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<u>定義</u>

$$P = \left(\frac{\partial}{\partial t} + \Delta\right)\left(2\Delta f - |\nabla f|^2 + R\right) - 2\langle \nabla(2\Delta f - |\nabla f|^2 + R), \nabla f\rangle$$

$$\begin{cases} u = (4\pi\tau)^{-\frac{n}{2}} e^{-f} \\ \Box^* u = 0 \end{cases}$$

<u>命題 8</u>

$$P = 2|Ric|^2 + 4\langle Ric, \nabla\nabla f\rangle + 2|\nabla\nabla f|^2$$

↑
これが目標。

<u>補題 9</u>

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \Longrightarrow \frac{\partial}{\partial t} R_{g(t)} = \Delta R + 2|Ric|^2$$

<u>補題 10</u>

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \Longrightarrow \left(\frac{\partial}{\partial t}\Delta_{g(t)}\right)(f) = 2\langle Ric, \underbrace{\nabla\nabla f}_{\substack{\shortparallel \\ Hess(f)}}\rangle$$

<u>補題 11</u>

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

$$\frac{\partial}{\partial t} \underbrace{|\nabla f|^2}_{\substack{\shortparallel \\ g^{ij}\nabla_i f \nabla_j f.}} = 2Ric(\nabla f, \nabla f) + 2\left\langle \nabla\left(\frac{\partial f}{\partial t}\right), \nabla f\right\rangle$$

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証明

$$\frac{\partial}{\partial t}|\nabla f|^2 = \frac{\partial}{\partial t}\left(g^{ij}\nabla_i f \nabla_j f\right)$$

$$= \underbrace{\left(\frac{\partial}{\partial t} g^{ij}\right)}_{\substack{\shortparallel \\ 2R^{ij} = 2g^{ik}g^{jl}R_{kl}}} \nabla_i f \nabla_j f + g^{ij}\nabla_i\left(\frac{\partial f}{\partial t}\right)\nabla_j f + g^{ij}\nabla_i f \nabla_j\left(\frac{\partial f}{\partial t}\right)$$

$$= 2R^{ij}\nabla_i f \nabla_j f + 2g^{ij}\nabla_i\left(\frac{\partial f}{\partial t}\right)\nabla_j f$$

$$= 2\mathrm{Ric}(\nabla f, \nabla f) + 2\left\langle \nabla\left(\frac{\partial f}{\partial t}\right), \nabla f\right\rangle$$

($2\langle \mathrm{Ric}, \nabla f \nabla f\rangle$ とも書く)

補題12 (ボホナー公式)

$$\Delta|\nabla f|^2 = 2\langle \nabla\Delta f, \nabla f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f) + 2|\nabla\nabla f|^2$$

証明

$$\Delta = g^{ij}\nabla_i\nabla_j, \quad |\nabla f|^2 = g^{lk}\nabla_l f \nabla_k f$$

なので、

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$$\Delta|\nabla f|^2 = g^{ij}\nabla_i\nabla_j\left(g^{lk}\nabla_l f\nabla_k f\right)$$

$$= g^{ij}g^{lk}\nabla_i\nabla_j\left(\nabla_l f\nabla_k f\right)$$

$$= g^{ij}g^{lk}\nabla_i\left(\nabla_j\nabla_l f\nabla_k f + \nabla_l f\nabla_j\nabla_k f\right)$$

$$= g^{ij}g^{lk}\left(\nabla_i\nabla_j\nabla_l f\nabla_k f + \underbrace{\nabla_j\nabla_l f\nabla_i\nabla_k f}\right.$$

$$\left. + \underbrace{\nabla_i\nabla_l f\nabla_j\nabla_k f} + \nabla_l f\nabla_i\nabla_j\nabla_k f\right)$$

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$$= 2g^{ij}g^{lk}(\nabla_i\nabla_j\nabla_l f\nabla_k f + \nabla_j\nabla_l f\nabla_i\nabla_k f)$$

$$= \underbrace{2g^{ij}g^{lk}((\nabla_i\nabla_j)\nabla_l f\nabla_k f)}_{\text{以下, これをさらに計算する.}} + 2|\nabla\nabla f|^2$$

公式 (Ricci-identity, commutation formula)

$$(\nabla_i\nabla_j)\nabla_l f - \nabla_l(\nabla_i\nabla_j)f = \underbrace{R^p_{lij}}_{\text{曲率テンソル } Rm_{(1,3)} \text{ の成分}}\nabla_p f$$

$$\Downarrow$$

$$(\nabla_i\nabla_j)\nabla_l f = \nabla_l(\nabla_i\nabla_j f) + R^p_{lij}\cdot\nabla_p f$$

$$= 2g^{ij}g^{lk}(\nabla_l\nabla_i\nabla_j f + R^p_{lij}\cdot\nabla_p f)\nabla_k f + 2|\nabla\nabla f|^2$$

$$= 2g^{ij}g^{lk}\nabla_l\nabla_i\nabla_j f\cdot\nabla_k f + 2g^{ij}g^{lk}R^p_{lij}\nabla_p f\nabla_k f + 2|\nabla\nabla f|^2$$

$$= 2g^{lk}\nabla_l \underbrace{g^{ij}\nabla_i\nabla_j f}_{\Delta f}\cdot\nabla_k f + 2g^{ij}g^{lk}g^{pm}R_{lijm}\nabla_p f\cdot\nabla_k f + 2|\nabla\nabla f|^2$$

R_{lijm}
$= -R_{iljm}$
$= R_{ilmj}$

$$= 2\langle\nabla\Delta f, \nabla f\rangle + \underbrace{2g^{ij}g^{lk}g^{pm}R_{ilmj}}_{g^{ij}R_{ilmj} = R_{lm}}\nabla_p f\nabla_k f + 2|\nabla\nabla f|^2$$

$$= 2\langle\nabla\Delta f, \nabla f\rangle + 2\underbrace{g^{lk}g^{pm}R_{lm}}_{= R^{kp}}\nabla_p f\nabla_k f + 4|\nabla\nabla f|^2$$

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$$= 2\langle \nabla\Delta f, \nabla f\rangle + 2Ric(\nabla f, \nabla f) + 2|\nabla\nabla f|^2$$

となり，ボホナー公式が出る。

[補題13 $\Delta(e^{-f}) = (-\Delta f + |\nabla f|^2)e^{-f}$

$\because$ $\Delta = g^{ij}\nabla_i\nabla_j$

$$\Delta(e^{-f}) = g^{ij}\nabla_i\nabla_j(e^{-f})$$
$$= g^{ij}\nabla_i(-\nabla_j f e^{-f})$$
$$= g^{ij}(-\nabla_i\nabla_j f e^{-f} - \nabla_j f(-\nabla_i f)e^{-f})$$
$$= -g^{ij}\nabla_i\nabla_j f e^{-f} + g^{ij}\nabla_j f\nabla_i f e^{-f}$$
$$= -\Delta f\cdot e^{-f} + |\nabla f|^2 e^{-f}$$
$$= (-\Delta f + |\nabla f|^2)e^{-f}$$

[補題14 $\Box^* = -\frac{\partial}{\partial t} - \Delta + R$， $\Box^* u = 0$ とし，

$u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ とするとき

$$\left(\frac{\partial}{\partial t} + \Delta\right)f = |\nabla f|^2 - R + \frac{n}{2\tau} \qquad \frac{\partial\tau}{\partial t} = 1$$

証明 $\Box^* u = 0$ なので

$$0 = \Box^* u$$
$$= \left(-\frac{\partial}{\partial t} - \Delta + R\right)\left[(4\pi\tau)^{-\frac{n}{2}}e^{-f}\right]$$
$$= -\frac{\partial}{\partial t}\left((4\pi\tau)^{-\frac{n}{2}}e^{-f}\right) + (-\Delta + R)\left((4\pi\tau)^{-\frac{n}{2}}e^{-f}\right)$$

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$$= \left(-\frac{\partial}{\partial t}(4\pi\tau)^{-\frac{n}{2}} e^{-f} - (4\pi\tau)^{-\frac{n}{2}}\left(\frac{\partial}{\partial t} e^{-f}\right) + (4\pi\tau)^{-\frac{n}{2}}\left[(-\Delta+R)e^{-f}\right]\right.$$

$$= -\frac{n}{2\tau} \underbrace{(4\pi\tau)^{-\frac{n}{2}} e^{-f}}_{u} - (4\pi\tau)^{-\frac{n}{2}}\left[\left(\frac{\partial}{\partial t}+\Delta\right)e^{-f}\right]$$

$$+ \underbrace{(4\pi\tau)^{-\frac{n}{2}} R e^{-f}}_{=u}.$$

$$= -\frac{n}{2\tau} u - (4\pi\tau)^{-\frac{n}{2}}\left(\frac{\partial}{\partial t} e^{-f} + \Delta e^{-f}\right) + Ru$$

$$= -\frac{n}{2\tau} u - (4\pi\tau)^{-\frac{n}{2}}\left(\underbrace{\frac{\partial}{\partial t} e^{-f}}_{-\frac{\partial f}{\partial t}e^{-f}} - \Delta f e^{-f} + |\nabla f|^2 e^{-f}\right) + Ru$$

$$= -\frac{n}{2\tau} u + (4\pi\tau)^{-\frac{n}{2}}\left(\frac{\partial f}{\partial t} + \Delta f - |\nabla f|^2\right)e^{-f} + Ru$$

$$= -\frac{n}{2\tau} u + u \times \left(\frac{\partial f}{\partial t} + \Delta f - |\nabla f|^2\right) + Ru.$$

両辺をuでわると

$$0 = -\frac{n}{2\tau} + \frac{\partial}{\partial t} + \Delta f - |\nabla f|^2 + R$$

図

$\downarrow$ 両辺に $2\Delta f$ をたして，∇ を作用させる

補題15 $\nabla\left(3\Delta f - |\nabla f|^2 + R + \frac{\partial f}{\partial t}\right) = 2\nabla\Delta f.$

以上を使って命題8を証明できる.

命題8 $P = 2|Ric|^2 + 4\langle Ric, \nabla\nabla f\rangle + 2|\nabla\nabla f|^2$

証明 $\Box^* = -\frac{\partial}{\partial t} - \Delta + R$，$u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$

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$$P = \left(\frac{\partial}{\partial t} + \Delta\right)(2\Delta f - |\nabla f|^2 + R) - 2\langle \nabla(2\Delta f - |\nabla f|^2 + R), \nabla f\rangle$$

$$= 2\left(\frac{\partial}{\partial t} + \Delta\right)(\Delta f) - \left(\frac{\partial}{\partial t} + \Delta\right)(|\nabla f|^2) + \left(\frac{\partial}{\partial t} + \Delta\right)R$$

$$- 2\langle \nabla(2\Delta f - |\nabla f|^2 + R), \nabla f\rangle$$

$$= 2\left(\left(\frac{\partial}{\partial t}\Delta\right)(f) + \Delta\left(\frac{\partial f}{\partial t}\right)\right) + 2\Delta(\Delta f) - \frac{\partial}{\partial t}|\nabla f|^2 - \Delta|\nabla f|^2$$

$$+ \frac{\partial}{\partial t}R + \Delta R - 2\langle \qquad \rangle$$

$$= 2\underbrace{\left(\frac{\partial}{\partial t}\Delta\right)}_{\text{補題10}}(f) + \underbrace{2\Delta\left(\left(\frac{\partial}{\partial t} + \Delta\right)(f)\right)}_{\text{補題14}} - \underbrace{\frac{\partial}{\partial t}|\nabla f|^2}_{\text{補題11}} - \underbrace{\Delta|\nabla f|^2}_{\text{補題12}}$$

$$+ \underbrace{\frac{\partial}{\partial t}R}_{\text{補題9}} + \Delta R - 2\langle \qquad \rangle$$

$$= \cdots$$

$$= 4\langle Ric, \nabla\nabla f\rangle + 2\underbrace{\Delta|\nabla f|^2}_{\text{補題12}} - 4Ric(\nabla f, \nabla f) - 2|\nabla\nabla f|^2$$

$$- 2\langle \underbrace{\nabla\left(3\Delta f - |\nabla f|^2 + R + \frac{\partial}{\partial t}\right)}_{\text{補題15}}, \nabla f\rangle$$

$$= 4\langle Ric, \nabla\nabla f\rangle + 2|\nabla\nabla f|^2 + 2|Ric|^2$$ 証

もとにもどる.

$$u^{-1}\square^*(wu) = 2\Delta f + |\nabla f|^2 + R - \underbrace{\left(\frac{\partial}{\partial t} + \Delta\right)f}_{\text{補題14}} - \tau \underbrace{P}_{\text{命題8}}$$

であった.

$$u^{-1}\square^*(wu) = 2\Delta f + |\nabla f|^2 + R - \left(|\nabla f|^2 - R + \frac{n}{2\tau}\right) - \tau P.$$

$$= 2\Delta f + |\nabla f|^2 + R - |\nabla f|^2 + R - \frac{n}{2\tau} - \tau P$$

$$= 2\Delta f + 2R - \frac{n}{2\tau} - \tau P$$

$$= 2(\Delta + R)f - \frac{n}{2\tau} - \tau P$$

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命題16

$$u^{-1}\square^*(wu) = 2(\Delta+R)f - \frac{n}{2\tau} - \tau P$$
$$P = 2|Ric|^2 + 4\langle Ric, \nabla\nabla f\rangle + 2|\nabla\nabla f|^2$$

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$$u^{-1}\square^*(wu) = -2\tau\left|Ric + \nabla\nabla f - \frac{g}{2\tau}\right|^2$$

命題17

$$\left|Ric+\nabla\nabla f-\frac{g}{2\tau}\right|^2 = |Ric|^2 + 2\langle Ric, \nabla\nabla f\rangle - \frac{1}{\tau}R + |\nabla\nabla f|^2 - \frac{1}{\tau}\Delta f + \frac{n}{4\tau^2}$$

T_{ij} に対し
$|T|^2 := g^{ij}g^{kl}T_{ik}T_{jl}$.

証明

$$\left|\underset{ij}{Ric} + \underset{ij}{\nabla\nabla f} - \underset{ij}{\frac{g}{2\tau}}\right|^2 = g^{ij}g^{kl}\left(R_{ik}+\nabla_i\nabla_k f - \frac{g_{ik}}{2\tau}\right)\left(R_{jl}+\nabla_j\nabla_l f-\frac{g_{jl}}{2\tau}\right)$$

$$= g^{ij}g^{kl}\Big(R_{ik}R_{jl} + R_{ik}\nabla_j\nabla_l f - \frac{1}{2\tau}R_{ik}g_{jl} + R_{jl}\nabla_i\nabla_k f + \nabla_i\nabla_k f\nabla_j\nabla_l f - \frac{g_{jl}}{2\tau}\nabla_i\nabla_k f - \frac{g_{ik}}{2\tau}R_{jl} - \frac{1}{2\tau}g_{ik}\nabla_j\nabla_l f + \frac{1}{4\tau^2}g_{ik}g_{jl}\Big)$$

$$= \underbrace{R^{jl}R_{jl}}_{|Ric|^2} + \underbrace{R^{jl}\nabla_j\nabla_l f}_{\langle Ric,\nabla\nabla f\rangle} - \frac{1}{2\tau}\underbrace{g^{ij}g^{kl}R_{ik}g_{jl}}_{R.}$$

あとはレポートとする.

$$\Downarrow$$

定理Aが従う.

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⑧

3. W-entropy (Perelman)

定義18 (W-entropy)

(M, g) リーマン多様体 (n次元)

$\mathcal{R}_M$: M上のリーマン計量全体

$\mathbb{R}^+$: 正の実数全体

$C^\infty(M)$: M上の C^∞級関数全体

$$W : \mathcal{R}_M \times C^\infty(M) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

$$(g, f, \tau) \longmapsto W(g, f, \tau)$$

$$W(g, f, \tau) = \int_M \left[\tau (R + |\nabla f|^2) + f - n \right] d\mu_g$$

W-entropy という.

Rはスカラー曲率

$R = g^{ij} R_{ij}$

fが重要な役割をはたす.

命題19

$$W(g, f, \tau) = \int_M wu$$

Perelman's Harnack quantity.