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集中講義
(戸田さん (IPMU))

X … smooth projective C.Y 3-fold $/\mathbb{C}$

$\mathcal{L} \longrightarrow X$: ample line bundle

$v \in \Gamma := \mathrm{Im}\,(ch : K(X) \to H^*(X, \mathbb{Q}))$

$B\mathbb{C}^* \curvearrowright \mathcal{M}^s_v \hookrightarrow \mathcal{M}^{ss}_v(\mathcal{L}) \hookrightarrow \mathrm{Coh}\, X.$

$\mathrm{g.c.d.}\{\chi_{v,\mathcal{L}}(n) : n \in \mathbb{Z}\} = 1$

$\Rightarrow \mathcal{M}^s_v(\mathcal{L}) = \mathcal{M}^{ss}_v(\mathcal{L}) = [M_v(\mathcal{L}) / \mathbb{C}^*]$

proj. var $/\mathbb{C}$

trivial

(☆)

Fact : $[E] \in M_v(\mathcal{L})$ i.e. E : $\mathcal{L}$-stable, $ch\,E = v$

$$\Longrightarrow T_{[E]} M_v(\mathcal{L}) = \mathrm{Ext}^1(E, E)$$

$u \in T_{[E]} M_v(\mathcal{L})$

$$\begin{tikzcd} \mathrm{Spec}\,\mathbb{C}[\varepsilon]/\varepsilon^2 \arrow[r, "u"] \arrow[d, hook] & M_v(\mathcal{L}) \\ \mathrm{Spec}\,\mathbb{C}[\varepsilon]/\varepsilon^3 \arrow[ur, "\exists u'"'] & \end{tikzcd} \qquad \bullet \longmapsto [E]$$

$\exists\, \mathcal{K} : \mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E)$

s.t. $\mathcal{K}(u) = 0 \iff u$ extends to 2nd order

expected dim. of $M_v(\mathcal{L})$ of $[E]$

$= \dim \mathrm{Ext}^1(E, E) - \dim \mathrm{Ext}^2(E, E)$

$\mathrm{Ext}^2(E,E) \cong \mathrm{Ext}^1(E, E)^\vee$ Serre duality

$= 0.$

⇓ Behrend - Fantechi, Li

$\exists$ 0-dim vir. cycle

$[M_v(\mathcal{L})]^{vir} \in A_0(M_v(\mathcal{L}), \mathbb{Z})$

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<u>Def</u> ☆ の仮定の下，

$$DT_v(\mathcal{L}) := \deg([M_v(\mathcal{L})]) \in \mathbb{Z}$$

<u>Ex</u>: $M_v(\mathcal{L})$: non-singular

$$\Longrightarrow \mathcal{O}bs := \bigcup_{[E]\in M_v(\mathcal{L})} \mathrm{Ext}^2(E,E) \underset{s\,:\,0\text{-section.}}{\overset{\text{vect. bundle } \Omega_{M_v(\mathcal{L})}}{\rightleftarrows}} M_v(\mathcal{L})$$

$$[M_v(\mathcal{L})]^{vir} = \text{“}s(M_v(\mathcal{L})) \cap s(M_v(\mathcal{L}))\text{” in } \mathcal{O}bs$$

$$= \int e(\Omega_{M_v(\mathcal{L})}) = (-1)^{\dim M_v(\mathcal{L})} \chi(M_v(\mathcal{L}))$$

<u>Thm</u> (Joyce-Song)

$\forall p \in M_v(\mathcal{L})$,

$p \in {}^{\exists}U \underset{\substack{\text{analitic}\\ \text{nbd}}}{\subset} M_v(\mathcal{L})$

・ゲージ理論を使って証明.

$$U \hookrightarrow {}^{\exists}V \;\cdots\; \text{cpx mfd}$$
$$\downarrow f \;\cdots\; \text{hol. fcn}$$
$$\mathbb{C}$$

s.t. $U = \{df = 0\}$

$$\dim V = \dim \mathrm{Ext}^1(E,E) \quad p=[E]$$

Behrend function

Define $\nu : M_v(\mathcal{L}) \longrightarrow \mathbb{Z}$ by

$$\nu(p) = (-1)^{\dim V}(1 - \chi(M_p(f))) \quad \text{well-defined}$$

$0 < \varepsilon \ll \delta \ll 1$

$$M_p(f) = \{ z \in V : \|z-p\| \leqq \delta, \; f(z) = f(p) + \varepsilon \}$$

norm of V

f の p における Milnor fiber

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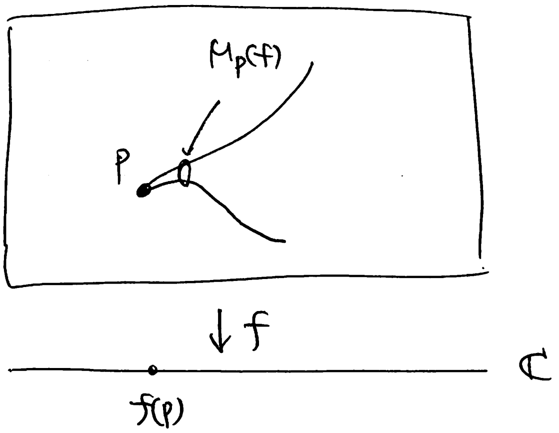


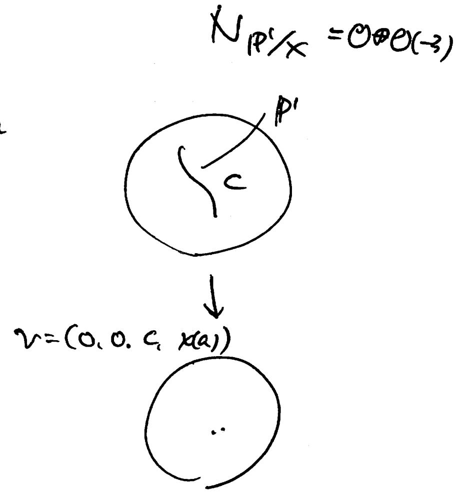
Thm (Behrend)

$$DT_v(\mathcal{L}) = \int \nu \cdot d\chi := \sum_{m \in \mathbb{Z}} m\chi(\nu^{-1}(m))$$

$M_v(\mathcal{L})$: smooth

$$\Longrightarrow \nu \equiv (-1)^{\dim M_v(\mathcal{L})}$$

$$\int \nu \cdot d\chi = (-1)^{\dim M_v(\mathcal{L})} \chi(M_v(\mathcal{L}))$$



Ex: Assume $M_v(\mathcal{L}) = \mathrm{Spec}\ \mathbb{C}[t]/t^n$

$$\begin{array}{ccc} M_v(\mathcal{L}) \hookrightarrow & \mathbb{C} \ni & z \\ & \downarrow f & \downarrow \\ M_0(f) & \mathbb{C} \ni & z^n \end{array}$$

$= (n+1)$ - points

$$\nu = (-1)^1 (1 - (n+1))$$
$$= n$$
$$= \text{length } \mathcal{O}_{M_v(\mathcal{L})}$$

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<u>Thm</u> (Thomas)

$DT_v(\mathcal{L})$ は X の complex str. の変形で不変

- Rank one theory

$$v = (1, 0, -\beta, -n) \in H^0 \oplus H^2 \oplus \underset{\underset{H_2}{\cup\!\!|}}{H^4} \oplus \underset{\underset{\mathbb{Z}}{\cup\!\!|}}{H^6}$$

$\Rightarrow$ ⛤ は常に成立

$$M_v(\mathcal{L}) = \left\{ I_C : \begin{array}{l} \dim C \leq 1 \\ \mathrm{ch}\, I_C = (1, 0, -\beta, -n) \end{array} \right\}$$

$$= \left\{ C \hookrightarrow X : \begin{array}{l} \dim C \leq 1 \\ [C] = \beta \\ \chi(\mathcal{O}_C) = n \end{array} \right\} \cdots \text{Hilbert scheme}$$

$$I_{n,\beta} := DT_{(1,0,-\beta,-n)}(\overset{\text{ample}}{\mathcal{L}}) \in \mathbb{Z}$$

$$DT_\beta(X) := \sum_{n \in \mathbb{Z}} I_{n,\beta}\, q^n$$

$$DT(X) := \sum_\beta DT_\beta(X)\, t^\beta$$

Ex ① $\beta = 0 \Rightarrow I_{n,0}$ # of length $= n$ 0-dim subschemes in X.

Local version $X = \mathbb{C}^3 \curvearrowleft T = (\mathbb{C}^*)^3$

$$I_{n,0}(\mathbb{C}^3) := \int_{\mathrm{Hilb}_n \mathbb{C}^3} \nu \cdot d\chi.$$

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T-localization

$$I_{n,0}(\mathbb{C}^3) = \int_{(\mathrm{Hilb}_n \mathbb{C}^3)^T} \nu \cdot d\chi.$$

$$(\mathrm{Hilb}_n \mathbb{C}^3)^T \overset{1:1}{\longleftrightarrow} \text{3-dim Young diagram } P \subset \mathbb{Z}^3_{\geq 0}$$
$$|P| = n$$

$$\underset{\substack{\text{gen. by monomial} \\ \cap \\ \mathbb{C}[x,y,z]}}{I_Z} \longmapsto P = \{(a,b,c) : x^a y^b z^c \notin I_Z\}$$

$$\nu|_{\mathrm{Hilb}_n \mathbb{C}^3} \equiv (-1)^n$$

$$DT_0(\mathbb{C}^3) = \sum_n (-1)^n \, \#\text{ of 3-dim Young図 } P,\ |P| = n \quad = M(-q)$$

$$M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n} \quad \longleftarrow \quad \text{物理でよく出る.}$$

Thm : (Behrend-Fantechi, Levine-Pandharipande, Li)

$$DT_0(X) = M(-q)^{\chi(X)}.$$

② $f : X \longrightarrow Y$, $C_0 \simeq \mathbb{P}^1$

$$\begin{matrix} X & \longrightarrow & Y \\ \cup & & \cup \\ C_0 & \longmapsto & 0 \end{matrix} \qquad N_{C_0/X} = \mathcal{O}(-1)^{\oplus 2}$$

Thm (Behrend-Bryan)

$$\sum I_{n,m[C_0]} q^n t^m = \underbrace{\boxed{M(-q)^{\chi(X)}}}_{= DT_0(X)} \prod_{m \geq 1} (1-(-q)^n t)^m$$

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$$DT'(X) := DT(X) / M(-q)^{\chi(X)}$$

$$= \sum_{\beta} DT'_{\beta}(X)\, t^{\beta}$$

Laurent series in q

• さっきの Ex

$$DT'_{[C_0]}(X) = q - 2q^2 + 3q^3 - \cdots = \frac{q}{(1+q)^2}$$

Conj (MNOP)

(i) $DT'_{\beta}(X)$ is the Laurent expansion of rational fcn. of q invariant under $q \longleftrightarrow \frac{1}{q}$.

(ii) $\exp\left(\sum N^{GW}_{g,\beta} \lambda^{2g-2} t^{\beta}\right) = DT'(X)$ $\quad q = -e^{i\lambda}$

GW inv.   makes sense by (i)

• Stable pair

Def A stable pair is (F, s)

- F : pure 1-dim
- $s : \mathcal{O}_X \longrightarrow F$: surj. on dim 1.

$P_n(X, \beta) :=$ moduli space of stable pairs (F, s), $[F] = \beta$, $\chi(F) = n$

proj. var.

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$$P_{n,\beta} := \int_{P_n(X,\beta)} \nu \cdot d\chi \quad \in \mathbb{Z}$$

$$PT(X) := \sum P_{n,\beta}\, q^n t^\beta = \sum_\beta PT_\beta(X)\, t^\beta$$

Conj (PT)

$$DT'_\beta(X) = PT_\beta(X)$$

(F, s) ··· stable pair

$$\Rightarrow \overset{I^\bullet}{\overbrace{\Big(\cdots \to 0 \to \underset{\deg = 0}{\mathcal{O}_X} \xrightarrow{s} \underset{\deg = 1}{F} \to 0 \to \cdots\Big)}} \in D^b\mathrm{Coh}\,X$$

Thm (Toda (Euler number version), Bridgeland (2010))

$\exists N_{n,\beta} \in \mathbb{Q}$, $\exists L_{n,\beta} \in \mathbb{Q}$

(i) $N_{n,\beta} = N_{-n,\beta}$, $L_{n,\beta} = L_{-n,\beta}$.

(ii) $N_{n+d,\beta} = N_{n,\beta}$, $\exists d > 0$ if $\beta \neq 0$.

(iii) $L_{n,\beta} = 0$ for $|n| \gg 0$.

s.t.
$$DT(X) = \prod_{\substack{n>0 \\ \beta>0}} \exp\left((-1)^{n-1} N_{n,\beta}\, q^n t^\beta\right)^n \left(\sum L_{n,\beta}\, q^n t^\beta\right)$$

$$PT(X) = \prod_{n>0} \exp\left((-1)^{n-1} N_{n,0}\, q^n\right)^n \cdot PT(X)$$

Cor · MNOP (i)
　 · PT conj
は正しい.

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$$N_{n,\beta} = DT_{(0,0,\beta,n)}(\mathcal{L})$$

$\uparrow$

☆を満たせば $N_{n,\beta} \in \mathbb{Z}$.

(i.e. $n=1$)

☆を満たさない.

e.g. $\mathfrak{M}^{s}_{v}(\mathcal{L}) \subsetneq \mathfrak{M}^{ss}_{v}(\mathcal{L})$

$\Longrightarrow \int_{\mathfrak{M}^{ss}_{v}(\mathcal{L})} \nu \, d\chi$ や $[M^{ss}_{v}(\mathcal{L})]^{vir}$ は

定義できない.

e.g. $M^{ss}_{v}(\mathcal{L}) = [\mathrm{Spec}\,\mathbb{C} / GL_n\mathbb{C}]$

$\Longrightarrow \chi(M^{ss}_{v}(\mathcal{L})) = \dfrac{1}{\chi(GL_n\mathbb{C})}$ "0".

$\nearrow$

代わりにHall代数を使って"log"をとる.

$L_{n,\beta} = DT_{(1,0,-\beta,-n)}(\tau)$

$\uparrow$

$\exists$ stab cond. on $D^b \mathrm{Coh}\, X$.