Computer Graphics in Minimal Surface Theory

Shoichi Fujimori

Abstract We give a summary of the computer-aided discoveries in minimal surface theory. In the latter half of the 20th century, the global properties of complete minimal surfaces of finite total curvature were investigated. Proper embeddedness of a surface is one of the most important properties amongst the global properties. However, before the early 1980s, only the plane and catenoid were known to be properly embedded minimal surfaces of finite total curvature. In 1982, a new example of a complete minimal surface of finite total curvature was found by C. J. Costa. He did not prove its embeddedness, but it was seen to satisfy all known necessary conditions for the surface to be embedded, and D. Hoffman and W. Meeks III later proved that the surface is in fact embedded. Computer graphics was a very useful aid for proving this. In this paper we introduce this interesting story.

Keywords computer graphics, minimal surfaces.

1 Introduction

A *minimal surface* in Euclidean three-space $\mathbb{R}^3$ is a surface at which each point of the surface has a neighborhood that is the surface of least area with respect to its boundary. Such soap films (not soap bubbles trapping air) are physical models of minimal surfaces. Minimal surface theory is one of the classical subjects in differential geometry.

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The global theory of minimal surfaces has generated much interesting literature in the past three decades. One reason for this is related to the development of computers. Using computers we can try numerical experiments and visualize the surfaces. This gives mathematicians the ability to find essential properties of minimal surfaces, and then establish those properties mathematically.

In this paper we exhibit some examples of how computer graphics have contributed to the development of minimal surface theory.

Although we do include some computer graphics of minimal surfaces in this paper, there are numerous places where one can find a wide variety of computer graphics. In particular, the web page [10] maintained by Matthias Weber is a great resource.

2 The history of minimal surfaces, 18th and 19th centuries

In this section we briefly review the history of minimal surface theory in the 18th and 19th centuries by giving examples. For further details of this history, we refer to [3, 7].

The flat plane is the simplest and oldest example of a minimal surface. The first example of minimal surfaces other than the plane is the catenoid found by L. Euler in 1744 as a minimal surface of revolution (Euler called the surface an "alysseid", but after Plateau's work the surface has been called a catenoid). See the left hand side of Fig. 1. In 1776, J. B. M. C. Meusnier found the next example, the helicoid, as a ruled minimal surface. See the right hand side of Fig. 1. Note that the helicoid is a singly periodic surface. Meusnier noticed that the definition of a minimal surface is equivalent to a surface having a vanishing geometric quantity which is known today as the *mean curvature*.

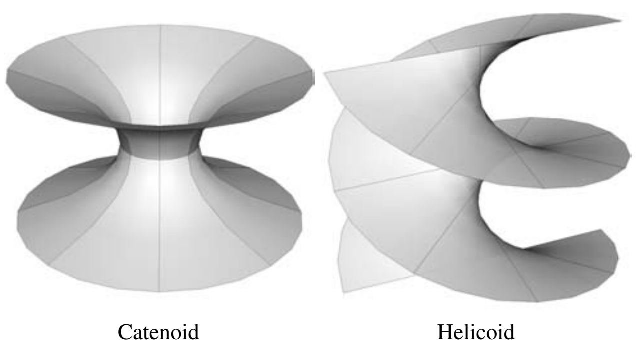


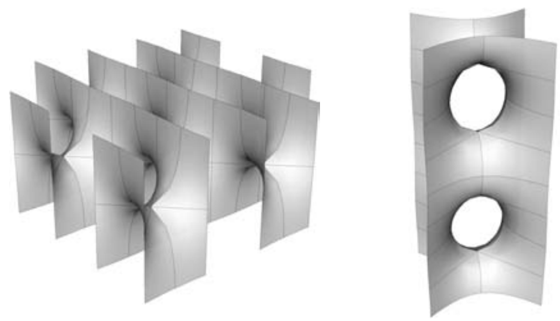
Fig. 1 The catenoid and helicoid.

If the surface is given as a graph $z = f(x,y)$, then it is a minimal surface if and only if f satisfies the following (quasilinear elliptic) partial differential equation

$$(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy} = 0 \tag{1}$$

which is today called the *minimal surface equation*.

So finding minimal surfaces is equivalent to finding solutions of the minimal surface equation (1). But because of the difficulties in solving it, in the 18th century no other minimal surfaces were found, as far as I know, and in 1835 H. F. Scherk found new examples. The most famous example Scherk found is a doubly periodic minimal surface. See the left hand side of Fig. 2. Scherk found this example as a translation surface. Another famous surface by Scherk is a singly periodic minimal surface. See the right hand side of Fig. 2.



Doubly periodic Singly periodic

Fig. 2 Scherk surfaces.

In the latter half of the 19th century, an important connection between minimal surfaces and complex analysis was obtained by great mathematicians such as K. Weierstrass, A. Enneper, H. A. Schwarz, B. Riemann, and others. One of the strongest tools for minimal surface theory is the following representation formula, which is today called the *Weierstrass representation*.

Theorem 1 (Weierstrass representation). *Let M be a Riemann surface, g a meromorphic function, and η a holomorphic 1-form on M so that $(1+|g|^2)^2\eta\bar{\eta}$ gives a positive definite metric on M. Then*

$$f(p) = \mathrm{Re}\int_{p_0}^{p} \left(1-g^2,\, i(1+g^2),\, 2g\right)\eta \qquad (p \in M) \tag{2}$$

is a conformally immersed minimal surface into $\mathbb{R}^3$. The stereographic projection of the Gauss map, that is, the unit normal, of f is equal to g. Moreover, f is single-valued on M if and only if

$$\mathrm{Re}\oint_{\ell}\left(1-g^2,\, i(1+g^2),\, 2g\right)\eta=\mathbf{0} \tag{3}$$

for any closed curve ℓ on M. Conversely, any minimal surface can be obtained in this manner.

This formula is extremely useful, because with it we can avoid finding a solution of the minimal surface equation (1), which is a second order partial differential equation, and we only need to take path integrals of complex analytic functions instead.

Using this formula, many interesting examples have been found. This formula is still one of the most powerful tools for minimal surface theory.

To close this section, we give two more examples found in this period. One is by Riemann. Riemann considered a minimal surface foliated by parallel circles, and then found a singly periodic minimal surface which is today called the *Riemann's staircase*. See the left hand side of Fig. 3.

The other is by Schwarz. Schwarz gave explicit parametrizations of minimal surfaces bounded by some special quadrilaterals in $\mathbb{R}^3$. He then applied the technique which is today called the *Schwarz reflection principle*, and he and his student E. R. Neovius constructed five examples of triply periodic minimal surfaces. The right hand side of Fig. 3 is called the *Schwarz primitive surface*.

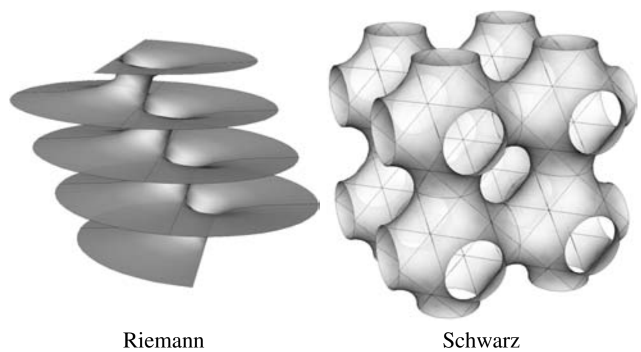


Fig. 3 Riemann's staircase and the Schwarz primitive surface.

Some beautiful images of minimal surfaces drawn by Meusnier, Schwarz, and Neovius themselves can be found in [11].

3 The Plateau problem

Between the late 19th century and the first half of the 20th century, the main topic of minimal surface theory was to solve the *Plateau problem*, that is, for any given

Jordan curve in $\mathbb{R}^3$, to ask whether there exists an area minimizing surface with that Jordan curve as its boundary. This problem was affirmatively solved by J. Douglas and T. Rado in 1930. They did not give an explicit solution, like Schwarz did for a given quadrilateral, but, using abstract arguments like in the Bourbaki style, they solved the problem. Due to the non-explicit methods, drawing images of minimal surfaces went out of fashion until the 1980s.

4 The global properties

In the latter half of the 20th century, the global properties of complete (as a metric space) minimal surfaces were investigated. Let $f : M \to \mathbb{R}^3$ be a conformally immersed minimal surface, and K and dA the Gaussian curvature and area element of the surface, respectively. Then the total curvature of the surface is given by

$$\int_M K dA,$$

which is either a non-positive real number or $-\infty$ for minimal surfaces.

In 1964, R. Osserman [8] showed the following theorem for a complete minimal surface of *finite total curvature*:

Theorem 2. *Let $f : M \to \mathbb{R}^3$ be a complete minimal surface of finite total curvature. Then the Riemann surface M is biholomorphic to a compact Riemann surface $\overline{M}$ of genus n $(n \geq 0)$ excluding a finite number of points $\{p_1, \ldots, p_k\}$ $(k \geq 1)$. Moreover, the Gauss map g, which is a meromorphic function on M, of f extends meromorphically to $\overline{M}$. Furthermore, the total curvature of f satisfies*

$$\frac{1}{2\pi} \int_M K dA \leq \chi(M) - k = -2(n + k - 1), \tag{4}$$

where $\chi(M)$ is the Euler characteristic of M.

The inequality (4) is called the *Osserman inequality*, which is a sharpening of the Cohn-Vossen estimate

$$\frac{1}{2\pi} \int_M K dA \leq \chi(M)$$

for the total curvature of a complete surface. The condition for equality to hold in (4) was given by L. Jorge and W. Meeks III [6] in 1983.

5 The Costa-Hoffman-Meeks surfaces

The examples explained in Sect. 2 are all properly embedded, that is, each of them is free of self-intersections, and the inverse image of any compact set in $\mathbb{R}^3$ into the associated Riemann surface (in which the minimal surface is defined) is compact. Such surfaces are called *properly embedded minimal surfaces*. As far as this author knows, the examples explained in Sect. 2 were all of the known properly embedded minimal surfaces, before the early 1980s. We remark that only the plane and catenoid are of finite total curvature in the examples in Sect. 2. In fact, no properly embedded minimal surfaces of finite total curvature other than the plane and the catenoid were known before the beginning of the 1980s.

In 1983, R. Schoen [9] showed that the catenoid was the unique properly embedded minimal surface of finite total curvature which is homeomorphic to a compact surface with two points removed. Also in 1983, L. Jorge and W. Meeks III [6] showed that there do not exist properly embedded minimal surfaces of finite total curvature which are homeomorphic to a sphere with three, four, or five points removed.

So the following question naturally occurs:

Question 1. Do there exist properly embedded minimal surfaces of finite total curvature other than the plane and the catenoid?

We now follow [4] to describe how this problem was solved, and how computer graphics helped to solve it.

In 1982, C. J. Costa at IMPA in Rio de Janeiro constructed a new minimal surface which is homeomorphic to a torus with three points removed, using the Weierstrass elliptic $\wp$ function. Costa showed that his example is properly immersed, has total curvature -12π (finite!), and satisfies all known necessary conditions for the surface to be embedded. See [2]. We call a punctured neighborhood of an omitted point an *end*. Costa's surface has three ends. Two ends grow logarithmically in the z direction with $z \to \pm\infty$, like a catenoid, while the other end is asymptotic to the xy-plane. We quote from D. Hoffman [4]:

> The idea of having one "flat" end between two "catenoid" ends occurred to Costa in an unusual way. ... at the movies at Rio, watching a documentary about the preparations of some "Samba School" for the dance competitions at Carnival time, he saw a dancer with an outlandish headdress that was made to look like two crows—one head up, the other head down—with their wings meeting in an expanding circle in the middle. This gave him the idea to try to create a plane-like end between the catenoid ends.

Although Costa did vaguely have a resemblance of the surface in his mind, nobody knew what the surface would look like, nor if it were embedded.

Hoffman and Meeks (with a graphics programmer J. Hoffman) has succeeded to produce computer graphics of the Costa surface, to investigate geometric features of the surface. See Fig. 4. They were able to reveal that the Costa surface is highly symmetric. In fact, the intersection of the Costa surface and the xy-plane consists of two straight lines meeting at a right angle at the origin, and the intersection of the surface with the xz-plane and yz-plane consists of three geodesics respectively. See Fig. 5. As a result, the Costa surface can be decomposed into eight congruent pieces, and this is the key to prove the embeddedness of the Costa surface mathematically. Hoffman described this in [4]:

> To prove the surface is embedded is now reduced to the question of whether each piece is embedded. In fact, we were able to show that each piece is a graph over an appropriate plane in $\mathbb{R}^3$. The computer graphics were quite helpful in checking the computation concerning which planes were the correct planes for projection.

In this way, they could prove that the Costa surface is indeed a properly embedded minimal surface of finite total curvature, other than the plane and the catenoid.

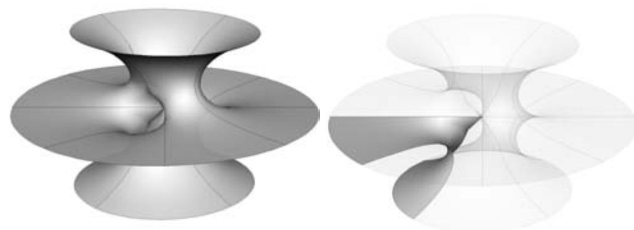


Fig. 4 The Costa surface (left) and its fundamental piece (right).

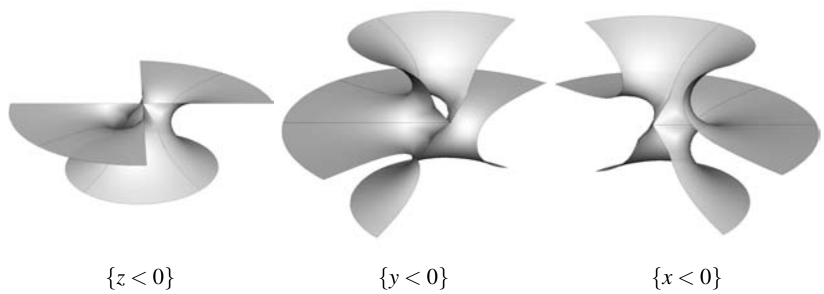


Fig. 5 Halves of the Costa surfaces.

Moreover, they succeeded to generalize the Costa surface to an arbitrary value for the genus. That is, for an arbitrary positive integer n, there exists a properly embedded minimal surface of finite total curvature which is homeomorphic to a compact surface of genus n with three points removed. See Fig. 6. These minimal surfaces are now called the *Costa-Hoffman-Meeks surfaces*. For details, we refer to [5].

Thereafter, many new minimal surfaces were discovered in quick succession, and minimal surface theory progressed explosively.

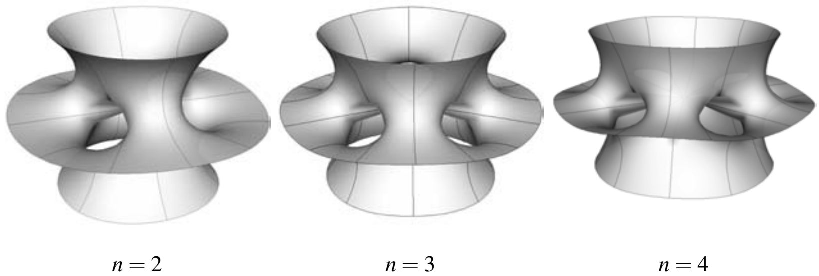


Fig. 6 The Costa-Hoffman-Meeks surfaces of genus n.

6 The Weierstrass data

Here we list the data for minimal surfaces we have seen in this paper. Solving the integral (2) for these data, one can obtain parametrizations of the surfaces.

(i) The catenoid (non-periodic): $M=\mathbb{C}\setminus\{0\}$, $g=z$, $\eta=\dfrac{dz}{z^2}$.

(ii) The helicoid (singly periodic): $M=\mathbb{C}\setminus\{0\}$, $g=z$, $\eta=i\dfrac{dz}{z^2}$.

(iii) Scherk surface (doubly periodic): $M=(\mathbb{C}\cup\{\infty\})\setminus\{1,-1,i,-i\}$,

$$g=z, \qquad \eta=\frac{dz}{z^4-1}.$$

(iv) Scherk surface (singly periodic): $M=(\mathbb{C}\cup\{\infty\})\setminus\{1,-1,i,-i\}$,

$$g=z, \qquad \eta=i\frac{dz}{z^4-1}.$$

(v) Riemann's staircase (singly periodic):

$$M=\big\{(z,w)\in(\mathbb{C}\cup\{\infty\})^2 \mid w^2=z(z^2-1)\big\}\setminus\{(0,0),(\infty,\infty)\},$$

$$g=z, \qquad \eta=\frac{dz}{zw}.$$

(vi) Schwarz primitive surface (triply periodic):

$$M=\big\{(z,w)\in(\mathbb{C}\cup\{\infty\})^2 \mid w^2=z^8+14z^4+1\big\},$$

$$g=z, \qquad \eta=\frac{dz}{w}.$$

(vii) Costa-Hoffman-Meeks surface of genus n (non-periodic):

$$M = \left\{(z,w) \in (\mathbb{C}\cup\{\infty\})^2 \mid w^{n+1} = z^n(z^2-1)\right\} \setminus \{(1,0), (-1,0), (\infty,\infty)\},$$

$$g = \frac{c}{w}, \qquad \eta = \frac{w}{z^2-1}dz,$$

where c is the real constant defined by

$$c = \sqrt{\frac{2A}{B}}, \qquad A = \int_0^1 \left(\frac{t}{1-t^2}\right)^{n/(n+1)} dt, \qquad B = \int_0^1 \frac{dt}{\left(t^n(1-t^2)\right)^{1/(n+1)}}.$$

When $n = 1$, the surface coincides with the original Costa surface.

To verify the periodicity of surfaces, consider the following map.

$$\operatorname{Per}(f) := \left\{ \operatorname{Re} \oint_\ell \left(1-g^2, i\left(1+g^2\right), 2g\right)\eta \;\middle|\; \ell \in H_1(M,\mathbb{Z}) \right\}. \tag{5}$$

The periodicity can be determined in the following way:

- If $\operatorname{Per}(f) = \{\mathbf{0}\}$, that is, f satisfies the condition (3) for any closed curve ℓ on M, then $f : M \to \mathbb{R}^3$ is well-defined on M, that is, f is *non-periodic*.
- If there exists only one direction $\boldsymbol{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that

$$\operatorname{Per}(f) \subset \Lambda_1 := \{n\boldsymbol{v} \mid n \in \mathbb{Z}\},$$

  then f is *singly periodic*. In this case, f is well-defined in $\mathbb{R}^3/\Lambda_1 = \mathbb{R}^2 \times S^1$. (A surface invariant under *screw-motions* $\Lambda_1 + R$, where R is a rotation around an axis in the direction of Λ_1, is also singly periodic. See, for example, [1] and the references therein.)
- If there exist two linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ (with $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ uniquely determined) such that

$$\operatorname{Per}(f) \subset \Lambda_2 := \left\{ \sum_{j=1}^{2} n_j \boldsymbol{v}_j \;\middle|\; n_j \in \mathbb{Z} \right\},$$

  then f is *doubly periodic*. In this case, f is well-defined in $\mathbb{R}^3/\Lambda_2 = T^2 \times \mathbb{R}$.
- If there exist three linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ such that

$$\operatorname{Per}(f) \subset \Lambda_3 := \left\{ \sum_{j=1}^{3} n_j \boldsymbol{v}_j \;\middle|\; n_j \in \mathbb{Z} \right\},$$

  then f is *triply periodic*. In this case, f is well-defined in $\mathbb{R}^3/\Lambda_3 = T^3$.

Acknowledgements The author thanks Wayne Rossman for valuable comments.

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