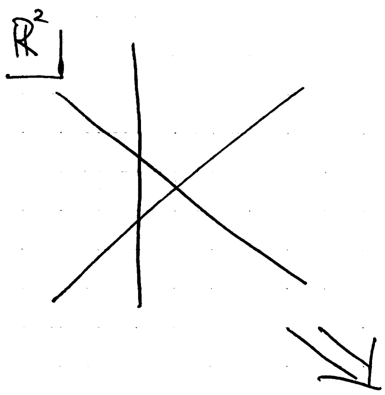
2011/11/15 ①

Minimal Stratifications for Line arrangements.

吉永さん (京大・理)
トポロジー・幾何学セミナー



$\xrightarrow{\otimes\mathbb{C}}$ $M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup H_i \otimes \mathbb{C}$

$\xleftrightarrow{??}$ トポロジー

min. "stratification"

Plan

1. Background & Motivation
2. Main result.
3. Proof.
4. π_1の表示 ↑ ArXiv: ---
5. π_2を理解できるか?

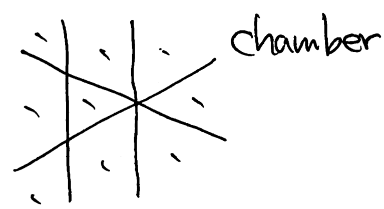
---

§1

$\mathcal{A} = \{H_1, H_2, \cdots, H_n\}$

$H_i = \{\alpha_i = 0\} \subset \mathbb{R}^\ell$: affine hyperplane

$ch(\mathcal{A}) = \{\text{chamber}\}$



$M(\mathcal{A}) := \mathbb{C}^\ell \setminus \bigcup H_i \otimes \mathbb{C}$

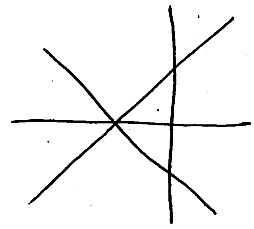
複素化した補集合

2011/11/15

②

Fact 1 (Zaslavsky)

$$\# ch(\mathcal{A}) = \sum_{k=0}^{\ell} b_k(M(\mathcal{A}))$$



$$\# bdd\text{-}ch(\mathcal{A}) = |\chi(M(\mathcal{A}))|$$

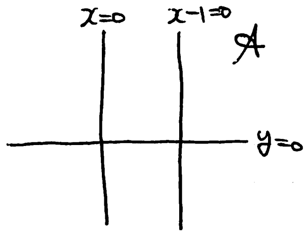
体積有限の chamber

$\sum b_i = 10$

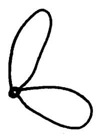
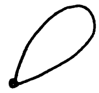
$\chi = 2$

例 $\ell = 2$.

$x(x-1)y = 0$



$$M(\mathcal{A}) = (\mathbb{C} \setminus \{0,1\}) \times (\mathbb{C} \setminus \{0\})$$

$\approx$  $\times$ 

(2次元の finite CW cpx)

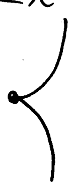
Fact 2 (Dimca - Papadima, Randell)

$M(\mathcal{A})$ is homotopy equiv. to a minimal ℓ-dim CW cpx X.

(ただし, X : minimal $\overset{\text{def}}{\Longleftrightarrow}$ $\# k\text{-cell} = b_k(X)$)

"$\geqq$" in general

※ $y^2 - x^3 = 0$

 は CW cpx に homotopy equiv. だが, minimal ではない.

2011/11/15
③

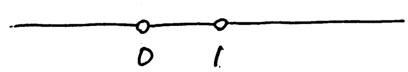
Rem Attaching map の記述は とても複雑.

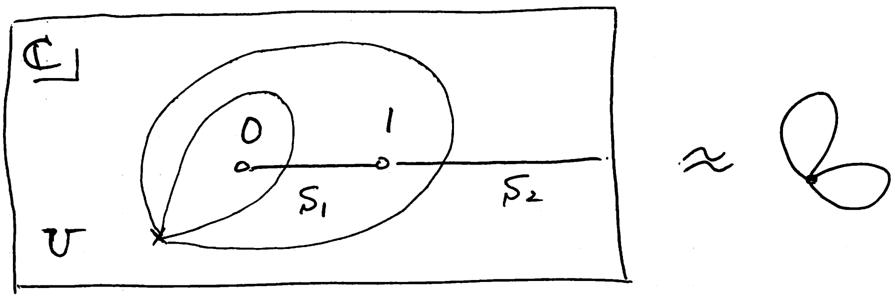
$$\left(\begin{array}{l} D^k \supset S^{k-1} \xrightarrow{\ \sigma\ } X_{k-1} \leftarrow \text{skeleton.} \\ \text{は homological には trivial} \\ \quad \text{homotopical には nontrivial (Yoshinaga)} \end{array} \right)$$

Motivating Example

$\mathbb{R} \setminus \mathcal{A}.$

$l=1,\ \mathcal{A} = \{0,1\} \subset \mathbb{R}.$





$M(\mathcal{A}) = \mathbb{C} \setminus \{0,1\}$

$S_1 := (0,1)$ ← open interval
$S_2 := (1,\infty)$ ←

$U := M \setminus S_1 \cup S_2$

$M = S_1 \sqcup S_2 \sqcup U$

- すべての piece は 可縮.
- lowest dim piece は chamber
- $\#(k\text{-codim piece}) = b_k(M)$
- Minimal CW cpx の dual とみなせる.

2011/11/15

4

Goal : generalizing to $l=2$.

Ex:

$$S_1 = \left\{ z \in \mathbb{C} \;\middle|\; \frac{z-1}{z} \in \mathbb{R}_{<0} \right\}$$

$$S_2 = \left\{ z \in \mathbb{C} \;\middle|\; \frac{-1}{z-1} \in \mathbb{R}_{<0} \right\}$$

§2 Main result

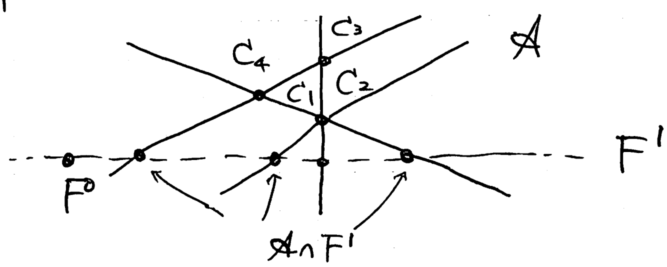
Setting : $l=2$, $\mathcal{A} = \{H_1, \cdots, H_n\}$, $H_i \subset \mathbb{R}^2$

$$M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup H_i \otimes \mathbb{C}$$

Fix a generic flag $F^0 \subset F^1 \subset F^2 = \mathbb{R}^2$

s.t.

- F^1 does not separate intersections of $\mathcal{A}$.
- F^0 —————— // —————— $\mathcal{A} \cap F^1$.



$ch_2(\mathcal{A}) = \{C_1, \cdots, C_4\}$

Def $ch_2(\mathcal{A}) = \{ C \in ch(\mathcal{A}) \mid C \cap F^1 = \emptyset \}$

Prop $|ch_2(\mathcal{A})| = b_2(M(\mathcal{A}))$

$\because$) $b_0 + b_1 + b_2 = |ch(\mathcal{A})|$ Zaslavsky

$b_0 = 1$, $b_1 = n$, $|ch(\mathcal{A})| = n+1+|ch_2(\mathcal{A})|$

$\blacksquare$

2011/11/15

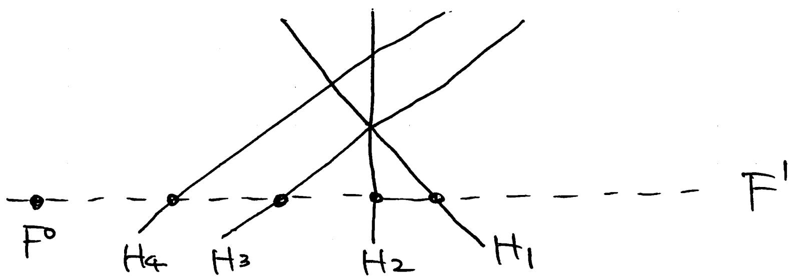
5

Assumption on numbering & sign

$H_1, \cdots, H_n$ は次をみたす numbering

$\underset{(\alpha_1=0)}{H_1}, \cdots, \underset{(\alpha_n=0)}{H_n}$

- $F^0 < (H_n \cap F') < (H_{n-1} \cap F') < \cdots < (H_1 \cap F')$ on F'



- $F^0 \in \{\alpha_i < 0\}$

  (記号 : $\alpha_0 := -1$)

Def $S_i := \left\{ z_i \in M(\mathcal{A}) \;\middle|\; \dfrac{\alpha_{i-1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\} \subset M(\mathcal{A})$

$(i = 1, 2, \cdots, n)$

Thm (Yoshinaga)

(0) S_i is an orientable 3-mfd.

(i) $S_i \pitchfork S_j$, $S_i \cap S_j = \bigsqcup C$

ただし, C runs $C \in ch_2(\mathcal{A})$ s.t. $\begin{cases} \alpha_i(C)\alpha_{i-1}(C) < 0 \\ \text{かつ} \\ \alpha_j(C)\alpha_{j-1}(C) < 0. \end{cases}$

(ii) $S_i^0 := S_i \setminus \bigcup_{C \in ch_2(\mathcal{A})} C$ is contractible.

(iii) $U := M \setminus \bigcup S_i$ is contractible 4-mfd.

2011/11/15

6

§3. idea of proof

$$\mathbb{C}^2 \xrightarrow{\sim} T\mathbb{R}^2$$

$$z = x + iy \longmapsto y \in T_x\mathbb{R}^2$$

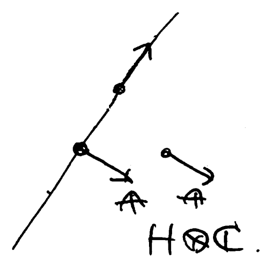
α : linear form / $\mathbb{R}$

$$\alpha(x_1, x_2) = a_1 x_1 + a_2 x_2 + b.$$

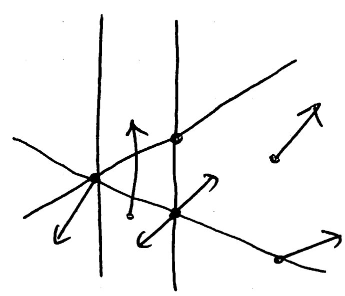
$$\alpha(x+iy) = a \cdot x + b + i a \cdot y.$$

$$= \alpha(x) + i a \cdot y.$$

$$x + iy \in H_\alpha \otimes \mathbb{C} \iff x \in H, \quad y \in T_x H.$$



$$M(\mathcal{A}) = \left\{ x + iy \;\middle|\; \begin{array}{l} x \in \mathbb{R}^2 \\ y \notin T_x H \text{ when } x \in \underset{\mathcal{A}}{\overset{\cap}{H}} \end{array} \right\}$$



Lemma $x + iy \in H \Rightarrow (x + ty) + iy \in H$ $(t \in \mathbb{R})$

(直線運動で $M(\mathcal{A})$ は保存)

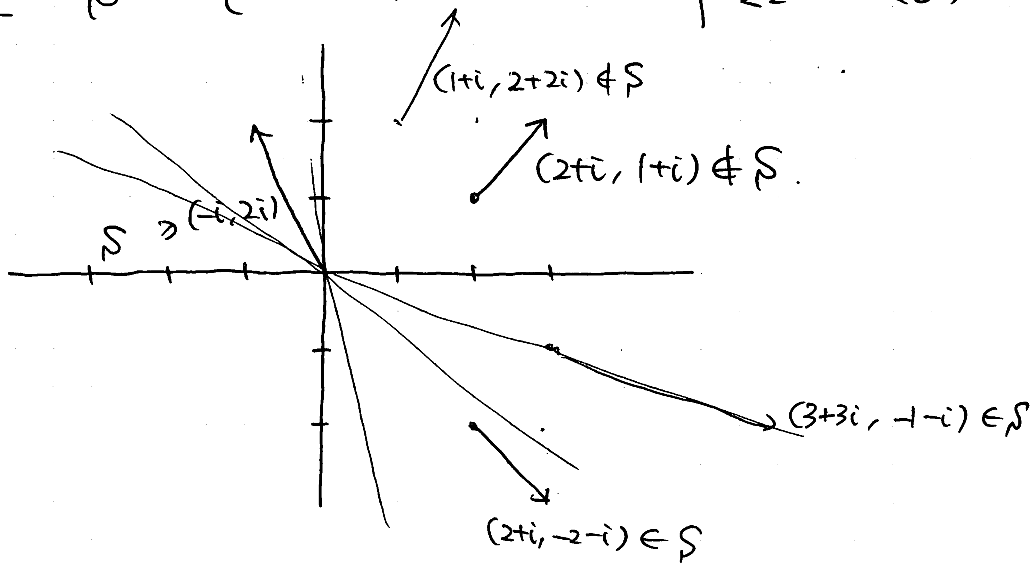
$\quad$ " $M(\mathcal{A}) \Rightarrow$ ———— " ———— $M(\mathcal{A})$

2011/11/15

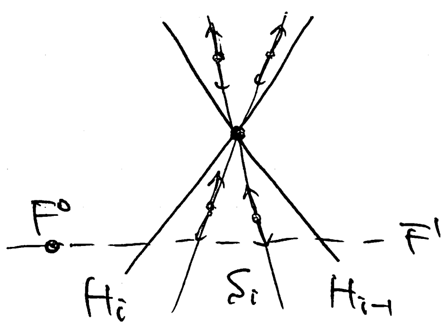
⑦

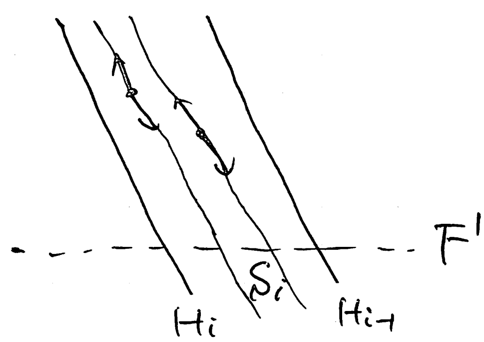
Description of S_i

Ex $S := \left\{ z=(z_1,z_2) \in (\mathbb{C}^*)^2 \;\middle|\; \frac{z_1}{z_2} \in \mathbb{R}_{<0} \right\}$



Prop $H_i \not\parallel H_{i-1}$ $\qquad\qquad$ $H_i \parallel H_{i-1}$

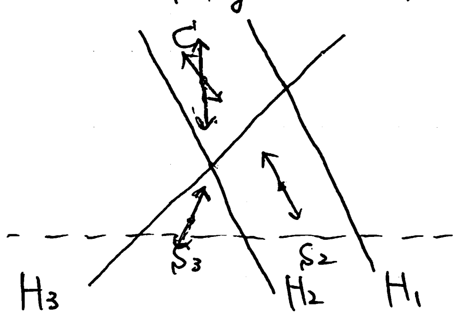




Lemma $x+iy \in S_i \Longrightarrow (x+ty)+iy \in S_i$

$\qquad$ 〃 $\in U \Longrightarrow$ 〃 $\in U$

(i) $S_i \pitchfork S_j$ の証明.



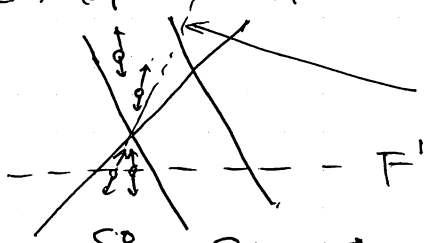
$S_i \cap S_j = C.$

$x \in S_i \cap S_j$

$\Rightarrow \begin{cases} x: 一致 \\ y=0 \end{cases} \Rightarrow x \in C.$

2011/11/15
8

(ii) S_i^0 : contractible



$S_3^0 = S_3 \setminus UC$

(iii) U : contractible (略)

§4 π_1

$$M = U \sqcup \left(\bigsqcup S_i^0\right) \sqcup \left(\bigsqcup C\right)$$

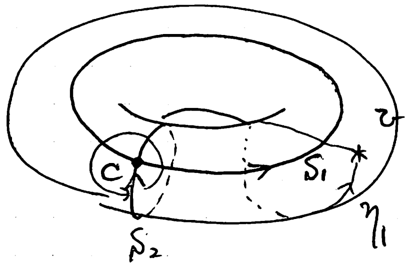
Stratification $\rightsquigarrow$ π_1

0-codim U $\rightsquigarrow$ $* \in U$

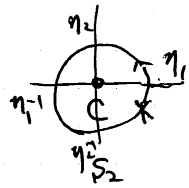
1-codim S_i $\rightsquigarrow$ η_i transversal loop.

2-codim C $\rightsquigarrow$ relations.

Ex



η_1 : well-defined in π_1



$\eta_1 \eta_2 \eta_1^{-1} \eta_2^{-1} = 1.$

§ Towards π_2 (??)

$\sigma : (D^2, \partial D^2) \longrightarrow (M(\mathcal{A}), *)$

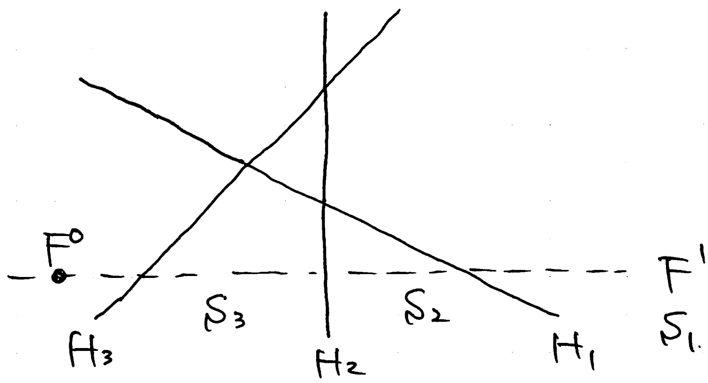
generic.

2011/11/15
⑨

$\sigma^{-1}(\bigcup_{i=1}^{n} S_i)$ = oriented S^1's の union

<u>予想(?)</u> σ の homotopy type は
Question $\sigma^{-1}(\cup S_i)$ で決まるか？

<u>例</u>

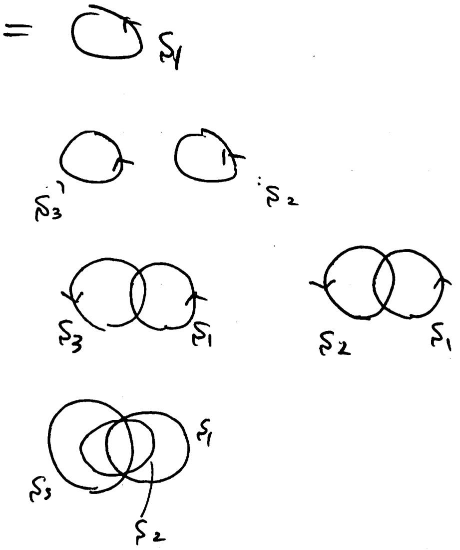


$M \simeq (T^3)^2$
$\simeq T^2 \setminus \{pt\}$

$\sigma : (D^2, S^1) \to (M, *)$, $U \ni *$, $* = F^0$

$\sigma^{-1}(\cup S_i) = \phi \Rightarrow \sigma(D^2) \subset U$
$\Rightarrow [\sigma] = 0$ in $\pi_2(M)$

$\sigma^{-1}(\cup S_i) =$



$\Rightarrow [\sigma] = 0$.

$\pi_2(M)$ is rank 1 module over $\mathbb{Z}\pi_1$
の生成元を上手くえらぶと $\sigma^{-1}(\cup S_i) =$