4次元多様体のコルクとその応用

(S. Akbulut (MSU) との共同研究)

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談話会

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§1. Intro.

Thm (Freedman) X, Y : simp. conn. closed C^∞- 4-mfd

$$Q_X \cong Q_Y \iff X \underset{\text{homeo}}{\approx} Y.$$

$$\left(\begin{array}{l} Q_X : H^2(X) \times H^2(X) \longrightarrow \mathbb{Z} \\ \text{intersection form} \quad (a, b) \longmapsto \langle a \cup b, [X] \rangle. \\ \text{対称双線型形式} \\ \text{unimodular} \iff \det Q_X = \pm 1 \end{array} \right)$$

<u>Fact</u> Q_X is determined by $b_2^+(X)$, $b_2^-(X)$, parity (odd, even)

$\Downarrow$

Donaldson's thm, 松本予想 (open)

$$\Rightarrow \quad X \underset{\text{homeo}}{\approx} \#_n \mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2} \quad \text{or} \quad \pm \left(\#_n (S^2 \times S^2) \#_m K3 \right)$$

$\exists$ exotic S^4, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $S^2 \times S^2$?

§2. Kirby diagram

<u>Fact</u> conn. closed C^∞ 4-mfd

$\underset{\text{diffeo}}{\cong}$ $D^4 \cup$ 1-handles $\cup$ 2-handles

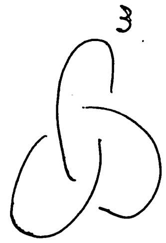
$\cup$ 3-handles $\cup$ a 4-handle.

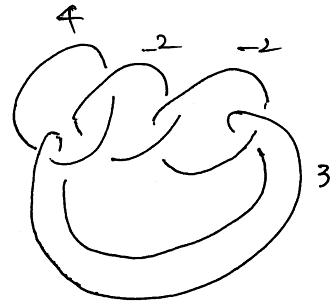
(i-handle : $(D^i \times D^{4-i}, \underbrace{\partial D^i \times D^{4-i}}_{\text{接着領域}})$

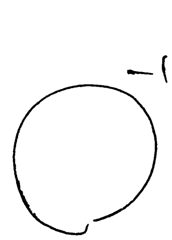
- Kirby diagram := 各ハンドルの接着領域を $\partial D^4 = \mathbb{R}^3 \cup \{\infty\}$ に描いたもの

- 2-handle : 接着領域 $S^1 \times D^2$

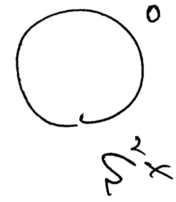
$S^1 \times 0$ と framing $\in \mathbb{Z}$





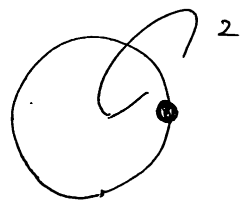


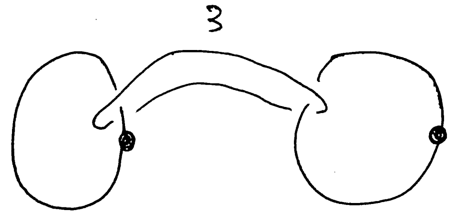
$\mathbb{C}P^2 - D^4$

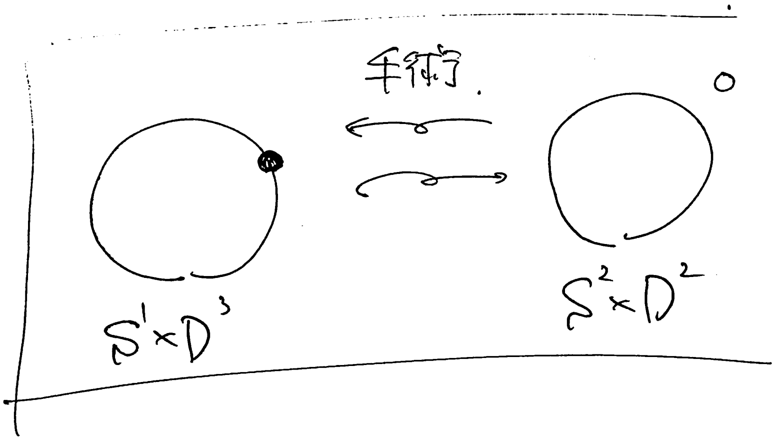


$S^2 \times D^2$

- 1-handle







§3 corks

C: contractible C^∞ 4-mfd.

$\tau: \partial C \longrightarrow \partial C$: involution

Thm (Freedman, Boyer)

τ extends to a self-homeo of C.

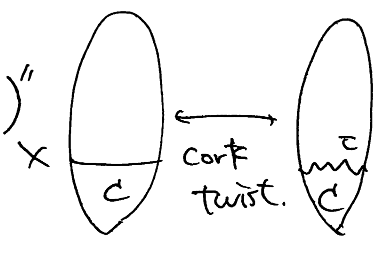
Def (1) (C,τ) : cork

$\underset{\text{def}}{\Longleftrightarrow}$ τ cannot extend to any self-diffeo of C

(2) X: C^∞ 4-mfd, $C\subset X$.

"cork twist of X (along (C,τ))"

$:= (X-C)\underset{\tau}{\cup} C$



(3) (C, τ) is a cork of X

$\Longleftrightarrow$ 「cork twist of X」 is (homeo but) not diffeo to X.

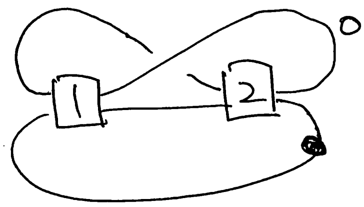
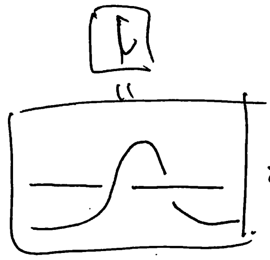
Thm (Matveyev et. al. '96)

X, Y : simp. conn. closed C^∞ 4-mfd

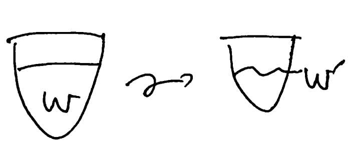
$\underset{\text{homeo}}{X \approx Y} \Longrightarrow {}^\exists$ cork (C, τ) s.t.

Y is diffeo to a cork twist of X along (C, τ)

<u>Example</u>

<u>Def</u> $W :=$   $\times i$

Mazur mfd

$f : \partial W \longrightarrow \partial W$ $\left(\begin{matrix} \bullet \longrightarrow \circ \\ \circ \longrightarrow \bullet \end{matrix} \right)$ 

$\cong W$

Thm (Akbulut '91, Bižaca − Gompf '96)

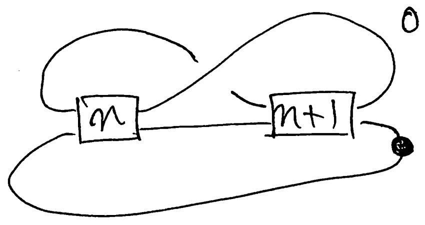
(W, f) is a cork of $E(n)\#\overline{\mathbb{CP}^2}$ $(n \geq 2)$

elliptic surface
$\chi(E(n)) = 12n$

($\because$ $E(n)\#\overline{\mathbb{CP}^2} \xleftrightarrow[\text{cork twist}]{} (2n-1)\mathbb{CP}^2 \# 10n\overline{\mathbb{CP}^2}$)

diffeoでない

<u>New example</u>

$W_n :=$ 

$f_n : \partial W_n \longrightarrow \partial W_n$

Thm (Akbulut − Yasui)

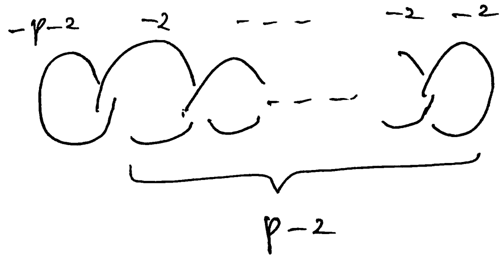
(W_n, f_n) is a cork of $E(2n)\#\overline{\mathbb{CP}^2}$ $(n \geq 1)$

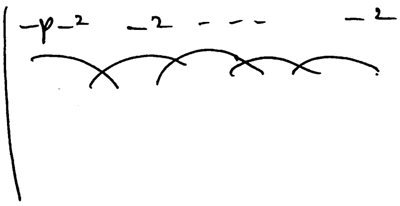
Q. corkとC^∞ strのより詳しい関係は?

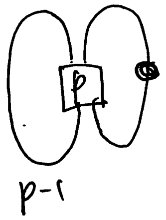
Q. corkと「良い」手術と関連づけよ.
サイバーグ・ウィッテン不変量 etc.

§. Disjointly embedded corks

Rational blowdown (Fintushel - Stern)

Def $C_p :=$ 



$B_p :=$ 

rational ball.

$$H_*(B_p ; \mathbb{Q}) \cong H_*(D^4 ; \mathbb{Q})$$

$$\partial C_p = \partial B_p$$

Def X : C^∞ 4-mfd, $C_p \subset X$.

$$X_{(p)} := (X - C_p) \underset{\partial}{\cup} B_p$$

∂のはり合わせ方によらない.

is called the rational blowdown of X.

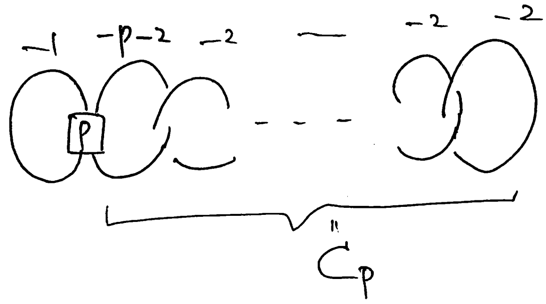
Lem

$$b_2^+(X_{(p)}) = b_2^+(X)$$

$$b_2^-(X_{(p)}) = b_2^-(X) - \underbrace{(p-1)}_{b_2^-(C_p)}$$

Rem $X_{(p)} \# (p-1)\overline{\mathbb{CP}^2} \underset{\text{homeo}}{\approx} X$?

とは限らない.

Df $D_p :=$ 

Thm (Akbulut - Yasui)

X : C^∞ 4-mfd. , $D_p \subset X$.

$\Rightarrow \overline{W}_{p-1} \overset{\exists}{\hookrightarrow} D_p$ s.t.

$X_{(p)} \# (p-1)\overline{\mathbb{CP}^2}$ is diffeo to the cork twist of X along $(\overline{W}_{p-1}, f_{p-1})$.

Thm (Akbulut - Yasui)

$\forall n \in \mathbb{N}$, $^\exists$corks $(C_1, \tau_1), \cdots, (C_n, \tau_n)$

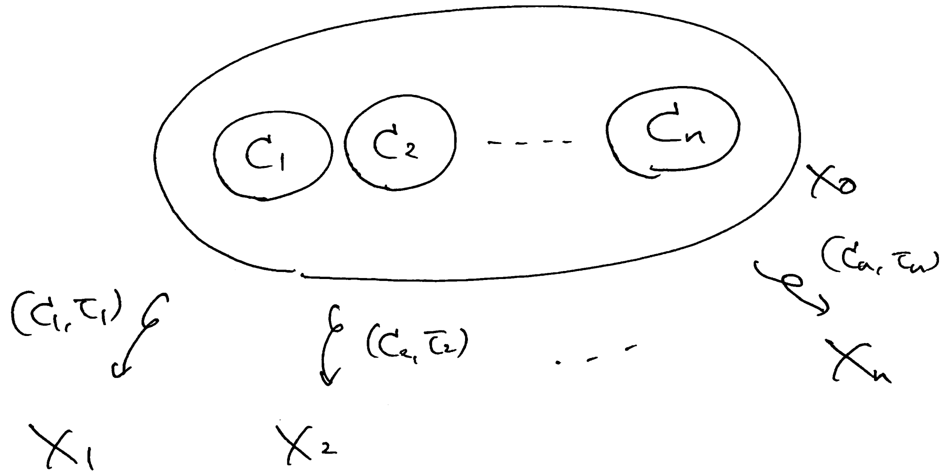
$^\exists X_0$: simp. conn. closed C^∞ 4-mfd

s.t. (1) $\forall i$, $C_i \overset{\exists}{\hookrightarrow} X_0$.

($X_i :=$ cork twist of X_0 along (C_i, τ_i))

(2) X_i and X_j are homeo but not diffeo $(i \neq j)$

(3) $C_1, \cdots, C_n$ are disjoint in X_0



Thm (Akbulut - Yasui)

$\exists$ simp. conn. closed C^∞ 4-mfd X_i. $(i \in \mathbb{N} \cup \{0\})$

$\exists$ cork (C, τ) s.t.

(1) X_i and X_j are homeo but not diffeo $({}^\forall i \neq {}^\forall j)$

(2) X_i is a cork twist of X_0 along (C, τ)