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安定性条件と数え上げ不変量

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Plan :

1. (Semi) stable sheaves
2. Donaldson - Thomas 不変量
3. Hall 代数と generalized DT invariants
4. 三角圏上の (弱) 安定性条件
5. K3 曲面上の不変量

X … smooth proj. C.Y. 3-fold $/\mathbb{C}$

i.e. $K_X = \mathcal{O}_X$, $H^1(\mathcal{O}_X) = 0$.

e.g. $X = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4_{\mathbb{C}}$

DT invariant

= " # of holomorphic vector bundles on X " ← 微分幾何的

安定な 連接層 ←-- "特異点をもった 正則ベクトル束" を表す 代数的対象

(Thomas, 1998)

Rank = 1 DT invariant

= curve counting invariant on X.

Gromov - Witten invariant $\overset{\text{conj}}{=}$ DT
$\in \mathbb{Q}$ (以下 GW) $\in \mathbb{N}$

MNOP 予想

(2004, Maulik - Nekrasov - Okounkov - Pandharipande)

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2008
Joyce - Song, Kontsevich - Soibelman

$\Downarrow$

- generalized DT invariant
- wall-crossing formula

$\Downarrow$

MNOP予想への応用.

∘ (Semi) stable sheaves

X ⋯ smooth projective variety $/\mathbb{C}$

$\mathcal{L} \longrightarrow X$: ample line bundle on X.

$$E \in Coh(X) \rightsquigarrow \underbrace{\chi(E \otimes \mathcal{L}^{\otimes n})}_{\text{Hilbert polynomial}} := \sum (-1)^i \dim H^i(X, E \otimes \mathcal{L}^n)$$

$$\overset{\text{Riemann-Roch}}{=} a_d n^d + a_{d-1} n^{d-1} + \cdots \in \mathbb{Q}[n]$$

$$= \int_X ch(E \otimes \mathcal{L}^{\otimes n}) \cdot t \, dt.$$

$a_d \neq 0$, $d = \dim \operatorname{Supp} E$

Ex: $\dim X = 1$, $\deg \mathcal{L} = 1$.

$$\Rightarrow \chi(E \otimes \mathcal{L}^{\otimes n}) = r \cdot n + d + r(1-g)$$

$g :=$ genus of X.
$r :=$ Eのrank
$d :=$ Eのdegree

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$\chi(E\otimes\mathcal{L}^{\otimes n})$ … Eのchern characterで書ける.

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Coh}(X) \arrow[r] \arrow[d, "\mathrm{ch}"'] & \mathbb{Q}[n] \\ H^*(X,\mathbb{Q}) \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

$$E \longmapsto \chi(E\otimes\mathcal{L}^{\otimes n})$$

$$H^*(X,\mathbb{Q}) \ni v \longmapsto \chi_{v,\mathcal{L}}(n)$$

s.t. $\chi(E\otimes\mathcal{L}^{\otimes n}) = \chi_{\mathrm{ch}E,\mathcal{L}}(n)$

Def Define

$$\mathbb{Q}[n]\setminus\{0\} \longrightarrow \mathbb{Q}[n]\setminus\{0\}$$
$$\chi \longmapsto \overline{\chi}$$

by $\underset{(\neq 0)}{a_d} n^d + a_{d-1}n^{d-1} + \cdots \longmapsto n^d + \dfrac{a_{d-1}}{a_d}n^{d-1} + \cdots$

Def $E\in\mathrm{Coh}(X)$ is $\mathcal{L}$-(semi) stable

$\Longleftrightarrow$ (i) E is pure (i.e. $\nexists F\subsetneq E$, $\dim\mathrm{Supp}\,F < \dim\mathrm{Supp}\,E$)

(ii) $\forall\, 0\neq F\subsetneq E$

$$\overline{\chi}_{\mathrm{ch}F,\mathcal{L}}(n) \underset{(\leqq)}{<} \overline{\chi}_{\mathrm{ch}E,\mathcal{L}}(n) \qquad n\gg 0$$

[$\left(\dfrac{a_{d-1}}{a_d}, \dfrac{a_{d-2}}{a_d}, \cdots, \dfrac{a_0}{a_d}\right)\in\mathbb{R}^d$, 辞書式順序で不等号, と同値]

Ex① $\dim X = 1$, E : torsion-free

$$\Rightarrow \overline{\chi}_{\mathrm{ch}E,\mathcal{L}}(n) = n + \frac{\deg E}{\mathrm{rank}\,E} + 1 - g$$

$\mu(E)$: Eのslope

E is $\mathcal{L}$-(semi) stable $\Leftrightarrow$ $\forall\, 0\neq F\subsetneq E$, $\mu(F) \underset{(\leqq)}{<} \mu(E)$

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$X = \mathbb{P}^1$

E is $\mathcal{L}$-semistable $\Longleftrightarrow$ $E = \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ $\quad {}^{\exists} a \in \mathbb{Z},\ {}^{\exists} r \in \mathbb{Z}_{\geq 1}$

E is $\mathcal{L}$-stable $\Longleftrightarrow$ $E = \mathcal{O}_{\mathbb{P}^1}(a)$, ${}^{\exists} a \in \mathbb{Z}$

(Grothendieck)

② $X = \mathbb{P}^2$

$\Rightarrow$ $\Omega_{\mathbb{P}^2}$ は $\mathcal{O}_{\mathbb{P}^2}(1)$-stable (Exercise)

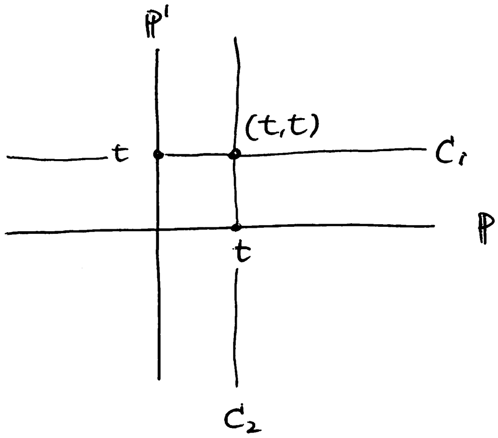
$T_{\mathbb{P}^2}$

③ $X = \mathbb{P}^1 \times \mathbb{P}^1$

$t \in \mathbb{P}^1$ fix

$C_1 := \mathbb{P}^1 \times \{t\}$

$C_2 := \{t\} \times \mathbb{P}^1$



$\mathrm{Ext}^1_X(\mathcal{O}_{C_2}, \mathcal{O}_{C_1}) = \mathbb{C}$.

$\Downarrow$

${}^{\exists}$ non-trivial exact sequence $0 \to \mathcal{O}_{C_1} \to E \to \mathcal{O}_{C_2} \to 0$

$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)$ $\quad a, b \in \mathbb{Z}_{>0}$.

$\overline{\chi}_{\mathrm{ch}\mathcal{O}_{C_1}, \mathcal{L}}(n) = n + \frac{1}{a}$

$\overline{\chi}_{\mathrm{ch}\mathcal{O}_{C_2}, \mathcal{L}}(n) = n + \frac{1}{b}$

$$E \text{ is } \begin{cases} \mathcal{L}\text{-stable} & \Longleftrightarrow \frac{1}{a} < \frac{1}{b} \Longleftrightarrow b < a \\ \mathcal{L}\text{-semistable} & \Longleftrightarrow b \leq a \\ \text{not } \mathcal{L}\text{-semistable} & \Longleftrightarrow b > a \end{cases}$$

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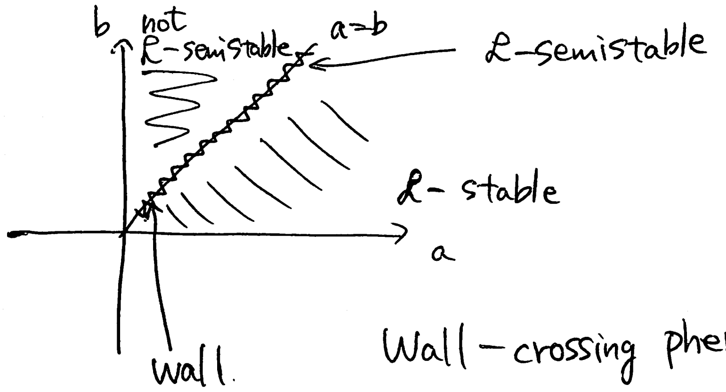

Wall-crossing phenomenon.

④ $\dim X = 3$.

$E \in \mathrm{Coh}\, X$, $\mathrm{rank}\, E = 1$.

Then, E is $\mathcal{L}$-stable Ⓐ

$\longleftrightarrow$ E is $\mathcal{L}$-semistable Ⓑ

$\longleftrightarrow$ E is torsion-free Ⓒ

$\longleftrightarrow$ $E \simeq M \otimes I_C$. Ⓓ

M : line bundle on X

$C \hookrightarrow X$ $\dim C = 1$

$I_C \hookrightarrow \mathcal{O}_X$: ideal sheaf of C

∵) Ⓐ $\Rightarrow$ Ⓑ $\Rightarrow$ Ⓒ

Ⓒ $\Rightarrow$ Ⓓ : $0 \to E \to E^{\vee\vee} \to F \to 0$.

isom on codim 1.

$E = M \otimes I_C$, $E^{\vee\vee} = M$. $\dim \mathrm{Supp}\, F \leq 1$.

rank 1, reflexive sheaf

$\Rightarrow$ line bundle ∎

Ⓓ $\Rightarrow$ Ⓐ

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⑤ X : smooth proj. 3-fold $/\mathbb{C}$.

$\mathrm{Pic}\, X = \mathbb{Z}[\mathcal{O}_X(1)]$

e.g. $X = \mathbb{P}^3$, $X \hookrightarrow \mathbb{P}^4$ gen. hypersurf.

<u>Thm</u> (Jardim) 2005 くらい

Suppose

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0.$$

- α injective
- β surjective
- $\beta \circ \alpha = 0$.

$\Rightarrow$ $\underbrace{K := \mathrm{Ker}\,\beta}_{\text{rank} = 2+c}$, $\underbrace{E := \mathrm{Ker}\,\beta / \mathrm{Im}\,\alpha}_{\text{rank} = 2}$ are $\mathcal{O}_X(1)$-stable

vector bundles on X

<u>e.g.</u> $X \hookrightarrow \mathbb{P}^4$ deg $= d$, gen. hypersurf.

$[x_0 : \cdots : x_4]$: homogenous coord. of $\mathbb{P}^4$

$$B_1 = \underbrace{\begin{pmatrix} x_0 & x_1 & & & \\ & x_0 & x_1 & & \\ & & \ddots & & \\ & & & x_0 & x_1 \end{pmatrix}}_{c+1} \Big\} c$$

$$B_2 = \underbrace{\begin{pmatrix} x_2 & x_3 & & & \\ & x_2 & x_3 & & \\ & & \ddots & & \\ & & & x_2 & x_3 \end{pmatrix}}_{c+1} \Big\} c$$

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$$A_1 = \underbrace{\begin{pmatrix} x_1 & & & \\ x_0 & x_1 & & \\ & x_0 & \ddots & \\ & & \ddots & x_1 \\ & & & x_0 \end{pmatrix}}_{c} \Big\} c+1$$

$$A_2 = \underbrace{\begin{pmatrix} x_3 & & & \\ x_2 & x_3 & & \\ & x_2 & \ddots & \\ & & \ddots & x_3 \\ & & & x_2 \end{pmatrix}}_{c} \Big\} c+1$$

$$\beta := (B_1 \ \ B_2), \quad \alpha := \begin{pmatrix} A_2 \\ -A_1 \end{pmatrix}$$

○ Moduli spaces (stack)

$$\mathrm{Coh}\,X : Sch/\mathbb{C} \longrightarrow \text{groupoid}$$

object of $\mathrm{Coh}\,X(S)$

$$\{ \mathcal{E} \in \mathrm{Coh}(X \times S) : \text{flat}/S \}$$

$$\mathrm{Isom}(\mathcal{E}_1, \mathcal{E}_2) := \{ \mathcal{E}_1 \xrightarrow{\cong} \mathcal{E}_2 \}$$

Fact　Coh X は局所有限型の Artin stack である.

(i.e. $^{\exists}M$: scheme, $M \xrightarrow{\exists} \mathrm{Coh}X$ smooth mor)

$$\mathrm{Coh}\,X = \coprod_{v \in \Gamma} \mathrm{Coh}_v X.$$

$$\Gamma := \mathrm{Im}(ch : \mathrm{Coh}\,X \longrightarrow H^*(X, \mathbb{Q}))$$

$\mathrm{Coh}_v X \hookrightarrow \mathrm{Coh}\,X$: substack of $E \in \mathrm{Coh}X$, $ch\,E = v$

⊙ $\Gamma := (ch : K(X) \longrightarrow H^*(X, \mathbb{Q}))$

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$\mathcal{L} \in \operatorname{Pic} X$

$$\mathfrak{M}^{s}_{v}(\mathcal{L}) \underset{\text{open}}{\hookrightarrow} \mathfrak{M}^{ss}_{v}(\mathcal{L}) \underset{\text{open}}{\hookrightarrow} \mathcal{C}oh\, X$$

substack of $\mathcal{L}$-semistable E, $\operatorname{ch} E = v$

substack of $\mathcal{L}$-stable E, $\operatorname{ch} E = v$

- separated

- finite type
- complete

<u>Fact</u> $\exists M_v(\mathcal{L})$: quasi-proj var. s.t.

$$\mathfrak{M}^{s}_{v}(\mathcal{L}) \longrightarrow M_v(\mathcal{L})$$

$B\mathbb{C}^*$-bundle ($\mathbb{C}^*$-gerb)

$\| $

$[\operatorname{Spec} \mathbb{C} / \mathbb{C}^*]$

Assume $\quad g.c.d \{ \chi_{v,\mathcal{L}}(n) : n \in \mathbb{Z} \} = 1.$

$v \in \Gamma$ (特に $v \in \Gamma$ は primitive)

$$\Rightarrow \mathfrak{M}^{s}_{v}(\mathcal{L}) = \mathfrak{M}^{ss}_{v}(\mathcal{L}) = [M_v(\mathcal{L}) / \mathbb{C}^*]$$

trivial action.

i.e. ① $\forall$ $\mathcal{L}$-semistable $= \mathcal{L}$-stable

② $\exists$ univ. family on $\underline{M_v(\mathcal{L})}$

- fine moduli space
- projective scheme