TOROIDAL SURGERY ON HYPERBOLIC KNOTS

MASAKAZU TERAGAITO

Abstract. For a hyperbolic knot K in S^3, at most finitely many Dehn surgeries yield non-hyperbolic 3-manifolds. As a typical case of such an exceptional surgery, a toroidal surgery is one that yields a closed 3-manifold containing an incompressible torus. The slope corresponding to a toroidal surgery is known to be integral or half-integral. In this note, we review most known results concerning toroidal surgery, and present some new results and conjectures.

1. Introduction

Let K be a knot in the 3-sphere S^3, and let $E(K)$ be its exterior. For a slope r (the isotopy class of an essential simple closed curve) on $\partial E(K)$, the closed manifold obtained from r-surgery is denoted by $K(r)$. In the usual way ([28]), slopes are parameterized by the set $\mathbf{Q} \cup \{1/0\}$. Here, $1/0$ corresponds to the meridian slope. If a slope corresponds to an integer, then it is said to be *integral*, otherwise *non-integral*.

Suppose that K is hyperbolic, that is, the complement $S^3 - K$ admits a Riemannian metric of constant sectional curvature -1 which is complete and of finite volume. Then Thurston's hyperbolic Dehn surgery theorem implies that all but finitely many surgeries yield hyperbolic manifolds [32]. These exceptional cases are called exceptional surgeries. A closed 3-manifold is said to be *toroidal* if it contains an incompressible torus. If $K(r)$ is toroidal, then such surgery is referred to as a toroidal surgery, and the corresponding slope is also called a toroidal slope. Assuming the Geometrization Conjecture, if $K(r)$ fails to be hyperbolic then it is either reducible, toroidal, or a small Seifert fibered manifold ([13]). The famous cabling conjecture states that the first case never happen. For many classes of knots, this conjecture is confirmed. But the other two cases are not empty. The simplest example is the figure-eight knot. This has exactly three integral toroidal slopes 0, -4 and 4. The slope 0 is equal to the boundary slope of any Seifert surface. (In general, the closed surface obtained by capping a minimal genus Seifert surface off with a meridian disk of the attached solid torus is an incompressible, non-separating surface by Gabai [10]. Hence any genus one knot has toroidal slope 0.) Also, ± 4-surgery gives a graph manifold which is the union of two Seifert fibered manifolds over the disk with two exceptional fibers. For the figure-eight knot, it is known that the slopes $\pm 1, \pm 2, \pm 3$ yield small Seifert fibered manifolds. There is another interesting conjecture that only integral slope can yield a Seifert fibered manifold from hyperbolic knots ([13]).

In this note, we focus on toroidal surgery on hyperbolic knots. Although many known results and new ones are stated, their proofs will be omitted. In the rest of this section, we review some known facts concerning toroidal surgery.

The most important result is the next one showing a strong constraint on toroidal slopes.

Theorem 1.1 (Gordon-Luecke [14, 15]). *If $r = m/n$ is a toroidal slope for a hyperbolic knot in S^3, then $|n| \le 2$.*

2000 *Mathematics Subject Classification.* Primary 57M50; Secondary 57M25.

This means that a toroidal slope is either integral or half-integral. The half-integral case will be discussed in the next section.

Boyer and Zhang [1] showed that only 2-bridge knots and pretzel knots with three strands admit toroidal slopes among alternating knots, and the slope is an integer divisible by 4. Furthermore, according to the classification of Dehn surgery on 2-bridge knots by Brittenham and Wu [3], most 2-bridge knots do not have toroidal surgery as follows:

Theorem 1.2 (Brittenham-Wu [3])**.** *Let K be a hyperbolic 2-bridge knot in S^3. If K admits a toroidal slope r, then K corresponds to a continued fraction $[b_1, b_2] = 1/(b_1 - 1/b_2)$. Furthermore,*

1. *If $|b_1|, |b_2| > 2$ and both b_1 and b_2 are even, then $r = 0$.*
2. *If $|b_1|, |b_2| > 2$ and b_1 is odd, b_2 is even, then $r = 2b_2$.*
3. *If $b_1 = 2n$ ($|n| > 1$) and $b_2 = \pm 2$ (i.e., K is a twist knot but the figure-eight knot), then $r = 0, \mp 4$.*
4. *If $b_1 = 2$ and $b_2 = -2$ (i.e., K is the figure-eight knot), then $r = 0, \pm 4$.*

In (1), (3) and (4) of this theorem, the slope 0 obviously comes from a genus one Seifert surface. The other toroidal slopes come from the boundary slope of once-punctured Klein bottle bounded by the knot. Also, Patton [26] gave a description of what types of manifolds arise from toroidal Dehn surgery on 2-bridge knots. In particular, all are graph manifolds.

As noted in the next section, all hyperbolic knots with non-integral toroidal surgery have tunnel number one. In contrast with this, Eudave-Muñoz and Luecke [9] showed

Theorem 1.3 (Eudave-Muñoz and Luecke [9])**.** *For any integer n, there exists a hyperbolic knot K_n in S^3 such that K_n admits an integral toroidal slope and its tunnel number is at least n.*

In [25], Osoinach gave a remarkable construction and showed

Theorem 1.4. *There exist infinitely many hyperbolic knots K_i for $i = 1, 2, \dots$ in S^3 such that $K_i(0)$ is homeomorphic to the toroidal manifold obtained from 0-surgery on the connected sum of two figure-eight knots. Moreover, there is no upper bound for the minimal numbers $\{t_i\}$ of intersection between the core of the attached solid torus of $K_i(0)$ and an incompressible torus in $K_i(0)$.*

Also, Osoinach constructed an infinite family of hyperbolic knots such that their 0-surgeries give the same hyperbolic 3-manifold. This is the first result that infinitely many knots can yield the same manifold by a surgery of the same non-trivial slope.

There are some results on toroidal Seifert fibered surgery. First, Boyer and Zhang [2] proved that such slope must be integral. Recently, Gordon and Luecke [17], and Eudave-Muñoz [8] independently found infinitely many examples of hyperbolic knots which admit such surgery. Their examples yield Seifert fibered manifolds over the projective plane with two exceptional fibers. It is still open whether a Seifert fibered manifold over the sphere with more than three exceptional fibers can arise by surgery on a hyperbolic knot, or not. In the opposite direction, Motegi [24] showed that if the order of the symmetry group $\mathrm{Sym}^*(K)$ is greater than two, for example, if K admits two strong inversions, or a strong inversion and a cyclic period, then the knot K has no toroidal Seifert fibered surgery.

Miyazaki and Motegi [23] studied toroidal surgery on periodic knots. They showed that a toroidal surgery on a hyperbolic knot with period two is integral, and that a hyperbolic, periodic knot K with period $p > 2$ admits a toroidal slope r if and only if K has genus one and $p = 3$, $r = 0$.

2. Non-integral case

Eudave-Muñoz [6] constructed an infinite family of hyperbolic knots $k(l,m,n,p)$ that admit non-integral toroidal surgeries. In fact, either $n=0$ or $p=0$, and thus there are essentially two families $k(l,m,n,0)$ and $k(l,m,0,p)$ with three parameters. (There are some prohibited values for these parameters.) The most famous example is the $(-2,3,7)$-pretzel knot which is $k(3,1,1,0)$. He gave a family of prime tangles $B(l,m,n,p)$ of two strings such that some sums with rational tangles produce a trivial knot and a double composite knot. Taking a double branched cover of the tangle, the knot exterior $k(l,m,n,p)$ is obtained. Although there is an explicit description of the knots using a surgery presentation [6], that is a little complicated.

The Eudave-Muñoz knots are strongly-invertible, tunnel number one, and fibered [8]. Each of them admits a non-integral toroidal slope r as follows [8]:

(1) $r = l(2m-1)(1-lm) + n(2lm-1)^2 - \frac{1}{2}$ for $k(l,m,n,0)$;
(2) $r = l(2m-1)(1-lm) + p(2lm-l-1)^2 - \frac{1}{2}$ for $k(l,m,0,p)$.

The r-surgery yields a toroidal manifold which contains an incompressible torus meeting the attached solid torus in two meridian disk. Also, the resulting manifold is the union of two Seifert fibered manifolds over the disk with two exceptional fibers.

Surprisingly, Gordon and Luecke [16] proved that these are the only knots with non-integral toroidal surgeries.

Theorem 2.1 (Gordon-Luekce [16]). *Let K be a hyperbolic knot in S^3 that admits a non-integral toroidal surgery. Then K is one of the Eudave-Muñoz knots $k(l,m,n,p)$ and the toroidal slope is r described above.*

As Eudave-Muñoz wrote in [6], there are some repetitions among his knots. For example, $k(2,-1,n,0) = k(-3,-1,n,0) = k(2,2,0,n)$. Hence it would be a good problem to classify the knots.

Relating to this, we hit on a simple question: What is the simplest hyperbolic knot with non-integral toroidal slope? We here adopt the bridge index to measure the complexity of knots. Then it is well known that no 2-bridge knot admits non-integral toroidal surgery [18]. Independently of Theorem 2.1, we can show

Theorem 2.2 (Ishigami-Teragaito [22]). *The $(-2,3,7)$-pretzel knot is the only hyperbolic 3-bridge knot that admits non-integral toroidal surgery.*

The argument goes as follows. For a 3-bridge hyperbolic knot K (with a non-integral toroidal slope r), its 3-bridge position is a thin position. Hence, by Gabai's lemma [10], there is a level sphere $\widehat{Q}$ which intersects the punctured torus T, coming from an incompressible torus $\widehat{T}$ in $K(r)$, only in loops and essential arcs. The standard construction ([4]) gives a pair of labelled graphs G_Q and G_T on $\widehat{Q}$ and $\widehat{T}$, respectively. In particular, G_Q has just 6 vertices of degree 4, and G_T has just two vertices of degree 12. We analyze this pair combinatorially and topologically, and finally conclude that there are only two possibilities for the configuration of the pair. Then each configuration implies that the knot is the $(-2,3,7)$-pretzel knot (or its mirror image). In addition, the technique used here can be easily applied to show that no 2-bridge knot admits non-integral toroidal surgery.

Combined Theorem 2.2 with [33], we have

Corollary 2.3. *The $(-2,3,7)$-pretzel knot is the only pretzel knot that admits non-integral toroidal surgery.*

3. Bound in terms of genus

For non-integral case, it seems to be clear that not all half-integers appear as a toroidal slope for some Eudave-Muñoz knot. However, for integral case, we can show

Theorem 3.1 ([29]). *For any integer r, there exists a hyperbolic knot K in S^3 such that $K(r)$ is toroidal. Furthermore, K has bridge index at most 3 and tunnel number one.*

The construction is based on Dean's doubly-Seifert fibered construction [5]. In fact, our knots are twisted torus knots, and the resulting toroidal manifolds are graph manifolds, which are the union of two Seifert fibered manifolds over the disk with two exceptional fibers.

Although there is no universal bound for integral toroidal slopes, we propose the next conjecture based on many known examples.

Conjecture 3.2. *If r is an integral toroidal slope for a hyperbolic knot K in S^3, then $|r| \leq 4g(K)$, where $g(K)$ is the genus of K. Furthermore, if $|r| = 4g(K)$, then K bounds a once-punctured Klein bottle whose boundary slope is r.*

For example, the $(-2, 3, 7)$-pretzel knot has toroidal slopes 16 and 20. Recall that this knot has genus 5 and bounds a once-punctured Klein bottle whose boundary slope 20 as one of the checkerboard surfaces of its standard diagram as a pretzel knot. (Also, this knot bounds another once-punctured Klein bottle with boundary slope 16. But this is a very special phenomenon. See [20, 27].)

As the first step in this direction, we showed that the conjecture is true for two important classes of knots.

Theorem 3.3 ([30]). *Let K be a genus one hyperbolic knot in S^3. If r is a toroidal slope for K, then r is an integer and $|r| = 0, 1, 2$ or 4. Furthermore, if $|r| = 2$ or 4, then $K(r)$ contains an incompressible torus meeting the attached solid torus in two meridian disks. Also, if $|r| = 4$, then K is a twist knot and it bounds a once-punctured Klein bottle whose boundary slope is r.*

We note that the integers 0 and 4 are realized as toroidal slopes of the figure-eight knot, and the pretzel knot $9_{46} = P(-3, 3, 3)$ has toroidal slope 2, which yields a graph manifold. But it is open that the integer 1 can be realized by a genus one hyperbolic knot.

Theorem 3.4 ([30]). *Let K be an alternating hyperbolic knot in S^3. If r is a toroidal slope for K, then r is an integer and $|r| \leq 4g(K)$. Furthermore, if the equality holds, then K bounds a once-punctured Klein bottle whose boundary slope is r.*

Since then, we have confirmed Conjecture 3.2 for genus two knots, non-two-bridge Montesinos knots up to 10 crossings and Eudave-Muñoz's $K(\ell, n)$ [7].

The present status is the following.

Theorem 3.5. *Let K be a hyperbolic knot in S^3 with an integral toroidal slope r. Let t denote the minimal number of intersection between an incompressible torus in $K(r)$ and the core of the attached solid torus. Then if $t \neq 4$, then $|r| \leq 4g(K)$. If $t = 4$, then $|r| \leq 6g(K) - 3$.*

Note that t is even, since $\widehat{T}$ is separating when $r \neq 0$. When $t \geq 6$, the argument similar to that of [14] works well. The case $t = 2$ is complicated and lengthy. The temporary upper bound for case $t = 4$ is easy to get by using only Scharlemann cycles of length two. We expect that this part will be improved by a deeper analysis.

Daniel Matignon at Université de Provence informed that he got a similar upper bound.

In most examples of integral toroidal surgery, we see $t = 2$, indeed. But [7] gave infinitely many hyperbolic knots $K(\ell, n)$ with $t = 4$, and as stated in Theorem 1.4, t can be arbitrarily high.

Ichihara [19] gave such an upper bound in a broader category.

Theorem 3.6 (Ichihara [19])**.** *Let K be a hyperbolic knot in S^3. If $|r| > 3 \cdot 2^{7/4} g(K)$, then $K(r)$ is an irreducible 3-manifold with infinite and word-hyperbolic fundamental group.*

It is known that an irreducible 3-manifold with infinite and word-hyperbolic fundamental group is neither toroidal nor Seifert fibered. Thus if r is toroidal, then $|r| \leq 3 \cdot 2^{7/4} g(K) \approx 10.09\, g(K)$,

As a supporting evidence of Conjecture 3.2, we have a similar result on Klein bottle surgery.

Theorem 3.7 (Ichihara-Teragaito [21])**.** *Let K be a hyperbolic knot in S^3. If $K(r)$ contains a Klein bottle, then r is an integer divisible by four and $|r| \leq 4g(K)$. Furthermore, if $|r| = 4g(K)$, then K bounds a once-punctured Klein bottle whose boundary slope is r.*

This upper bound is sharp for each genus. Theorem 1.2 (2) shows such examples.

4. The number of toroidal surgeries and distance

In the paper [6] that gave an infinite family of hyperbolic knots $k(l, m, n, p)$ with non-integral toroidal surgeries, Eudave-Muñoz [6] proposed the following:

Conjecture 4.1. *A hyperbolic knot in S^3 admits at most three toroidal surgeries.*

As stated in Section 1, the figure-eight knot admits exactly three toroidal slopes 0 and ± 4. Among $k(l, m, n, p)$, the $(-2, 3, 7)$-pretzel knot, which is $k(3, 1, 1, 0)$, has three toroidal slopes $16, 20$ and $37/2$. Let us consider a hyperbolic knot K which is not an Eudave-Muñoz knot. If K is not the figure-eight knot, then K admits at most 6 integral toroidal surgeries by [11]. (For, there are only four hyperbolic 3-manifolds that admit two toroidal slopes with distance at least 6. Among them, the figure-eight knot exterior is the only one that can be embedded in the 3-sphere. Hence, if r and s are (integral) toroidal slopes for the knot exterior $E(K)$, then $|r - s| \leq 5$.) Also, any Eudave-Muñoz knot admits at most 5 toroidal slopes. (Use [11] again, and the fact that this knot has two integral (atoroidal) Seifert fibering slopes [6].) Thus any hyperbolic knot admits at most 6 toroidal slopes.

We show

Theorem 4.2 ([31])**.** *Let K be a hyperbolic knot in S^3, which is not the figure-eight knot. If K admits two integral toroidal slopes r and s, then $|r - s| \leq 4$.*

The argument is based on the combinatorial and geometrical analysis of a pair of labelled graphs coming from the intersection between two essential punctured tori. It heavily depends on the fact that both slopes are integral.

As a corollary of Theorem 4.2, we obtain a better upper bound for the number of toroidal slopes.

Corollary 4.3. *A hyperbolic knot in S^3 admits at most 5 toroidal surgeries.*

As a refinement of Conjecture 4.1, we propose

Conjecture 4.4. *If a hyperbolic knot in S^3 admits three toroidal surgeries, then it is either the figure-eight knot or the $(-2, 3, 7)$-pretzel knot.*

The next is another corollary of Theorem 4.2. This follows immediately from Theorem 2.1. Here, $\Delta(r, s)$ denotes the minimal geometric intersection number between two slopes r and s.

Corollary 4.5. *If a hyperbolic knot K in S^3 admits two toroidal slopes r and s with $\Delta(r, s) = 5$, then K is an Eudave-Muñoz knot.*

Among Eudave-Muñoz knots, the family $k(2, -1, n, 0)$, $n \neq 1$, seems to be the only one that realizes the distance 5, but this is an open question.

For the case $\Delta = 4$, we have another conjecture.

Conjecture 4.6. *If a hyperbolic knot K in S^3 admits two toroidal slopes r and s with $\Delta(r, s) = 4$, then K is either the $(-2, 3, 7)$-pretzel knot or a twist knot.*

Any twist knot $C[2n, \pm 2]$ in Conway's notation, except the trefoil, has toroidal slopes 0 and ± 4 as seen in Theorem 1.2.

The author would like to thank Mario Eudave-Muñoz for pointing out an error in the manuscript.

References

1. S. Boyer and X. Zhang, *Cyclic surgery and boundary slopes*, in Geometric topology (Athens, GA, 1993), 62–79, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.
2. S. Boyer and X. Zhang, *On Culler-Shalen seminorms and Dehn filling*, Ann. of Math. **148** (1998), 737–801.
3. M. Brittenham and Y. Q. Wu. *The classification of exceptional surgeries on 2-bridge knots*, Comm. Anal. Geom. **9** (2001), 97–113.
4. M. Culler, C. McA. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300.
5. J. Dean, *Small Seifert-fibered Dehn surgery on hyperbolic knots*,. Algebr. Geom. Topol. **3** (2003), 435–472.
6. M. Eudave-Muñoz, *Non-hyperbolic manifolds obtained by Dehn surgery on hyperbolic knots*, Geometric topology (Athens, GA, 1993), 35–61, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.
7. M. Eudave-Muñoz, *4-Punctured tori in the exteriors of knots*, J. Knot Theory Ramifications **6** (1997), 659–676.
8. M. Eudave-Muñoz, *On hyperbolic knots with Seifert fibered Dehn surgeries*, Topology Appl. **121** (2002), 119–141.
9. M. Eudave-Muñoz and J. Luecke, *Knots with bounded cusp volume yet large tunnel number*, J. Knot Theory Ramifications **8** (1999), 437–446.
10. D. Gabai, *Foliations and the topology of 3-manifolds, III*, J. Differential Geom. **26** (1987), 479–536.
11. C. McA. Gordon, *Boundary slopes of punctured tori in 3-manifolds*, Trans. Amer. Math. Soc. **350** (1998), 1713–1790.
12. C. McA. Gordon, *Dehn filling: a survey*, Knot theory (Warsaw, 1995), 129–144, Banach Center Publ. **42**, Polish Acad. Sci., Warsaw, 1998.
13. C. McA. Gordon, *Small surfaces and Dehn filling*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), 177–199 (electronic), Geom. Topol. Monogr. **2**, Geom. Topol. Publ., Coventry, 1999.
14. C. McA. Gordon and J. Luecke, *Dehn surgeries on knots creating essential tori, I*, Comm. Anal. Geom. **3** (1995), 597–644.
15. C. McA. Gordon and J. Luecke, *Dehn surgeries on knots creating essential tori, II*, Comm. Anal. Geom. **8** (2000), 671–725.
16. C. McA. Gordon and J. Luecke, *Non-integral toroidal Dehn surgeries*, preprint. (available from Gordon's website.)
17. C. McA. Gordon and J. Luecke, a private communication.
18. A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79** (1985), 225–246.
19. K. Ichihara, *Exceptional surgeries and genera of knots*, Proc. Japan Acad. Ser. A Math. Sci. **77** (2001), 66–67.

20. K. Ichihara, M. Ohtouge and M. Teragaito, *Boundary slopes of non-orientable Seifert surfaces for knots*, Topology Appl. **122** (2002), 467–478.
21. K. Ichihara and M. Teragaito, *Klein bottle surgery and genera of knots*, Pacific J. Math. **210** (2003), 317–333.
22. H. Ishigami and M. Teragaito, *The simplest hyperbolic knots with non-integral toroidal surgery*, J. Knot Theory Ramifications **12** (2003), 1023–1039.
23. K. Miyazaki and K. Motegi, *Toroidal surgery on periodic knots*, Pacific J. Math. **193** (2000), 381–396.
24. K. Motegi, *Dehn surgeries, group actions and Seifert fiber spaces*, Comm. Anal. Geom. **11** (2003), 343–389.
25. J. Osoinach, *Manifolds obtained by Dehn surgery on infinitely many distinct knots in* S^3, dissertation, the University of Texas at Austin, 1998.
26. R. Patton, *Incompressible punctured tori in the complements of alternating knots*, Math. Ann. **301** (1995), 1–22.
27. E. Ramírez-Losada and L. Valdez-Sánchez, *Once-punctured Klein bottles in knot complements*, to appear in Topology Appl.
28. D. Rolfsen, *Knots and links*, Mathematics Lecture Series, **7**, Publish or Perish, Inc., Berkeley, Calif., 1976.
29. M. Teragaito, *Toroidal surgeries on hyperbolic knots*, Proc. Amer. Math. Soc. **130** (2002), 2803–2808.
30. M. Teragaito, *Toroidal surgeries on hyperbolic knots, II*, Asian J. Math. **7** (2003), 139-146.
31. M. Teragaito, *Distance between toroidal surgeries on hyperbolic knots in the 3-sphere*, preprint.
32. W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University, 1978.
33. Y. Q. Wu, *Dehn surgery on arborescent knots*, J. Differential Geom. **3** (1996), 171–197.

Department of Mathematics and Mathematics Education, Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-hiroshima 739-8524, Japan
E-mail address: `teragai@hiroshima-u.ac.jp`