2011/11/4

①

集中講義
(戸田さん(IPMU))

・三角圏の(弱)安定性条件.

$\mathcal{D}$: triangulated category

e.g. $\mathcal{D} = D^b \mathrm{Coh} X$　　X: variety
$\uparrow$
bounded complex in $\mathrm{Coh}\, X$

Γ: free abelian group of finite rank

$cl: K(\mathcal{D}) \longrightarrow \Gamma$: fixed group hom

$\{0\} = \Gamma_{-1} \subsetneqq \Gamma_0 \subsetneqq \Gamma_1 \subsetneqq \cdots \subsetneqq \Gamma_N = \Gamma$: fixed filtration

Γ_i / Γ_{i-1} : free abelian gp.

e.g. X: smooth proj. var. $/\mathbb{C}$ of $\dim = d$

$\Gamma := \mathrm{Im}\left(ch : K(X) \to H^*(X, \mathbb{Q})\right)$

$\left(= K(X)/\ker ch\right)$

$$\begin{array}{c} cl : K(\mathcal{D}) \longrightarrow \Gamma \\ \shortparallel \\ ch \end{array}$$

$\Gamma_i = \Gamma \cap H^{\geq 2d-2i}(X, \mathbb{Q})$

$\Gamma_0 = \Gamma \cap H^{2d}$ ←--- point.

$\Gamma_1/\Gamma_0 = \Gamma \cap H^{2d-2}$ -- curve

$\left(\Gamma_N/\Gamma_{N-1} \gg \cdots \gg \Gamma_1/\Gamma_0\right)$

2011/11/4

2

Def A weak stability condition on $\mathcal{D}$ is $(Z, \mathcal{A})$,

- $Z = \{Z_i\}_{i=0}^{N} \in \prod_{i=0}^{N} \mathrm{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$
- $\mathcal{A} \hookrightarrow \mathcal{D}$: heart of bounded t-str.

abelian cat.

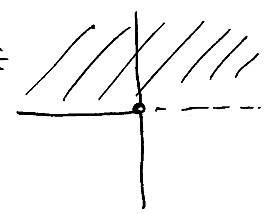
(e.g. $\mathrm{Coh}\,X \hookrightarrow D^b\mathrm{Coh}\,X$)

(i) $0 \neq E \in \mathcal{A}$,

Take $-1 \leqq i \leqq N$ s.t. $cl(E) \in \Gamma_i \setminus \Gamma_{i-1}$

$\rightsquigarrow$ $[cl(E)] \in \Gamma_i/\Gamma_{i-1}$ satisfies

$Z(E) := Z_i([cl(E)]) \in$



(i) $\Rightarrow$ $E \in \mathcal{A}$ is Z-(semi)stable

$\Leftrightarrow$ $\forall\, 0 \to F \to E \to G \to 0$ exact in $\mathcal{A}$,

we have

$$\arg Z(F) \underset{(\leqq)}{<} \arg Z(G)$$

(ii) $\forall\, 0 \neq E \in \mathcal{A}$, $\exists$ filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$

s.t. $F_i := E_i/E_{i-1}$ is Z-semistable

$\arg Z(F_i) > \arg Z(F_{i+1})$ $\forall i$

unique if it exists (up to isom)

Harder - Narasimhan filtration.

$N = 0$

$\Rightarrow$ Bridgeland's stability condition

2011/11/4

③

Ex ① C: smooth proj. curve

$D = D^b \operatorname{Coh} C$

$\Gamma = K(C)/\operatorname{Ker} \operatorname{ch}$

$N = 0$.

- $\mathcal{A} = \operatorname{Coh} C \hookrightarrow D$
- $Z : \Gamma \longrightarrow \mathbb{C}$

$$E \longmapsto -\deg E + \sqrt{-1}\operatorname{rank} E.$$

② X … smooth proj var. $/\mathbb{C}$

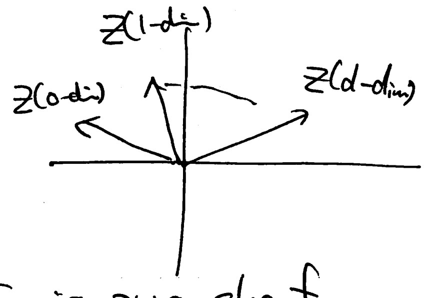
$D = D^b \operatorname{Coh} X$

$\Gamma = \operatorname{Im} \operatorname{ch}$.

$\Gamma_i = \Gamma \cap H^{\geq 2d-2i}$

- $\mathcal{A} = \operatorname{Coh} X \hookrightarrow D$
- $Z = \{Z_i\}_{i=0}^{N}$, $Z_i : \underset{\Gamma \cap H^{\geq 2d-2i}}{\Gamma_i} / \Gamma_{i-1} \longrightarrow \mathbb{C}$, $v \longmapsto \left(\int_X v \cdot \omega^i\right) e^{i\pi\phi_i}$

$1 \geq \phi_0 > \phi_1 > \cdots > \phi_d > 0$.



$(Z, \mathcal{A})$: weak stab. cond.

$E \in \operatorname{Coh} X$ is Z-semistable $\Leftrightarrow$ E is pure sheaf.

Remark ① $\dim X \geq 2 \Rightarrow \nexists (Z, \operatorname{Coh} X)$: stability condition ($N=0$)

② $\dim X \geq 3$ X: C.Y.

$\Longrightarrow$ 安定性条件の例は知られていない.

2011/11/4

4

$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$: the set of "good" weak stability condition

$\uparrow$ $\begin{cases} \text{local finiteness} \\ \text{support property} \end{cases}$

<u>Thm</u> (Bridgeland)

$\exists$ not topology on $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ s.t.

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}) \longrightarrow \prod_{i=0}^{N} \mathrm{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$$
$$(Z, \mathcal{A}) \longmapsto Z.$$

is a local homeo.

特に $\forall$ conn. component $\subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ は complex manifold.

- Category of D0 D2 D6 bound states :

D_{2i} brane $\longleftrightarrow$ i-dim sheaf

$$\mathcal{D}_X := \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1} X \rangle_{tr} \hookrightarrow D^b \mathrm{Coh}\, X$$

$\uparrow$

Smallest triangulated subcat. which contains $\mathcal{O}_X$, $\mathrm{Coh}_{\leq 1} X$

$\mathrm{Coh}_{\leq 1} X = \{ F \in \mathrm{Coh}\, X \mid \dim \mathrm{Supp}\, F \leq 1 \}$

- $C \hookrightarrow X$: curve

$\rightsquigarrow I_C \in \mathcal{D}_X$.

$\because$)

$$\underset{\substack{\cap \\ \mathcal{D}_X}}{\mathcal{O}_C(-1)} \to \underset{\substack{\cap \\ \mathcal{D}_X}}{I_C} \to \underset{\substack{\cap \\ \mathcal{D}_X}}{\mathcal{O}_X} \to \mathcal{O}_C$$

- (F, s) : PT stable pair $\Rightarrow$ $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{D}_X$.

2011/11/4

5

$$\Gamma := \overbrace{\mathbb{Z}}^{\Gamma_1/\Gamma_0} \oplus \overbrace{H_2(X,\mathbb{Z})_{\mathrm{free}} \oplus \mathbb{Z}}^{\Gamma_0}.$$

$$cl : K(D_X) \longrightarrow \Gamma$$

$$E \longmapsto (ch_0 E,\ ch_2 E,\ ch_3 E)$$

$$\Gamma_0 = \{0\} \oplus H_2(X,\mathbb{Z})_{\mathrm{free}} \oplus \mathbb{Z}$$

$$\Gamma_1 = \Gamma \rightsquigarrow \Gamma_1/\Gamma_0 \cong \mathbb{Z}.$$

D6 brane の質量 $\gg$ D0, D2

Fix ω : ample divisor

$0 < \theta < 1$

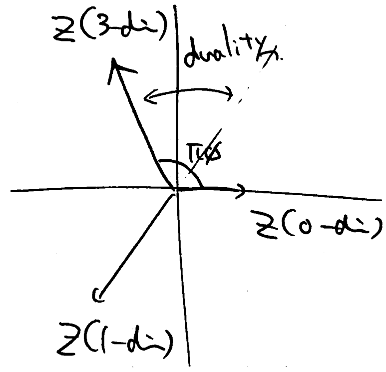
$$Z_\theta \in \prod_{i=0}^{1} \mathrm{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$$

$$Z_{\theta,0} : H_2 \oplus \mathbb{Z} \longrightarrow \mathbb{C}$$

$$(\beta, n) \longmapsto n - (\beta\cdot\omega)\sqrt{-1}$$

$$Z_{\theta,1} : \mathbb{Z} \longrightarrow \mathbb{C}$$

$$r \longmapsto r e^{i\pi\theta}$$



<u>Lem</u> (i) $\exists \mathcal{A}_X \hookrightarrow D_X$: heart of bounded t-str. s.t.

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1} X[-1] \rangle_{ex} \hookrightarrow D_X \hookrightarrow D^b \mathrm{Coh} X$$

$\langle S \rangle_{ex}$: smallest ext-closed subcat which contains S.

$$D_X \cap \mathrm{Coh} X \supset \langle \mathcal{O}_X, \mathrm{Coh}_{\le} X \rangle_{ex}$$

$I_C \in D_X \cap \mathrm{Coh} X$, $I_C \in \langle \mathcal{O}_X, \mathrm{Coh}_{\le} X \rangle_{ex}$ ← $\mathrm{Coh}_{\le} X[-1]$ で書ける

(ii) $(Z_\theta, \mathcal{A}_X) \in \mathrm{Stab}_{\Gamma_\bullet}(D_X)$

- Moduli stacks.

$\mathcal{M}_{n,\beta}(\theta)$: moduli stack of Z_θ-semistable $E \in \mathcal{A}_X$,

$\mathrm{cl}\, E = (1, -\beta, -n)$ $\qquad$ $\mathrm{cl}\big(\underset{\mathcal{A}_X}{\overset{\cap}{I_C}}\big) = (1, -[C], -\chi(\mathcal{O}_C))$

Prop (i) $\mathcal{M}_{n,\beta}(\theta)$ is an Artin stack of fin. type $/\mathbb{C}$.

(ii) $\theta \to 1$

$\Rightarrow$ $\mathcal{M}_{n,\beta}(\theta) \simeq [P_n(X,\beta)/\mathbb{C}^\times]$

trivial

moduli of PT stable pair.

(iii) $\exists$ isom $\mathcal{M}_{n,\beta}(\theta) \xrightarrow{\sim} \mathcal{M}_{-n,\beta}(1-\theta)$

$E \longmapsto R\mathcal{H}om(E, \mathcal{O}_X)$

(iv) $\mathcal{M}_{n,\beta}(\theta = \frac{1}{2}) = \emptyset$ for $|n| \gg 0$.

- Counting invariants

$DT_{n,\beta}(\theta)$: DT type invariant

$\underset{\text{defined from } \mathcal{M}_{n,\beta}(\theta)}{\uparrow}$ $\quad \overset{\cap}{\mathbb{Q}}$ $\quad$ # of Z_θ-stable $E \in \mathcal{A}_X$, $\mathrm{cl}\, E = (1, -\beta, -n)$

Prop (ii') $\Rightarrow$ $\theta \to 1$ $\Rightarrow$ $DT_{n,\beta}(\theta) = P_{n,\beta}$.

Def $L_{n,\beta} := DT_{n,\beta}(\theta = \frac{1}{2})$

Prop (iii) $\Rightarrow$ $L_{n,\beta} = L_{-n,\beta}$

(iv) $\Rightarrow$ $L_{n,\beta} = 0$ for $|n| \gg 0$.

2011/11/4

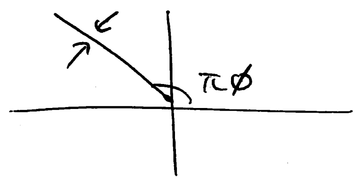
7

- Wall-crossing formula

$$DT(\theta) := \sum_{n,\beta} DT_{n,\beta}(\theta)\, q^n T^\beta \quad \in \widehat{\mathbb{Q}[\mathbb{Z}\oplus H_2]}$$

<u>Thm</u> $\forall\, 0 < \phi < \frac{1}{2}$.

$$\lim_{\theta\to\phi+0} DT(\theta) = \prod_{\substack{n>0 \\ \beta>0 \\ -n+(\omega\cdot\beta)\sqrt{-1}\,\in\,\mathbb{R}_{>0}e^{i\pi\phi}}} \exp\left((-1)^{n-1} N_{n,\beta}\, q^n T^\beta\right)^n$$



$$\lim_{\theta\to\phi-0} DT(\theta).$$

前回の記号で

$$N_{n,\beta} = DT_{(0,0,\beta,n)}(\mathcal{L})$$

<u>Fact</u> $N_{n,\beta}$ is independent of $\mathcal{L}$.

Apply Thm from $\theta\to 1$ to $\theta=\frac{1}{2}$.

<u>Cor</u>

$$PT(X) = \prod_{\substack{n>0 \\ \beta>0}} \exp\left((-1)^{n-1} N_{n,\beta}\, q^n t^\beta\right)^n \left(\sum L_{n,\beta}\, q^n t^\beta\right)$$

2011/11/4

8

- Stable pairs on local K3 surfaces (arXiv:1103.4230)

S : smooth proj. K3 $/\mathbb{C}$

i.e. $K_S = \mathcal{O}_S$, $H^1(\mathcal{O}_S) = 0$.

e.g. $X \underset{4次}{\hookrightarrow} \mathbb{P}^3$

$X := K_S = S \times \mathbb{C}$.

$n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z}) \simeq H_2(S, \mathbb{Z}) \simeq H^2(S, \mathbb{Z})$

$P_n(X, \beta)$: moduli space of stable pairs (F, s),

- F : supported on fibers of $X \longrightarrow \mathbb{C}$
- $[F] = \beta$, $\chi(F) = n$

quasi-proj. var.

$$\int_{[P_n(X,\beta)]^{vir}} 1 = 0.$$

S can be deformed to non-alg. one

Maulik - Pandharipande - Thomas (2010)

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{red}} 1 \in \mathbb{Z}$$

βが代数的である deformationで不変

Q $\left\{ \begin{array}{l} \cdot\ P_{n,\beta} = \int_{P_n(X,\beta)} \nu \, d\chi\ ? \\ \cdot\ \exists \text{ Hol. Chern-Simons fcn?} \end{array} \right.$

2011/11/4

9

Thm (MPT)

β : irreducible i.e. $\beta \neq \beta_1 + \beta_2$, $\beta_i > 0$

$\Rightarrow P_{n,\beta} = (-1)^{n-1} \underbrace{\chi(P_n(X,\beta))}_{\text{smooth}}$

Thm (Kawai - Yoshioka, 2000) coherent system

(i) If β is irreducible, then $P_n(X,\beta)$ depends only on n and $\beta^2 = 2h-2$, up to deformation invariance

$\rightsquigarrow P_n(X,h) := P_n(X,\beta)$ as top. sp.

(ii) $$\sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} \chi(P_n(X,h)) z^n q^{h-1} = \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^{-2} \frac{1}{\Delta(z,q)}$$

$$\Delta(z,q) = q \prod_{n=1}^{\infty} (1-q^n)^{20} (1-zq^n)^2 (1-z^{-1}q^n)^2$$

Q What is $\chi(P_n(X,\beta))$ when β is not nessesary irreducible?

A We can relate $\chi(P_n(X,\beta))$ to invariants counting semi stable sheaves on fibers of $\pi : X \to \mathbb{C}$.

• Invariants $J(r,\beta,n)$

ω : ample div. on S

$$\begin{array}{ccc} X = S \times \mathbb{C} & \xrightarrow{\pi} & \mathbb{C} \\ p \downarrow & & \\ S & & \end{array}$$

2011/11/4

10

$\mathrm{Coh}_\pi X := \{ F \in \mathrm{Coh}\, X \mid$ supported on fibers of $\pi \}$

$\cup$

$\exists\, p^*\omega$ - stability on $\mathrm{Coh}\, X$

↑
classical

Mukai lattice

$v \in \tilde{H}(S, \mathbb{Z}) := H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$

$\mathcal{M}_v^{ss}(\omega)$: moduli stack of $p^*\omega$-stable $F \in \mathrm{Coh}_\pi X$

$v(F) = v$
↑
Mukai vector

$$v : \mathrm{Coh}_\pi X \longrightarrow \tilde{H}(S, \mathbb{Z})$$
$$F \longmapsto ch(p_* F) \cdot \sqrt{td_S}$$
$$= (ch_0(p_*F),\ ch_1(p_*F),\ ch_2(p_*F) + ch_0(p_*F))$$

<u>Def</u> $J_v(\omega) := DT^X_{i_*(v \cdot \sqrt{td_S}^{-1})}(\omega) \in \mathbb{Q}$

↑
Euler number inv.
\# of $p^*\omega$-semistable $F \in \mathrm{Coh}_\pi X$, $v(F) = v$

$i : S \hookrightarrow X = S \times \mathbb{C}$

$x \longmapsto (x, 0)$

- $\gcd \{ \chi_{v,\omega}(m) \mid m \in \mathbb{Z} \} = 1$.

$\Rightarrow$ ω-semistable $= \omega$-stable

$\mathcal{M}_v^{ss}(\omega) = [M_v(\omega)/\mathbb{C}^*]$

↑
moduli of ω-stable sheaves on S
Mukai vector $= v$

2011/11/4

11

Thm (Mukai)

$M_v(\omega)$ is a holomorphic symplectic manifold

$\dim = \langle v, v \rangle + 2$

- $v_i = (r_i, \beta_i, n_i) \in \tilde{H}$, $i = 1, 2$.

$\rightsquigarrow \langle v_1, v_2 \rangle := \beta_1\beta_2 - n_1 r_2 - n_2 r_1$ (Mukai inner product)

Thm (Huybrechts, O'Grady, Yoshioka)

$$M_v(\omega) \underset{\substack{\text{deformation}\\ \text{equivalent}}}{\sim} \mathrm{Hilb}_{\langle v,v \rangle/2+1}(S)$$

$$\Rightarrow J_v(\omega) = \chi(M_v(\omega))$$
$$= \chi(\mathrm{Hilb}_{\frac{\langle v,v \rangle}{2}+1}(S))$$

v が primitive なら成立

Thm (Göttsche)

$$\sum_{n \geq 0} \chi(\mathrm{Hilb}_n S) q^n = \prod_{n \geq 1} \frac{1}{(1-q^n)^{\chi(S)}}$$

Lem. In general, $J_v(\omega)$ is independent of ω.

$\longrightarrow J_v := J_v(\omega) \in \mathbb{Q}$

Thm (Toda (2011))

$PT^{\chi}(X) := \sum_{n,\beta} \chi(P_n(X,\beta)) y^n z^\beta$ is written as

$$PT^{\chi}(X) = \prod_{\substack{r \geq 0\\ \beta > 0\\ n \geq 0}} \exp(J_{r,\beta,r+n}\, y^\beta z^n)^{n+2r} \cdot \prod_{\substack{r > 0\\ \beta > 0\\ n > 0}} \exp(J_{n,\beta,r+n}\, y^\beta z^{-n})^{n+2r}$$

2011/11/4

12

Thm $\Rightarrow$ Kawai - Yoshioka

β = irreducible

$\Rightarrow v = (r, \beta, n)$ is primitive

$n \geq 0$ fin. sum.

$$\chi(P_n(X, \beta)) = \sum_{r \geq 0} (n+2r) \underbrace{J(r, \beta, r+n)}_{\chi(\mathrm{Hilb} -)}$$

$$\chi(P_{-n}(X, \beta)) = \sum_{>0} \quad \vdots$$

+ Göttsche formula

+ modular form の計算 $\Rightarrow$ KY

Q. What is J_v when v is not primitive?

① $v = (0, \beta, n)$

$$J(0, \beta, n) \stackrel{?}{=} \sum_{\substack{k \geq 1 \\ k | (\beta, n)}} \frac{1}{k^2} J(0, \beta/k, 1)$$

$$\shortparallel$$

$$\chi(\mathrm{Hilb}_{\beta^2/2k^2+1}(S))$$

② Automorphic property:

Thm (Toda)

$\forall g \in O_{\mathrm{Hodge}}(\tilde{H}(S, \mathbb{Z}))$ $\leftarrow$ Hodge isometries

$J_{gv} = J_v$

$\tilde{H}^{2,0} = H^{2,0}$

$\tilde{H}^{1,1} = H^0 \oplus H^{1,1} \oplus H^4 \simeq \tilde{H}^{0,2} = H^{0,2}$

2011/11/4

13

M-Conj. $v \in \tilde{H}(S,\mathbb{Z})$: alg. class

$$\Downarrow$$

$$J_v = \sum_{\substack{k>1 \\ k|v}} \frac{1}{k^2} \chi(\mathrm{Hilb}_{\langle v/k,\, v/k\rangle/k+1}(S))$$

Conj + Thm

$$\Rightarrow PT^{\chi}(X) = \prod_{\substack{r\geq 0 \\ \beta>0 \\ n\geq 0}} (1-y^{\beta}z^{n})^{-(n+2r)\chi(\mathrm{Hilb}_{\frac{\beta^2}{2}-r(r+n)+1}(S))}$$

$$\times \prod_{\substack{r>0 \\ \beta>0 \\ n>0}} (1-y^{\beta}z^{n})^{-(n+2r)\chi(\mathrm{Hilb}_{\frac{\beta^2}{2}-r(r+n)+1}(S))}$$

Thm (Toda (2011))

(S, L) : gen. point of moduli of pol. K3 / $\mathbb{C}$

(L: pol)

$\Rightarrow J_{(0,pL,n)}$ satisfies M-conj if p : prime number

or

$p \leq 10$.

Remark $L^2=2$, $p=2$, $n=0$.

$$\Rightarrow \mathcal{M}^{ss}_{(0,2L,0)}(\omega) \overset{\text{birat}}{\dashleftarrow\dashrightarrow} \text{O'Grady's}$$

$\langle 0, 2L, 0\rangle^2 + 2 = 10$.

10-dim symplectic mfd.

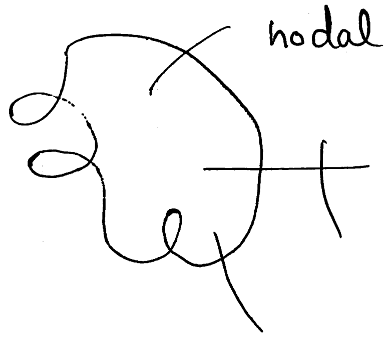
Mozgovoy

2011/11/4

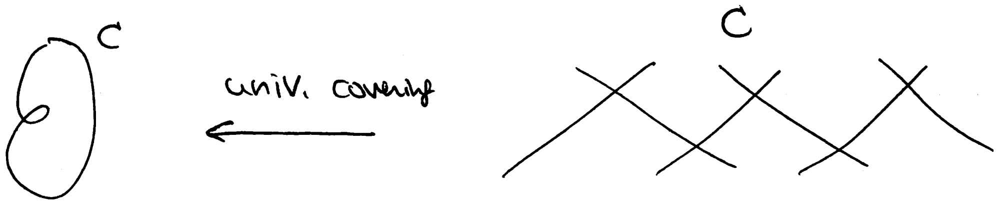
14

- $J_{(0,\beta,n)}$ に寄与する sheaf は rational curve に台をもつ.

- $\forall C \in |pL|$ rational curve has at worst nodal sing.



By taking univ. covering can assume that C is simply conn.



$\uparrow$

$\mathrm{Pic}\, C = \mathbb{C}^*$