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Obstructions to deforming curves on a uniruled 3-fold.

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§ Intro

$k = \bar{k}$, char $k \geq 0$

V : proj. 3-fold

$V \supset S$ (smooth surf.) $\supset C$, $V \supset C$ smooth curve

曲線 C の V 内での変形を曲面 S を用いて研究.

Hilbert sch.

$\mathrm{Hilb}^{sc} V = \{ C \subset V : \text{smooth curve} \}$ (irred.)

$H^0(C, N_{C/V})$ tang. sp.

$H^1(C, N_{C/V})$ obst. sp.

Ex. S : smooth cubic $\subset \mathbb{P}^3$

$S = \mathrm{Blow}_{6pt} \mathbb{P}^2 \supset E$: (-1)-$\mathbb{P}^1$ (直線)

h 超平面切断

$C \in |4h + 2E| \subset S \subset \mathbb{P}^3$

次数14, 種数24 (空間曲線)

para'd by W ($\subset \mathrm{Hilb}^{sc}_{14,24} \mathbb{P}^3$) of dim 56

$$\left(\dim |\mathcal{O}_{\mathbb{P}^3}(3)| + \dim |\mathcal{O}_S(C)| = 56 \right), \quad \dim |\mathcal{O}_{\mathbb{P}^3}(3)| = 19, \quad \dim |\mathcal{O}_S(C)| = 37$$

$h^0(N_{C/\mathbb{P}^3}) = 57$ (>56)

$\dim W \leq \dim_{[C]} \mathrm{Hilb}\, \mathbb{P}^3 \leq h^0(N_{C/\mathbb{P}^3})$, $\dim W = 56$, $h^0(N_{C/\mathbb{P}^3}) = 57$

Thm (Mumford '62) char $= 0$

- $\overline{W}$ is an irred. comp. of $(\mathrm{Hilb}\, \mathbb{P}^3)_{red}$
- $\mathrm{Hilb}\, \mathbb{P}^3$ is generically non-reduced along W.

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generalization by
Kleppe, Ellia, Gruson-Peskine, Froystad,
Vakil.

<u>Thm</u> (Mukai-Nasu, 2009) any char.

V : sm. proj. 3-fold, Suppose that

(1) $\mathbb{P}^1 \cong E \subset V$, $N_{E/V}$: global sect. で生成

(2) $E \subset {}^{\exists}S \subset V$ 中間曲面 (smooth)

$(E)^2_S = -1$, $H^1(N_{S/V}) = P_g(S) = 0$

$\Longrightarrow$ $\mathrm{Hilb}^{sc} V$ has infinitely many gen. non-reduced comp.

<u>Ex.</u> (1) V : Fano 3-fold, $-K_V = H + H'$: ample divs.

${}^{\exists}S \in |H|$ smooth surf

$S \not\cong \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, then ${}^{\exists}E \cong \mathbb{P}^1$, s.t. $E^2 = -1$

(cubic $V_3^{(3)} \subset \mathbb{P}^4$)

(2) $V \xrightarrow{\pi} F$ $\mathbb{P}^1$-bundle over smooth surf F, $P_g(F) = 0$

$$\begin{array}{ccc} V & \xrightarrow{\pi} & F \\ \cup & & \cup \\ S_1 \text{ : section} & & A \text{ : ample} \end{array}$$

$$\begin{array}{ccc} \text{(smooth) } S \subset |S_1 + \pi^* A| & \longrightarrow & F \\ \cup & & \Psi \\ E & \longrightarrow & pt. \end{array}$$

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§ Obstruction criterion

$C \subset S \subset V$ flag of (smooth) curve, surf, 3-fold

$\tilde{C} \subset V \times \operatorname{Spec} k[t]/\langle t^2 \rangle \longleftrightarrow \alpha \in H^0(N_{C/V})$
(1st order)

$\tilde{C}$ lifts to $\tilde{\tilde{C}}$ (2nd order) iff $ob(\alpha) \in H^1(N_{C/V})$, $ob(\alpha) = 0$

$\pi : N_{C/V} \longrightarrow N_{S/V}|_C$ proj

Def
$$\begin{cases} \pi_S(\alpha) = H^0(\pi)(\alpha) \in H^0(N_{S/V}|_C) \\ ob_S(\alpha) = H^1(\pi)(ob(\alpha)) \in H^1(N_{S/V}|_C) \end{cases}$$
exterior component (外成分)

☆ infinitesimal deformation with pole

($E \subset S \subset V$ flag

これ → $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$

1位の極　2E, 3E … 計算?
(2位) (3位)

☆ $ob(\alpha) \neq 0$ の十分条件

$\bigcirc \in H^0(N_{E/V}(E))$, $N_{E/V}(E) = N_{E/V} \otimes \mathcal{O}_E(E)$, $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^1}(-1)$

(☆)
$$\begin{array}{ccccccc}
H^0(N_{C/V}) \ni \alpha & & & & & & \\
\downarrow \quad \downarrow & & & & & & \\
H^0(N_{S/V}|_C) \ni \pi_S(\alpha) = v|_C & \longleftarrow & \exists\, v \text{ (仮定)} & \longrightarrow & v|_E & \not\ni & \text{(仮定) } \operatorname{Im}(\pi_E \otimes \mathcal{O}(E)) \\
\cap & & \cap & & \cap & & \\
H^0(N_{S/V}(E)|_C) & \longleftarrow & H^0(N_{S/V}(E)) & \longrightarrow & H^0(N_{S/V}(E)|_E) & \xleftarrow{\pi_E \otimes \mathcal{O}(E)} & H^0(N_{E/V}(E))
\end{array}$$

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Thm (Mukai – Nasu, 2009)

$C, E \subset S \subset V$ as above.

E : (-1)-$\mathbb{P}^1$ on S

$\alpha \in H^0(N_{C/V})$ satisfy (☆)

If ① $\Delta = C + K_V|_S - 2E$,

$(\Delta, E)_S = 0$.

② $H^0(S, \Delta) \twoheadrightarrow H^0(E, \Delta|_E) \simeq k$ 全射.

then, $ob_S(\alpha) \neq 0$.

☆ How to apply the criterion (sketch of pf.)

del pezzo $\begin{cases} V = V_3 \subset \mathbb{P}^4 & : \text{smooth cubic 3-fold} \\ \cup & \\ S = H \cap V_3 & : \text{smooth cubic surf} \end{cases}$

$\cup$

$E \leftarrow (-1)$-$\mathbb{P}^1$

↗ 一般

$C \in |-K_V|_S + 2E| \subset S$

↖ 次数 8, 種数 5

[Mumford の ex.
$-K_V = 4h$
$C - 4h - 2E = 0$.
$V = \mathbb{P}^3$]

Para'd by $W_C^{(16)} \subset \mathrm{Hilb}^{sc}_{8,5} V_3$

$h^0(N_{C/V}) = 17$ (> 16)

$(0 \to N_{C/S} \to N_{C/V} \to N_{S/V}|_C \to 0)$

$\underset{\underset{16}{\shortparallel}}{\dim W} \leq \dim_{[C]} \mathrm{Hilb}\, V \underset{\uparrow}{\leq} \underset{\underset{17}{\shortparallel}}{h^0(N_{C/V})}$

"<" を示す!!

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It suffices to prove that C is obstructed.

$$\underset{\substack{\| \\ -K_S}}{N_{S/V}}(E) \simeq -K_{S'}$$

S' = blow-down of S by E
↑
$\mathbb{P}^2$の5点 blow-up.

$$h^0(N_{S/V}(E)) = h^0(N_{S/V}) + 1$$

$v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$ (極付無限小変形)

$$\left[0 \to N_{E/S} \to N_{E/V} \xrightarrow{\pi_E} N_{S/V}|_E \to 0\right] \otimes \mathcal{O}_E(E)$$

$N_{E/S} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, $N_{E/V} \simeq \mathcal{O}^{\oplus 2}$, $N_{S/V}|_E \simeq \mathcal{O}(1)$, $\mathcal{O}_E(E) \simeq \mathcal{O}(-1)$

($N_{E/V} \simeq \mathcal{O}^{\oplus 2}$ ← Iskovskih (char = 0))

$$H^0(N_{E/V}(E)) \simeq H^0(\mathcal{O}(-1)^{\oplus 2})$$

Cの取り方より

$$\Delta = C + K_V|_S - 2E = 0.$$

(1), (2) の条件を満たす.

$\exists \alpha \longrightarrow v|_C \in H^0(N_{S/V}|_C)$

$\alpha \in H^0(C, N_{C/V})$, $H^0(C, N_{C/V}) \xrightarrow{\text{射}} H^0(N_{S/V}|_C)$

$C \cap E = \emptyset$

$$C \sim -K_V|_S + 2E = 2h + 2E$$

$$\left((C, E) = 2Eh + 2E^2 = 2 - 2 = 0 \right)$$

$\therefore \mathrm{ob}_S(\alpha) \neq 0$

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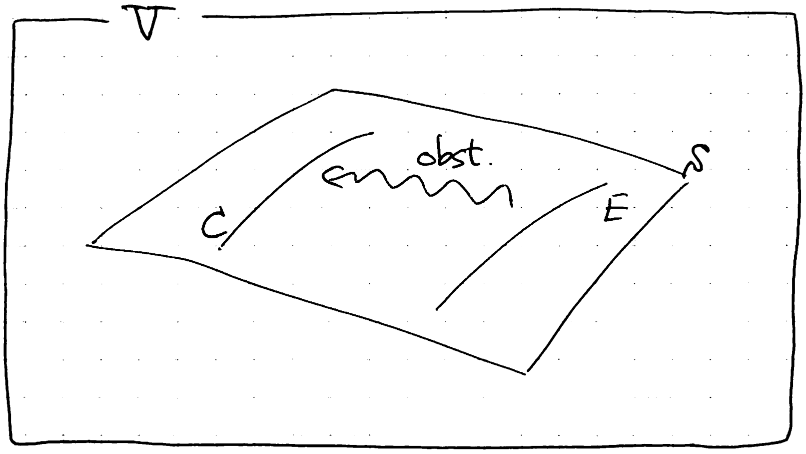

§ Higher dimensional example

$$E \subset S \subset V \subset \mathbb{P}^n$$

excep. div. $\rightsquigarrow$ C, $C \subset S$ (hypersurf.)

Construct an example of obstructed curve $C \subset S$ similar to Mumford's example.

$n \geq 2$.

$$\begin{array}{ccccc} V = \mathbb{P}^1 \times \mathbb{P}^n & \xrightarrow{\text{Segre}} & \mathbb{P}^{2n+1} \\ \cup & & \cup \\ E \subset S = V \cap H & \subset & \mathbb{P}^{2n} \\ \pi \downarrow \text{ blow-up along } L & & \\ \mathbb{P}^n \supset L^{n-2} & & \end{array}$$

C in S; $C' \subset \mathbb{P}^n$; C' and L disjoint

<u>Ex</u> $C' \subset \mathbb{P}^n$ 完全交叉

$C' = Q_1 \cap Q_2 \cap \cdots \cap Q_{n-1}$ $(\deg Q_i \geq 5\ (\forall i))$

$C' \cap L = \emptyset$

$C = \pi^{-1}(C') \subset S \implies \exists$ obst. def. $\tilde{C}$ of C in V

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§ Hom scheme への応用

X, V : variety

$$\mathrm{Hom}(X, V) = \{ f : X \to V \text{ morphism} \} \underset{\text{closed}}{\subset} \mathrm{Hilb}(X \times V)$$

Thm (Mukai - Nasu)

X : 種数 5 の一般曲線

V_3 : cubic 3-fold (一般 or フェルマー型)

$\Rightarrow$ $\mathrm{Hom}_8(X, V_3)$ has a generically non-reduced component T of dim $\underline{\underline{4}}$

image の degree

$$\begin{array}{ccc} \mathrm{Hilb}\, V_3 & & \\ \cup & & \\ W^{(16)} & \xrightarrow{\varphi} & \mathcal{M}_5 \\ \cup & & \ni \\ T \simeq \varphi^{-1}(X) & \longrightarrow & X \ (\text{一般}) \end{array}$$

($W^{(16)}$: non-red., $\mathcal{M}_5$: smooth)

Obs. cri の pf.

$$\alpha \in H^0(N_{C/V}) \simeq \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C)$$

$\downarrow$ coboundary map

$$\alpha \mapsto \delta(\alpha) \in \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C)$$

$$0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0$$

$$ob(\alpha) = \delta(\alpha) \cup \alpha \in \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \supset H^1(N_{C/V}) \overset{\text{実は}}{\ni} \delta(\alpha) \cup \alpha$$

$$\cup : \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \times \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

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ideal sheaf $\mathcal{I}_S \subset \mathcal{I}_C \subset \mathcal{O}_V$

$$
\begin{array}{ccccc}
\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) & \times & \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) & \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \\
\downarrow & & \ni \alpha & & \downarrow \\
\mathrm{Ext}^1(\mathcal{I}_S, \mathcal{I}_C) & \times & \text{——}/\!/\text{——} & \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_S, \mathcal{O}_C) \\
\cong & & & & \cong \\
H^1(\mathcal{I}_C \otimes \mathcal{O}_V(S)) & & \xrightarrow{\ \cup \alpha\ } & & H^1(\mathcal{O}_C(S))
\end{array}
$$

カップ積を計算.

↓

最終的には Serre duality の perfect coupling に帰着できる.