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①

| 集中講義
(FIZL (IPMUS))

X : smooth projective \mathbb{C} - γ . 3-fold / \mathbb{C}

$v \in \Gamma := \text{Im}(ch: K(X) \rightarrow H^*(X, \mathbb{Q}))$

$\mathcal{L} \rightarrow X$: ample line bundle

$$M_v^s(\mathcal{L}) \subset M_v^{ss}(\mathcal{L})$$

$$\textcircled{\star}: M_v^s(\mathcal{L}) = M_v^{ss}(\mathcal{L}) = [M_v(\mathcal{L}) / \mathbb{C}^*]$$

$$\downarrow$$

$$[E]$$

$$\text{Aut } E = \mathbb{C}^*$$

e.g. $\text{Aut}(\mathcal{O}_X^{\oplus r}) = GL_r(\mathbb{C})$

Today: define $DT_v(\mathcal{L}) \in \mathbb{Q}$ without $\textcircled{\star}$

- Grothendieck group of varieties

S : \mathbb{C} -variety

Def $K_0(\text{Var}/S) := \bigoplus \mathbb{Z} [P: \gamma \rightarrow S] / \sim$

- γ : quasi-proj. var.

$$[\gamma \xrightarrow{P} S] \sim [\mathbb{Z} \xrightarrow{P|_{\mathbb{Z}}} S] + [\gamma, \mathbb{Z} \xrightarrow{P|_{\mathbb{Z}}} S]$$

$$\mathbb{Z} \hookrightarrow \gamma$$

closed subvar.

$$[\gamma_1 \rightarrow S] \sim [\gamma_2 \rightarrow S] \text{ if } \gamma_1 \xrightarrow{\cong} \mathbb{Q} \xrightarrow{\sim} \gamma_2$$

$$\downarrow \quad \downarrow$$

$$S \quad S$$

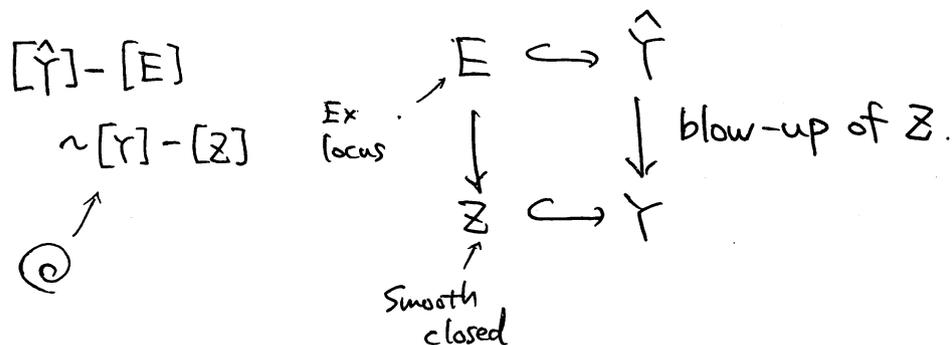
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$$K_0(\text{Var}) := K_0(\text{Var}/\text{Spec } \mathbb{C})$$

Fact: $K_0(\text{Var})$ is generated by $[Y] := [Y \rightarrow \text{Spec } \mathbb{C}]$
 Y : smooth proj. var. / \mathbb{C}

Relation is generated by



Virtual Poincaré polynomial

$$P_t: K_0(\text{Var}) \longrightarrow \mathbb{Z}[t]$$

Y : smooth proj. var.

$$\Rightarrow P_t(Y) := \sum (-1)^i b_i(Y) t^i$$

compatible with \textcircled{c}

i -th Betti 数

\forall quasi projective var. Y ,

$$\lim_{t \rightarrow 1} P_t(Y) = \chi(Y).$$

• Stack version:

\mathcal{S} : Artin stack / \mathbb{C} , ^{with} aff. geometric stabilizers

$$K_0(\text{St}/\mathcal{S}) := \bigoplus \mathbb{Q}[P: Y \rightarrow \mathcal{S}] / \sim$$

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• \mathcal{Y} = Artin stack of finite type / \mathbb{C}
 with geometric affine stabilizers

$\forall x \in \mathcal{Y}(\mathbb{C})$,
 $\text{Aut}(x)$: aff. alg. gp

• $[\mathcal{Y} \xrightarrow{f} \mathcal{S}] \sim [\mathcal{Z} \xrightarrow{p|_{\mathcal{Z}}} \mathcal{S}] + [\mathcal{Y} \setminus \mathcal{Z} \xrightarrow{p|_{\mathcal{Y} \setminus \mathcal{Z}}} \mathcal{S}]$
 $\mathcal{Z} \hookrightarrow \mathcal{Y}$ closed substack

• $[\mathcal{Y}_1 \sim \mathcal{P}] \sim [\mathcal{Y}_2 \sim \mathcal{S}]$ $\mathcal{Y}_1 \xrightarrow{\sim} \mathcal{Y}_2$
 $\downarrow \quad \downarrow$
 \mathcal{S}

Pull-back, Push-forward

$f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$: l-morphism of stacks
 $\uparrow \quad \uparrow$
 Artin stacks with aff. geom. stab.

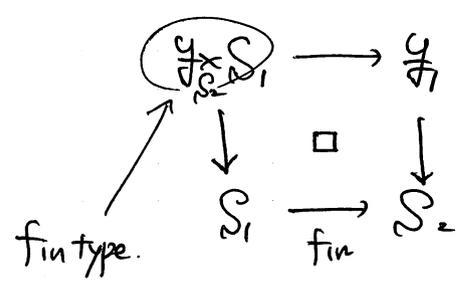
• $f_*: K_0(\text{St}/\mathcal{S}_1) \rightarrow K_0(\text{St}/\mathcal{S}_2)$

$f_*[\mathcal{Y} \xrightarrow{f} \mathcal{S}_1] = [\mathcal{Y}_2 \xrightarrow{f \circ p} \mathcal{S}_2]$

• f : fin. type

$\Rightarrow f^*: K_0(\text{St}/\mathcal{S}_2) \rightarrow K_0(\text{St}/\mathcal{S}_1)$

$f^*[\mathcal{Y} \xrightarrow{f} \mathcal{S}_2] = [\mathcal{Y} \times_{\mathcal{S}_2} \mathcal{S}_1 \rightarrow \mathcal{S}_1]$



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Lem: $\exists P_t = K_0(\text{St}) := K_0(\text{St}/\text{Spec } \mathbb{C}) \longrightarrow \mathbb{Q}(t)$

s.t. $P_t([Y/GL_n(\mathbb{C})]) = \frac{P_t(Y)}{P_t(GL_n(\mathbb{C}))}$

$Y \dots$ quasi-proj. var.

$GL_n(\mathbb{C}) \curvearrowright Y$.

morphism of \mathbb{Q} -vect. sp.

\therefore Kresch:

$K_0(\text{St})$ is gen. by $[Y/GL_n(\mathbb{C})]$

$\forall [Y] \in K_0(\text{St})$ is stratified by

$Y = \coprod [Y_i/GL_{n_i}(\mathbb{C})]$
quasi-proj. var.

$= \coprod [\tilde{Y}_i/GL_{\tilde{n}_i}(\mathbb{C})]$

D-M stack $\subset K_0(\text{St})$

e.g. $Y = [\text{Spec } \mathbb{C}/\mathbb{Z}/m\mathbb{Z}]$

$= [\mathbb{C}^{\times} \curvearrowright \mathbb{C}^{\times}]$
wt = m

$P_t(Y) = \frac{P_t(\mathbb{C}^{\times})}{P_t(\mathbb{C}^{\times})} = 1$

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• Hall algebra

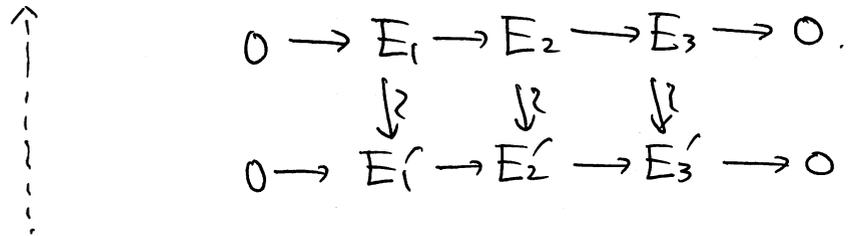
X : smooth proj. C- γ . 3-fold / \mathbb{C}

$\rightsquigarrow \text{Coh } X$

Def $H(X) := K_0(\text{St}/\text{Coh } X)$

$\uparrow \exists$ $*$ -product. based on Ringel-Hall alg.

\mathcal{E}_X : moduli stack of all exact seq in $\text{Coh } X$



Artin stack of local finite type / \mathbb{C}

$P_1: \mathcal{E}_X \rightarrow \text{Coh } X$

$E. \mapsto E_i$

$\mathcal{Z} \xrightarrow{\quad} \mathcal{E}_X \xrightarrow{P_2} \text{Coh } X$

$\downarrow \square (P_1, P_2) \downarrow \text{fin. type}$
 $\mathcal{Z}_1 \times \mathcal{Z}_2 \rightarrow \text{Coh } X \times \text{Coh } X$

$$x_i = \left(\bigoplus_{j=1}^i P_j \rightarrow \text{Coh } X \right)_{i=1,2}$$

$$x_1 \otimes x_2 \in K_0(\text{St}/\text{Coh } X \times \text{Coh } X)$$

Def: define

$*$: $H(X)^{\otimes 2} \rightarrow H(X)$ by

$$* = P_{2*} (P_{1*} P_{3*})^* \mathcal{L}$$

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Prop (Joyce)

* is an associative product with unit

$$[\text{Spec } \mathbb{C} \rightarrow \text{Coh } X]$$

$$\bullet \longmapsto 0 \in \text{Coh } X.$$

Ex: $E_1, E_2 \in \text{Coh } X.$

$$\alpha_i := [\text{Spec } \mathbb{C} \rightarrow \text{Coh } X] \in H(X)$$

$$\bullet \longmapsto E_i$$

$$\alpha_1 * \alpha_2 = \left[\left[\text{Ext}'(E_2, E_1) / \text{Hom}(E_2, E_1) \right] \rightarrow \text{Coh } X \right]$$

↑
trivial

$$u \in \text{Ext}'(E_2, E_1)$$

↓

$$0 \rightarrow E_1 \rightarrow E_u \rightarrow E_2 \rightarrow 0$$

$$u \longmapsto E_u$$

$$P_t = K_0(\text{St}) \rightarrow \mathbb{Q}(t)$$

$$\Rightarrow P_t = H(X) \rightarrow K_0(\text{St})_{\mathbb{Q}(t)}$$

$$(Y \rightarrow \text{Coh } X) \mapsto [Y]$$

• Elements $\delta_v(\mathcal{L}), \epsilon_v(\mathcal{L})$

$$\mathcal{L} \xrightarrow{\text{ample}} X, v \in \Gamma$$

$$\Rightarrow \delta_v(\mathcal{L}) := [m_v^{ss}(\mathcal{L}) \hookrightarrow \text{Coh } X] \in H(X)$$

Remark $P_t(\delta_v(\mathcal{L})) \in \mathbb{Q}(t)$

If $(*)$ is satisfied

$$\Rightarrow P_t(\delta_v(\mathcal{L})) = P_t([M_v(\mathcal{L})/\mathbb{C}^*]) = \frac{P_t(M_v(\mathcal{L}))}{P_t(\mathbb{C}^*)}$$

"t=1"

⑦

$$\chi(M_v(\mathcal{L})) = \lim_{t \rightarrow 1} (t^2 - 1) P_t(\delta_v(\mathcal{L}))$$

⊗ or "exists" is not, $\lim_{t \rightarrow 1}$ is not in \mathbb{C} .

Def
$$E_v(\mathcal{L}) := \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{v_1, \dots, v_l \in \mathbb{P}} \delta_{v_1}(\mathcal{L}) * \delta_{v_2}(\mathcal{L}) * \dots * \delta_{v_l}(\mathcal{L})$$

\uparrow
 $H(X)$

$\bar{\chi}_{v_i, \mathcal{L}}(n) = \bar{\chi}_{v, \mathcal{L}}(n)$
 $n_1 + \dots + n_l = v$

reduced Hilbert polynomial finite sum.

Note: $\otimes \Rightarrow E_v(\mathcal{L}) = \delta_v(\mathcal{L})$

i.e. $\forall p(n) \in \mathbb{Q}[n]$,

$$\sum_{v: \bar{\chi}_{v, \mathcal{L}}(n) = p(n)} E_v(\mathcal{L}) = \log \left(1 + \sum_{v: \bar{\chi}_{v, \mathcal{L}}(n) = p(n)} \delta_v(\mathcal{L}) \right) \in \widehat{H(X)}$$

Thm (Joyce)

$$\lim_{t \rightarrow 1} (t^2 - 1) P_t(E_v(\mathcal{L})) \stackrel{DT_v^{\chi}(\mathcal{L})}{=} \in \mathbb{Q} \text{ exists.}$$

" χ " ($m_v^{ss}(\mathcal{L})$) $\in \mathbb{Z}$.

$\exists \nu: \text{Coh } X \rightarrow \mathbb{Z}$: local constructible fun.

Behrend fun \uparrow local $r = [\{df=0\}/G]$ a \mathbb{P}^1 .

$$\begin{aligned} \nu|_{M_v(\mathcal{L})} \\ = -\nu|_{[M_v(\mathcal{L})/\mathbb{C}^*]} \end{aligned}$$

$$\nu \cdot H(X) \rightarrow H(X)$$

$$[\mathcal{Y} \xrightarrow{f} \text{Coh } X] \mapsto \sum_{m \in \mathbb{Z}} m [(\nu \circ f)^{-1}(m) \rightarrow \text{Coh } X]$$

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$$\underline{\text{Def}} \quad DT_2(L) := - \lim_{t \rightarrow 1} (t^2 - 1) P_t (v \cdot E_2(L)) \in \mathbb{Q}$$