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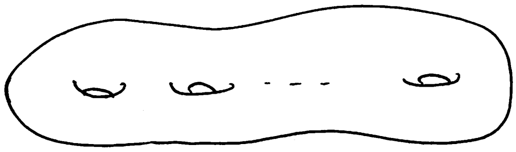
Donaldson - Thomas不変量と生成関数

談話会
(戸田さん(IPMU))

Algebraic curve = 1-dim projective variety $/\mathbb{C}$
(scheme)

e.g. Riemann surface



Calabi-Yau 3-fold X.

3-dim. smooth projective variety $/\mathbb{C}$

$K_X := \overset{3}{\Lambda} T_X^\vee \cong \mathcal{O}_X$

$c_1(X) = 0$.

e.g. $\{x_0^5 + x_1^5 + x_2^5 + x_0^5 + x_4^5 = 0\} \hookrightarrow \mathbb{P}^4_{\mathbb{C}}$

Want to "count" curve in X.

$\updownarrow$ Mirror Symmetry

Period integral of mirror manifold of X.

- $\exists$ 3 ways to count curves in X.

(i) Gromov-Witten theory (GW)

(ii) Donaldson-Thomas theory (DT) 1998

(iii) Pandharipande-Thomas theory (PT) 2007

(i) $\Leftrightarrow$ (ii) ? MNOP conj (2004)
Maulik-Nekrasov-Okounkov-Pandharipande

(ii) $\Updownarrow$ (iii) PT conj (2007)

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- Gromov–Witten theory

$g \geqq 0, \quad \beta \in H_2(X, \mathbb{Z})$

$$N^{GW}_{g,\beta} := \#\left\{ (C, f) : \begin{array}{l} C : \text{nodal curve of genus } g \\ f : C \to X, \ f_*[C] = \beta \\ \# \mathrm{Aut}(f) < \infty \end{array} \right\} \in \mathbb{Q}$$

$\uparrow$ Gromov–Witten invariant.

moduli = Deligne–Mumford stack.

- Donaldson–Thomas theory

$n \in \mathbb{Z}, \quad \beta \in H_2(X, \mathbb{Z})$

$$I_{n,\beta} := \#\left\{ C \hookrightarrow X : \begin{array}{l} \cdot \dim C \leqq 1 \\ \cdot [C] = \beta \\ \cdot \chi(\mathcal{O}_C) = n \end{array} \right\} \in \mathbb{Z}.$$

被約とは限らない.

Donaldson–Thomas invariant.

$$\cdot \ GW(X) := \exp\Big(\sum_{\substack{g \geqq 0 \\ \beta}} N^{GW}_{g,\beta} \lambda^{2g-2} t^{\beta} \Big)$$

$$\cdot \ DT(X) := \sum_{n,\beta} I_{n,\beta}\, q^n t^{\beta} \quad \Big(= \sum_{\beta} DT_{\beta}(X)\, t^{\beta} \Big)$$

$\uparrow$ series in q

Ex: $f : X \to Y$ bir. contr.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup & & \psi \\ \mathbb{P}^1 \simeq C_0 & \longmapsto & 0 \end{array}$$

$N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

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$$\Rightarrow \sum_{n,m} I_{n,m[C_0]}\, q^n t^m = \boxed{M(-q)^{\chi(X)}} \prod_{m\geq 1} \left(1-(-q)^m t\right)^n$$ (Behrend – Bryan)

$$DT_0(X) \qquad M(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n}$$

$$DT'(X) := DT(X)/DT_0(X) = \sum DT'_\beta(X)\, t^\beta$$

Laurent series in q

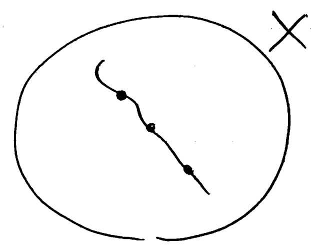
Ex2　Ex1と同じ状況で $DT'_{[C_0]}(X) = q - 2q^2 + 3q^3 - \cdots$

$$= \frac{q}{(1+q)^2}$$

- PT theory

Def　A stable pair is (F, s)

- F: pure 1-dim sheaf $\left(\begin{array}{l}\nexists Q \subset F \\ \dim \operatorname{Supp} Q = 0\end{array}\right)$
- $s: \mathcal{O}_X \to F$ surj in dim 1.



Ex　$C \hookrightarrow X$　smooth curve

$D \hookrightarrow C$　divisor

$F := \mathcal{O}_C(D)$.　$s: \mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \hookrightarrow \mathcal{O}_C(D)$

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$$P_{n,\beta} := \#\left\{ \text{Stable pairs } (F, s) \ : \ \begin{array}{c} [F] = \beta \\ \chi(F) = n \end{array} \right\} \in \mathbb{Z}.$$

$$PT(X) := \sum_{n,\beta} P_{n,\beta}\, q^n t^\beta \quad \left(=: \sum_\beta PT_\beta(X)\, t^\beta \right)$$

Conj (MNOP, PT)

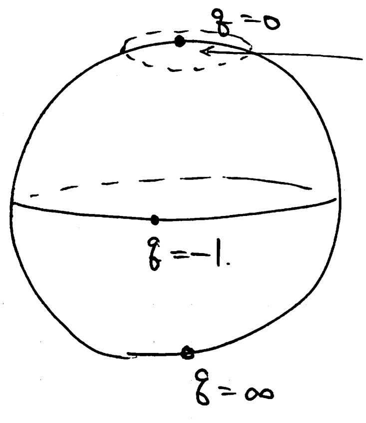
(i) $DT'_\beta(X) = PT_\beta(X)$

and they are rational fcn. of q, invariant under $q \leftrightarrow \frac{1}{q}$.

$$\frac{q}{(1+q)^2} = \frac{1/q}{(1+1/q)^2} = \frac{q}{(1+q)^2}$$

(ii) $GW(X) = DT'(X) = PT(X)$

via $q = -e^{i\lambda}$.



$q \leftrightarrow \frac{1}{q}$

$\Leftrightarrow \lambda \leftrightarrow -\lambda$.

Thm (Toda (Euler number version, 2008), Bridgeland (2010))

$\exists\, N_{n,\beta} \in \mathbb{Q}$, $L_{n,\beta} \in \mathbb{Q}$ satisfying

(i) $N_{n,\beta} = N_{-n,\beta}$, $L_{n,\beta} = L_{-n,\beta}$

(ii) $N_{n+d,\beta} = N_{n,\beta}$ for some $d > 0$, if $\beta \neq 0$

(iii) $L_{n,\beta} = 0$ for $|n| \gg 0$.

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s.t.

$$\bullet\ PT(X) = \prod_{\substack{n>0 \\ \beta>0}} \exp\left((-1)^{n-1} N_{n,\beta}\, q^n t^\beta\right)^n \cdot \left(\sum L_{n,\beta}\, q^n t^\beta\right)$$

$\updownarrow$

(β: effective class)

$$\bullet\ DT(X) = \underbrace{\prod_{n>0} \exp\left((-1)^{n-1} N_{n,0}\, q^n\right)^n}_{\substack{\parallel \\ M(-q)^{\chi(X)} = DT_0(X)}} PT(X)$$

<u>Cor</u> Conj (i) is true.

- $N_{n,\beta}$:= # of semistable sheaves F on X,

$$ch F = (0, 0, \beta, n) \in H^0 \oplus H^2 \oplus \underset{\substack{\cup\!\shortmid \\ H_2}}{H^4} \oplus H^6$$

- $L_{n,\beta}$:= # of semistable perverse coherent sheaves $E \in D^b Coh X$

$$ch E = (1, 0, -\beta, -n)$$

$$E \longmapsto R\mathcal{H}om(E, \mathcal{O}_X) \iff L_{n,\beta} = L_{-n,\beta}$$

Bezrukavnikov

Kashiwara

Conj (Gopakumar – Vafa)

$\exists\, n_{g,\beta} \in \mathbb{Z}$, $g \geq 0$, $\beta \in H_2$.

s.t. $$PT(X) = \prod_{\beta>0} \left\{ \prod_{j=1}^{\infty} \left(1-(-q)^j t^\beta\right)^{j n_{0,\beta}} \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} \left(1-(-q)^{g-1-k} t^\beta\right)^{(-1)^{k+g} n_{g,\beta} \binom{2g-2}{k}} \right\}$$

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<u>Cor</u> GV conj is equivalent to

$$N_{n,\beta} = \sum_{\substack{k \geq 1 \\ k \mid (n,\beta)}} \frac{1}{k^2} N_{1,\beta/k} \quad \cdots \quad (\star)$$

Thm (Toda (2011))

$(\star)$ is reduced to contribution of sheaves supported on tree of $\mathbb{P}^1$.