n_3} with $n_1 \geq n_2 \geq n_3 \geq 1$.

1. *If $n_3 \geq 3$, then $\mathrm{tcr}(G) = \infty$.*
2. *If $n_3 = 2$, then $\mathrm{tcr}(G) = \infty$, except $K_{2,2,2}$ with $\mathrm{tcr} = 0$.*
3. *If $n_3 = 1$, then $\mathrm{tcr}(G) = \infty$, except $K_{3,3,1}$, $K_{6,2,1}$, $K_{4,2,1}$, $K_{3,2,1}$, $K_{2,2,1}$ and $K_{n_1,1,1}$. For these exceptions,*

	$(3,3,1)$	$(6,2,1)$	$(4,2,1)$	$(3,2,1)$	$(2,2,1)$	$(n_1,1,1)$
tcr	1	2	1	1	0	0

Complete bipartite graphs

Theorem

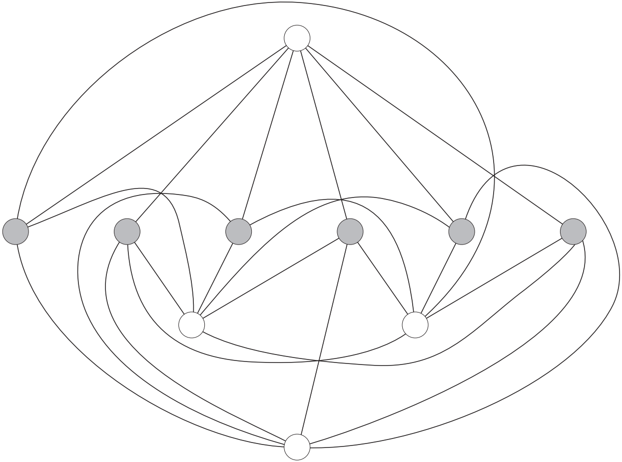
Let G be a complete bipartite graph K_{n_1,n_2} with $n_1 \geq n_2 \geq 1$.

1. *If $n_2 \leq 2$, then $\mathrm{tcr}(G) = 0$.*
2. *If $n_2 \geq 3$, then $\mathrm{tcr}(G) = \infty$, except $K_{3,3}$, $K_{4,3}$, $K_{6,3}$ and $K_{6,4}$.*

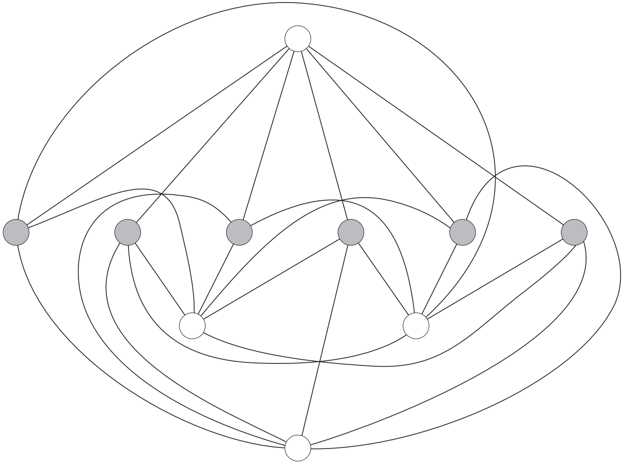
   For these exceptions,

	$(3,3)$	$(4,3)$	$(6,3)$	$(6,4)$
tcr	1	1	2	4

$K_{6,4}$



$K_{6,4}$



This shows $\mathrm{tcr}(K_{6,4}) \leq 4$. But, $\mathrm{cr}(K_{6,4}) = 12$ implies $\mathrm{tcr}(K_{6,4}) \geq 4$.

Tough graphs

Surprisingly, it is hard to show that $K_{5,4}$, $K_{4,4}$, $K_{5,3}$ and $K_{n,3}$ with $n \geq 7$ do not admit a drawing with only triple crossings.

It is good for neither one thing nor the other.

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Let $G = K_{5,4}$.
Assume G has a drawing with only k triple crossings.

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Assume G has a drawing with only k triple crossings.
By adding new vertices to triple crossings, we obtain a plane graph G' with $9 + k$ vertices and $20 + 3k$ edges.
Hence the faces of G' are 3-sided, except one 4-sided face, called the exceptional face.

A key

Let $V_1 = \{A, B, C, D\}$ and $V_2 = \{x_1, \ldots, x_5\}$ be the partite sets of G. The former is referred to as white vertices, and the latter as black vertices. The five edges at A are called A-lines. Similarly for others.

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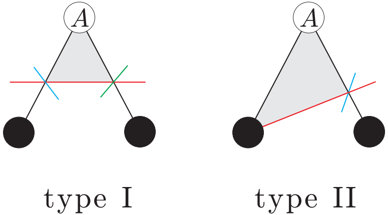
Since the exceptional face is incident with at most two white vertices, we can assume that vertex A is incident with only triangles.

Types of triangle

There are two types of triangles at A.



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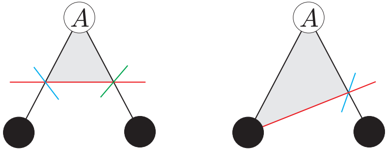
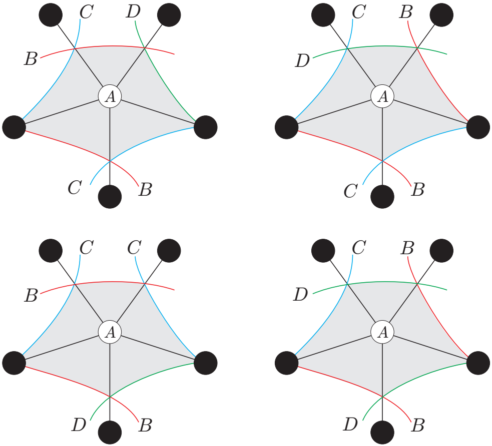


type I type II

Since type II triangles appear in pairs, the number of type II triangles at A is either 0, 2 or 4. We divide the proof, according to this number.

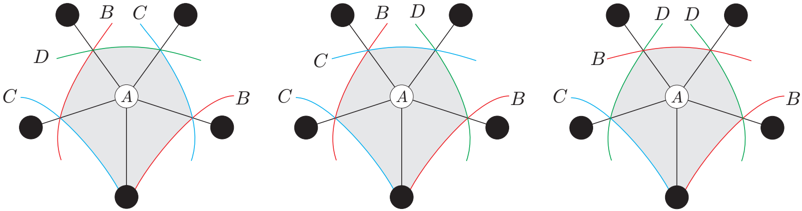
Four type II triangles at A

Up to symmetry and renaming, there are 4 subcases.



Two type II triangles at A

Up to symmetry and renaming, there are 3 subcases.



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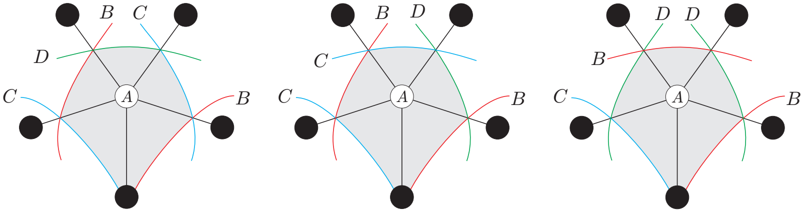
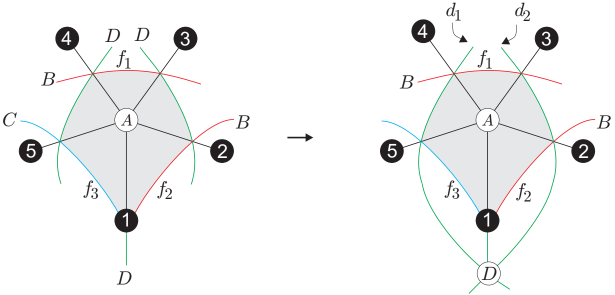


We demonstrate how the last configuration is excluded.

Demonstration 1



At least two among f_1, f_2, f_3 are 3-sided. But f_1 and f_2 cannot be 3-sided, simultaneously. Similarly for f_1 and f_3. Thus f_2 and f_3 are 3-sided.

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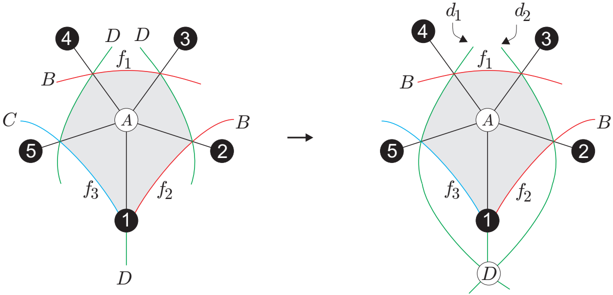
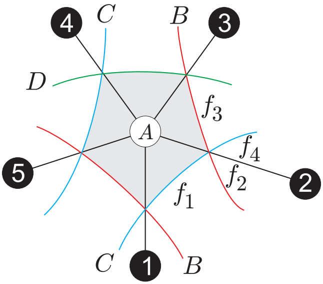


At least two among f_1, f_2, f_3 are 3-sided. But f_1 and f_2 cannot be 3-sided, simultaneously. Similarly for f_1 and f_3. Thus f_2 and f_3 are 3-sided.
Then d_1 goes to x_2 or x_3, and d_2 goes to x_4 or x_5, impossible.

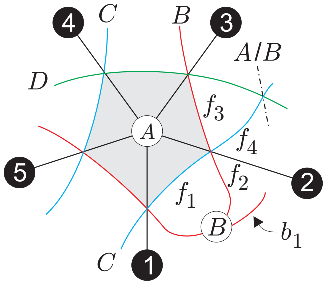
No type II triangles at A

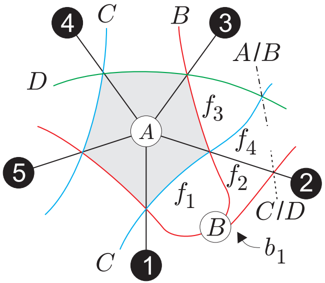
Up to symmetry and renaming, the local configuration at A is as below.



Demonstration 2

By symmetry, we can assume that $f_1, \ldots, f_4$ are 3-sided.





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2. Two faces of G' are 4-sided, and the others are 3-sided.

For the former case, there are 3 subcases, according to the number of type II triangles at A.

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For the former case, there are 3 subcases, according to the number of type II triangles at A.
For the latter case, there are 2 subcases:

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For the latter case, there are 2 subcases:

- all white vertices are incident with an exceptional face.

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For the former case, there are 3 subcases, according to the number of type II triangles at A.

For the latter case, there are 2 subcases:

- all white vertices are incident with an exceptional face.
- Some white vertex is not incident with an exceptional face.

$K_{n,3}$ with $n \geq 5$ and $n \neq 6$

There are 3 possibilities for exceptional faces of G':

1. G' has only one exceptional face, which is 6-sided.
2. G' has just two exceptional faces, which are 5-sided and 4-sided, resp.
3. G' has just three exceptional faces, which are 4-sided.

A comment

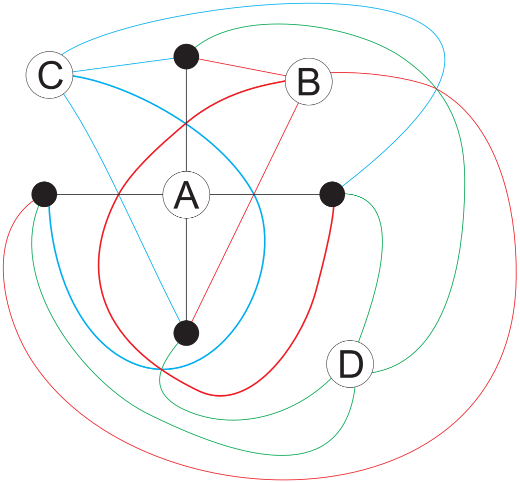
In this study, we require:

1. Two adjacent edges do not intersect;
2. Two edges intersect at most once.

It may be so strong that most complete multipartite graphs do not admit drawings with only triple crossings. If we relax it, then $K_{4,4}$, for example, admits a drawing with only triple crossings.

$K_{4,4}$

In this drawing of $K_{4,4}$, some two edges meet twice.



A generalization

For $n \geq 4$, we can define the n-fold crossing number similarly. In fact, all non-planar complete multipartite graphs do not admit drawings with only n-fold crossings.