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集中講義
(戸田さん (IPMU))

X : smooth projective C-Y. 3-fold $/\mathbb{C}$

$v \in \Gamma := \mathrm{Im}(ch : K(X) \longrightarrow H^*(X, \mathbb{Q}))$

$\mathcal{L} \longrightarrow X$: ample line bundle

$\mathcal{M}_v^s(\mathcal{L}) \subset \mathcal{M}_v^{ss}(\mathcal{L})$

$(\star)$: $\mathcal{M}_v^s(\mathcal{L}) = \mathcal{M}_v^{ss}(\mathcal{L}) = [M_v(\mathcal{L})/\mathbb{C}^*]$

$\quad \ni [E]$, $\mathrm{Aut}\, E = \mathbb{C}^*$

e.g. $\mathrm{Aut}(\mathcal{O}_X^{\oplus r}) = GL_r(\mathbb{C})$

Today : define $DT_v(\mathcal{L}) \in \mathbb{Q}$ without $(\star)$

- Grothendieck group of varieties

S : $\mathbb{C}$-variety

Def $K_0(\mathrm{Var}/S) := \bigoplus \mathbb{Z}[\rho : Y \to S]/\sim$

- Y : quasi-proj. var.
- $[Y \xrightarrow{\rho} S] \sim [Z \xrightarrow{\rho|_Z} S] + [Y \setminus Z \xrightarrow{\rho|_{Y\setminus Z}} S]$

  $Z \underset{\text{closed}}{\hookrightarrow} Y$ subvar.

- $[Y_1 \to S] \sim [Y_2 \to S]$ if $\exists$
$$\begin{tikzcd} Y_1 \arrow[rr, "\sim"] \arrow[dr] & \circlearrowleft & Y_2 \arrow[dl] \\ & S & \end{tikzcd}$$

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$$K_0(\mathrm{Var}) := K_0(\mathrm{Var}/\mathrm{Spec}\,\mathbb{C})$$

<u>Fact</u> : $K_0(\mathrm{Var})$ is generated by $[Y] := [Y \to \mathrm{Spec}\,\mathbb{C}]$

Y: smooth proj. var. $/\mathbb{C}$

Relation is generated by

$$[\hat{Y}] - [E] \sim [Y] - [Z] \quad ⓪$$

Ex locus

$$\begin{array}{ccc} E & \hookrightarrow & \hat{Y} \\ \downarrow & & \downarrow \text{ blow-up of } Z. \\ Z & \hookrightarrow & Y \end{array}$$

Smooth closed

Virtual Poincaré polynomial

$$P_t : K_0(\mathrm{Var}) \longrightarrow \mathbb{Z}[t]$$

Y: smooth proj. var.

i-th Betti数

$$\Rightarrow \quad P_t(Y) := \sum (-1)^i b_i(Y) t^i$$

compatible with ⓪

$\forall$ quasi projective var. Y,

$$\lim_{t \to 1} P_t(Y) = \chi(Y).$$

- Stack version :

$\mathcal{S}$: Artin stack $/\mathbb{C}$, with aff. geometric stabilizers

$$K_0(\mathrm{St}/\mathcal{S}) := \bigoplus \mathbb{Q}[\rho : \mathcal{Y} \to \mathcal{S}]/\sim$$

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- $\mathcal{Y}$: Artin stack of finite type $/\mathbb{C}$
  with geometric affine stabilizers

$\forall x \in \mathcal{Y}(\mathbb{C})$,
$Aut(x)$: aff. alg. gp

- $[\mathcal{Y} \xrightarrow{\rho} \mathcal{S}] \sim [\mathcal{Z} \xrightarrow{\rho|_{\mathcal{Z}}} \mathcal{S}] + [\mathcal{Y} \setminus \mathcal{Z} \xrightarrow{\rho|_{\mathcal{Y} \setminus \mathcal{Z}}} \mathcal{S}]$
  $\mathcal{Z} \hookrightarrow \mathcal{Y}$ closed substack

- $[\mathcal{Y}_1 \to \mathcal{S}] \sim [\mathcal{Y}_2 \to \mathcal{S}]$

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[rr, "\sim"] \arrow[dr] & \circlearrowleft & \mathcal{Y}_2 \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}$$

Pull-back, Push-forward

$f : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$: 1-morphism of stacks

Artin stacks with aff. geom. stab.

- $f_* : K_0(St/\mathcal{S}_1) \longrightarrow K_0(St/\mathcal{S}_2)$
  $f_*[\mathcal{Y} \xrightarrow{\rho} \mathcal{S}_1] = [\mathcal{Y}_2 \xrightarrow{f \circ \rho} \mathcal{S}_2]$

- f : fin. type
$\Rightarrow f^* : K_0(St/\mathcal{S}_2) \longrightarrow K_0(St/\mathcal{S}_1)$
  $f^*[\mathcal{Y} \xrightarrow{\rho} \mathcal{S}_2] = [\mathcal{Y} \times_{\mathcal{S}_2} \mathcal{S}_1 \to \mathcal{S}_1]$

$$\begin{tikzcd} \mathcal{Y} \times_{\mathcal{S}_2} \mathcal{S}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d] \\ \mathcal{S}_1 \arrow[r, "\text{fin}"'] & \mathcal{S}_2 \end{tikzcd}$$

fin type.

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<u>Lem</u>: $\exists P_t : K_0(St) := K_0(St/_{\mathrm{Spec}\,\mathbb{C}}) \longrightarrow \mathbb{Q}(t)$

s.t. $P_t([Y/GL_n(\mathbb{C})]) = \dfrac{P_t(Y)}{P_t(GL_n(\mathbb{C}))}$

morphism of $\mathbb{Q}$-vect. sp.

Y ... quasi-proj. var.

$GL_n(\mathbb{C}) \curvearrowright Y$.

$\because$) <u>Kresch</u>:

$K_0(St)$ is gen. by $[Y/GL_n(\mathbb{C})]$

$\forall [\mathcal{Y}] \in K_0(St)$ is stratified by

$\mathcal{Y} = \coprod [Y_i / GL_{n_i}(\mathbb{C})]$

quasi-proj. var.

$= \coprod [\tilde{Y}_i / GL_{\tilde{n}_i}(\mathbb{C})]$

D-M stack $\subset K_0(St)$

<u>e.g.</u> $\mathcal{Y} = [\mathrm{Spec}\,\mathbb{C} / \mathbb{Z}/m\mathbb{Z}]$

$= [\mathbb{C}^{\times} / \mathbb{C}^{\times}]$ wt $= m$

$P_t(\mathcal{Y}) = \dfrac{P_t(\mathbb{C}^{\times})}{P_t(\mathbb{C}^{\times})} = 1$

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- Hall algebra

X : smooth proj. C-Y. 3-fold $/\mathbb{C}$

$\rightsquigarrow \mathrm{Coh}\, X$

<u>Def</u> $H(X) := K_0(\mathrm{St}/\mathrm{Coh}\, X)$

$\exists$ $*$-product. based on Ringel-Hall alg.

$\mathcal{E}_X$: moduli stack of all exact seq in Coh X

$$\begin{array}{ccccccccc} 0 & \to & E_1 & \to & E_2 & \to & E_3 & \to & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \to & E_1' & \to & E_2' & \to & E_3' & \to & 0 \end{array}$$

Artin stack of local finite type $/\mathbb{C}$

$$P_i : \mathcal{E}_X \longrightarrow \mathrm{Coh}\, X.$$
$$E_\bullet \longmapsto E_i$$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & \mathcal{E}_X \arrow[r, "P_2"] \arrow[d, "(P_1,P_3)", "\text{fin. type}"'] & \mathrm{Coh}\, X \\ \mathcal{Y}_1 \times \mathcal{Y}_2 \arrow[r] & \mathrm{Coh}\, X \times \mathrm{Coh}\, X & \end{tikzcd}$$

$x_i = (\mathcal{Y}_i \xrightarrow{P_i} \mathrm{Coh}\, X)$
$i=1,2.$

$$x_1 \otimes x_2 \overset{\iota}{\mapsto} (x_1, x_2) \in K_0(\mathrm{St}/\mathrm{Coh}\, X \times \mathrm{Coh}\, X)$$

<u>Def</u> : define

$$* : H(X)^{\otimes 2} \longrightarrow H(X) \quad \text{by}$$

$$* = P_{2*}(P_1, P_3)^* \iota$$

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Prop (Joyce)

$*$ is an associative product with unit

$$[\mathrm{Spec}\,\mathbb{C} \longrightarrow \mathrm{Coh}\,X]$$
$$\bullet \longmapsto 0 \in \mathrm{Coh}\,X.$$

Ex: $E_1, E_2 \in \mathrm{Coh}\,X$.

$$x_i := [\mathrm{Spec}\,\mathbb{C} \longrightarrow \mathrm{Coh}\,X] \in H(X)$$
$$\bullet \longmapsto E_i$$

$$x_1 * x_2 = \left[[\mathrm{Ext}^1(E_2,E_1) / \mathrm{Hom}(E_2,E_1)] \longrightarrow \mathrm{Coh}\,X \right]$$

trivial

$u \in \mathrm{Ext}^1(E_2, E_1)$

$\wr$

$0 \to E_1 \to E_u \to E_2 \to 0$

$u \longmapsto E_u$

$P_t : K_0(St) \to \mathbb{Q}(t)$

$\Rightarrow P_t : H(X) \to K_0(St) \to \mathbb{Q}(t)$

$(\mathcal{Y} \to \mathrm{Coh}\,X) \mapsto [\mathcal{Y}]$

- Elements $\delta_v(\mathcal{L})$, $\epsilon_v(\mathcal{L})$

$\mathcal{L} \xrightarrow[\text{ample}]{} X$, $v \in \Gamma$

$$\Rightarrow \delta_v(\mathcal{L}) := [\mathcal{M}_v^{ss}(\mathcal{L}) \hookrightarrow \mathrm{Coh}\,X] \in H(X)$$

Remark $P_t(\delta_v(\mathcal{L})) \in \mathbb{Q}(t)$

If ✡ is satisfied

$$\Rightarrow P_t(\delta_v(\mathcal{L})) = P_t([M_v(\mathcal{L})/\mathbb{C}^*]) = \frac{P_t(M_v(\mathcal{L}))}{P_t(\mathbb{C}^*)}$$

$P_t(\mathbb{C}^*) = t^2 - 1$

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$$\chi(M_v(\mathcal{L})) = \lim_{t\to 1}(t^2-1)\,P_t(\delta_v(\mathcal{L}))$$

㊟が成立しなければ, $\lim_{t\to 1}$ は存在しない.

<u>Def</u>
$$\epsilon_v(\mathcal{L}) := \sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{v_1,\dots,v_l\in\Gamma \\ \bar{\chi}_{v_i,\mathcal{L}}(n)=\bar{\chi}_{v,\mathcal{L}}(n) \\ v_1+\dots+v_l=v}} \delta_{v_1}(\mathcal{L})*\delta_{v_2}(\mathcal{L})*\dots*\delta_{v_l}(\mathcal{L})$$

$\hat{H}(X)$

reduced Hilbert polynomial

finite sum.

<u>Note</u>: ㊟ $\Longrightarrow$ $\epsilon_v(\mathcal{L}) = \delta_v(\mathcal{L})$

i.e, $\forall\, p(n)\in\mathbb{Q}[n]$,

$$\sum_{v:\bar{\chi}_{v,\mathcal{L}}(n)=p(n)}\epsilon_v(\mathcal{L}) = \log\Big(1+\sum_{v:\bar{\chi}_{v,\mathcal{L}}(n)=p(n)}\delta_v(\mathcal{L})\Big) \in \widehat{H(X)}$$

<u>Thm (Joyce)</u>

$$\underbrace{\lim_{t\to 1}(t^2-1)\,P_t(\epsilon_v(\mathcal{L}))}_{=\,DT^{\chi}_v(\mathcal{L})} \in \mathbb{Q} \quad \text{exists.}$$

"χ"$(\mathfrak{M}^{ss}_v(\mathcal{L}))$ と思える.

$\exists\,\nu : \mathrm{Coh}\,X \longrightarrow \mathbb{Z}$: local constructable fcn.

Behrend fcn

local に $[\{df=0\}/G]$ の形.

$\boxed{\nu|_{M_v(\mathcal{L})} = -\nu|_{[M_v(\mathcal{L})/\mathbb{C}^*]}}$

$\nu\cdot H(X) \longrightarrow H(X)$

$$[\mathcal{Y}\xrightarrow{\rho}\mathrm{Coh}\,X] \mapsto \sum_{m\in\mathbb{Z}} m\,[(\nu\circ\rho)^{-1}(m)\longrightarrow \mathrm{Coh}\,X]$$

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TITLE

Def. $DT_v(\mathcal{L}) := -\lim_{t\to 1} (t^2-1) P_t(v \cdot \epsilon_v(\mathcal{L})) \in \mathbb{Q}$