A construction of a two-parameter family of triply periodic minimal surfaces

Norio Ejiri,* Shoichi Fujimori,† and Toshihiro Shoda‡

February 7, 2018

Abstract

In this paper, we first consider a four-parameter family of hyperelliptic Riemann surfaces of genus three and determine its Jacobian variety explicitly. After that, by the implicit function theorem, we construct a two-parameter family of triply periodic minimal surfaces.

1 Introduction

Properly immersed triply periodic minimal surfaces in $\mathbb{R}^3$ have been used for the description of an extensive field of structures, ranging from inorganic solids to lipid membranes and biochemical macromolecules. (See [6].) Many one-parameter families have been considered in physics, chemistry, and so on. (See [11].) Also, many families of triply periodic minimal surfaces have been constructed in mathematics. (See for instance [3], [5].) Most of their constructions are peculiar to the case of minimal surfaces in $\mathbb{R}^3$. In the present paper, we will construct a two-parameter family of triply periodic minimal surfaces, and our procedure can be applied to the higher codimensional case.

Recall that a minimal surface in $\mathbb{R}^3$ is said to be *periodic* if it is connected and invariant under a group Γ of isometries of $\mathbb{R}^3$ that acts properly discontinuously and freely. (See [8].) We usually call it a *triply periodic minimal surface* if Γ can be chosen to be a rank three lattice Λ in $\mathbb{R}^3$. The geometry

*Partly supported by JSPS Grant-in-Aid for Scientific Research (C) 15K04859.
†Partly supported by JSPS Grant-in-Aid for Scientific Research (C) 17K05219.
‡Partly supported by JSPS Grant-in-Aid for Scientific Research (C) 16K05134.

of a periodic minimal surface in $\mathbb{R}^3$ can usually be described in terms of the geometry of its quotient surface M in the flat three manifold $\mathbb{R}^3/\Gamma$. Thus a triply periodic minimal surface is a minimal surface in a flat 3-torus $\mathbb{R}^3/\Lambda$. Our main object of study here is properly immersed triply periodic minimal surfaces in $\mathbb{R}^3$, and so they can be replaced by compact minimal surface in flat 3-tori $\mathbb{R}^3/\Lambda$.

Let $f : M \to \mathbb{R}^3/\Lambda$ be a compact oriented minimal surface in a flat 3-torus. Using isothermal coordinates, M can be reconsidered as a Riemann surface, and we call f a *conformal minimal immersion.* The following theorem gives an explicit description for a conformal minimal immersion. (See for instance [7].)

Theorem 1.1 (Weierstrass representation formula)**.** *Let $f : M \to \mathbb{R}^3/\Lambda$ be a conformal minimal immersion. Then, up to translations, f can be represented by the following path-integrals*:

$$f(p) = \Re \int_{p_0}^{p} {}^t(\omega_1,\, \omega_2,\, \omega_3) \quad \mathrm{mod} \ \ \Lambda,$$

where p_0 is a fixed point on M and the ω_i are holomorphic differentials on M satisfying the following three conditions.

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 0, \tag{1.1}$$

$$\omega_1,\, \omega_2,\, \omega_3 \ \text{have no common zeros,} \tag{1.2}$$

$$\left\{ \Re \int_C {}^t(\omega_1,\, \omega_2,\, \omega_3) \,\middle|\, C \in H_1(M,\, \mathbb{Z}) \right\} \ \text{is a sublattice of } \Lambda. \tag{1.3}$$

Conversely, the real part of path-integrals of holomorphic differentials satisfying the above three conditions defines a conformal minimal immersion.

We now review some fundamental arguments for compact Riemann surfaces. (See p.91 in [2] for the details.) Given a compact Riemann surface M of genus g there is an associate complex torus, called the *Jacobian variety* of M, in the following way. Take a basis $\{\eta_j\}_{j=1}^g$ of the space of holomorphic differentials and a canonical homology basis $\{A_j,\, B_j\}_{j=1}^g$ on M, set ${}^t\eta = (\eta_1,\, \cdots,\, \eta_g)$ and consider

$$\Omega = \left(\int_{A_1} \eta \ \ \cdots \ \ \int_{A_g} \eta \ \ \int_{B_1} \eta \ \ \cdots \ \ \int_{B_g} \eta \right).$$

The Jacobian variety is defined by $\mathbb{C}^g/\Omega$, and moreover we obtain the holomorphic embedding $j : M \to \mathbb{C}^g/\Omega$ by $j(p) = \displaystyle\int_{p_0}^{p} \eta$, called the *Abel-Jacobi map*. It is well-known that the Abel-Jacobi map has the following universal property. (See [9] for instance.)

Theorem 1.2. *If $f : M \to \mathbb{R}^3/\Lambda$ is a conformal minimal immersion of a Riemann surface M into a flat 3-torus, then there exists a real homomorphism $h : \mathbb{C}^g/\Omega\,(\cong \mathbb{R}^{2g}/\Lambda) \to \mathbb{R}^3/\Lambda$ such that $f = h \circ j$.*

Remark 1.1. (1.3) in Theorem 1.1 holds if and only if

$$\operatorname{rank}_{\mathbb{Q}} \begin{pmatrix} \Re\displaystyle\int_{A_1}\omega & \cdots & \Re\displaystyle\int_{A_g}\omega & \Re\displaystyle\int_{B_1}\omega & \cdots & \Re\displaystyle\int_{B_g}\omega \end{pmatrix} = 3, \tag{1.4}$$

where ${}^t\omega = (\omega_1, \omega_2, \omega_3)$. (See [1].)

Hence we can construct a conformal minimal immersion of a compact Riemann surface into a flat 3-torus via the Abel-Jacobi map. In fact, for a compact Riemann surface, we carry out the following process: (i) determine its Jacobian variety, (ii) consider h in Theorem 1.2 and check (1.4). In this work, we consider the four-parameter family of hyperelliptic Riemann surfaces as follows. For $0 < a < 1$, $0 < r < 1$, $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$, let M be the hyperelliptic Riemann surface of genus 3 defined by

$$w^2 = (z - r)\left(z - \frac{1}{r}\right)(z + a)\left(z + \frac{1}{a}\right)\prod_{j=1}^{2}(z - e^{i\theta_j})(z - e^{-i\theta_j}).$$

We will choose suitable parameters for M so that (1.4) holds, and our main theorem is

Main Theorem. Using the above notation, there exists a two-parameter family of conformal minimal immersions into flat 3-tori, which is derived from the real part of the Abel-Jacobi map of M.

We finally remark that Theorem 1.1, Theorem 1.2, and (1.4) can be extended to the higher codimensional case. (See [7] and [10].) Thus the above procedure can be applied to the higher codimensional case.

The paper is organized as follows. In §2 we determine the Jacobian variety of M and give some relations for its periods, which are important tools

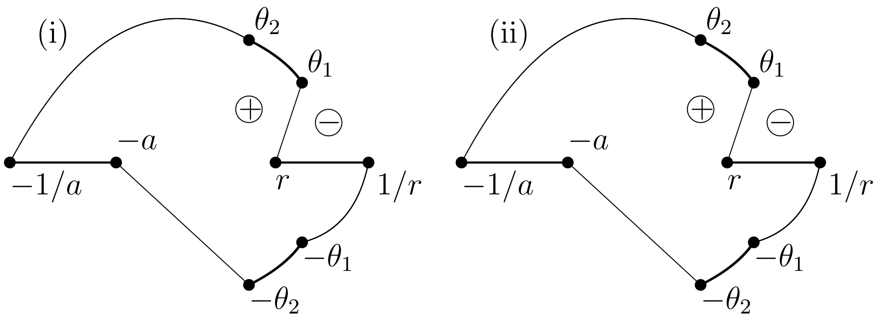


Figure 2.1: M as the two-sheeted branched covering of S^2

to show the existence of the desired parameters. In §3 we prove our main theorem through four subsections, and the technique developed in this section is about how to use the implicit function theorem for our complicated situations.

2 The Jacobian variety and relations for periods

We can write out a basis for the space of the holomorphic differentials on M given in the introduction: $\{(1-z^2)dz/w,\ i\,(1+z^2)dz/w,\ 2zdz/w\}$ (see p.255 in [4]).

Let j be the hyperelliptic involution on M defined by $j(z,\, w) = (z,\, -w)$, and it gives rise to the two-sheeted covering $M \to M/\langle j\rangle\,(= S^2)$ which is branched at the eight points $r,\ 1/r,\ -a,\ -1/a,\ e^{\pm i\theta_1},\ e^{\pm i\theta_2}$.

First we prepare two copies of the Riemann sphere $\overline{\mathbb{C}}$ and draw closed curves passing through the eight points. Each closed curve separates $\overline{\mathbb{C}}$ into two domains, and label "+" and "−" (see Figure 2.1). In particular, join r and $1/r$, $-a$ and $-1/a$, θ_1 and θ_2, $-\theta_1$ and $-\theta_2$ by the thick curves.

Slit the thick curves and glue "+" domain in (i) and "−" domain in (ii), "−" domain in (i) and "+" domain in (ii). We remark that the thin curves in Figure 2.1 correspond to the thin curves in Figure 2.2. M can be obtained by this procedure (see Figure 2.2).

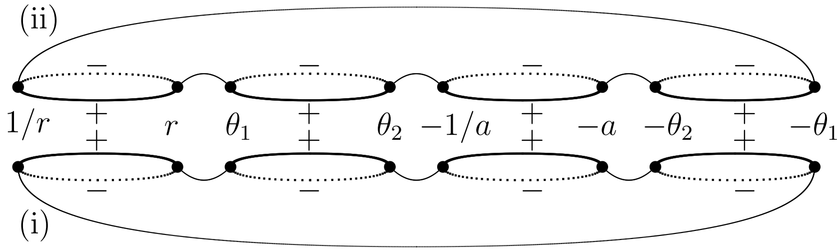


Figure 2.2: M by gluing (i) and (ii)

2.1 A canonical homology basis on M

In this subsection, we shall determine a canonical homology basis on M. We first introduce the following key paths.

$$C_1 = \left\{(z,\, w) = \left(t,\, i\left((t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right)^{\frac{1}{2}}\right) \middle|\, r \le t \le \frac{1}{r},\, \sqrt{*}>0\right\},$$

$$C_2 = \left\{(z,\, w) = \left(t,\, i\left((t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right)^{\frac{1}{2}}\right) \middle|\, r \le t \le 1,\, \sqrt{*}>0\right\},$$

$$C_3 = \{(z,\, w) = (e^{it},\, w(t)) \mid 0 \le t \le \theta_1,\, w(0) \in i\mathbb{R}_{>0}\},$$
$$C_4 = \{(z,\, w) = (e^{it},\, w(t)) \mid \theta_1 \le t \le \theta_2,\, z^2/w \ge 0\},$$
$$C_5 = \{(z,\, w) = (e^{it},\, w(t)) \mid \theta_2 \le t \le \pi,\, w(\pi) \in i\mathbb{R}_{>0}\},$$

$$C_6 = \left\{(z,\, w) = \left(t,\, i\left((r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right)^{\frac{1}{2}}\right) \middle|\, -1 \le t \le -a,\, \sqrt{*}>0\right\},$$

$$C_7 = \left\{(z,\, w) = \left(t,\, \left((r-t)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right)^{\frac{1}{2}}\right) \middle|\, -a \le t \le r,\, \sqrt{*}>0\right\}.$$

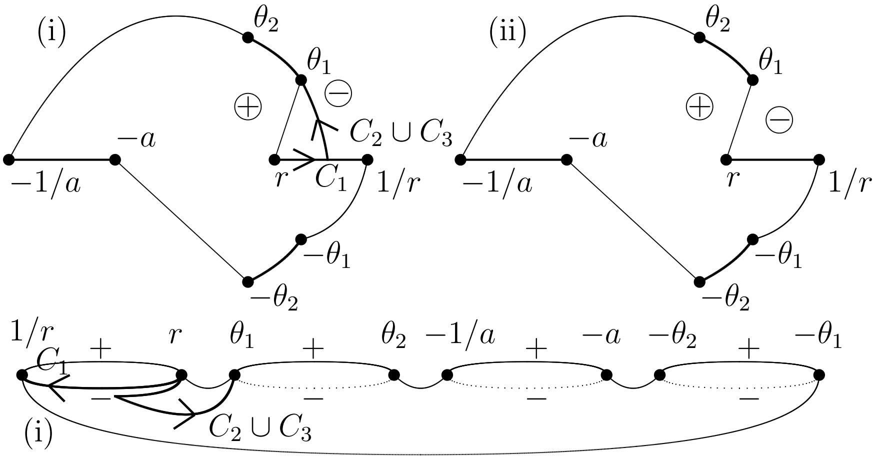


Figure 2.3: C_1 and $C_2 \cup C_3$ in M

The inequalities

$$t^2 - 2(\cos\theta_j)t + 1 = \begin{cases} (t-1)^2 + 2t(1-\cos\theta_j) > 0 & (t \geq 0) \\ (t+1)^2 - 2t(1+\cos\theta_j) > 0 & (t \leq 0) \end{cases}$$

imply $C_2 \cap C_3 \neq \emptyset$ and $C_5 \cap C_6 \neq \emptyset$.

Now we choose C_1 as in Figure 2.3, and we will determine other paths. From $C_1 \cap C_2 \neq \emptyset$, $C_2 \cup C_3$ can be obtained as in Figure 2.3.

For suitable m, n, $\ell \in \{0, 1\}$, $(C_2 \cup C_3) + j^m(C_4) + j^n(C_5 \cup C_6) + j^\ell(C_7)$ must be homotopic to 0, write $(C_2 \cup C_3) + j^m(C_4) + j^n(C_5 \cup C_6) + j^\ell(C_7) \sim 0$. To determine m, n, $\ell \in \{0, 1\}$, we calculate periods as follows. Straightforward calculations yield

$$\int_{C_2} \frac{1-z^2}{w} dz = -i \int_r^1 \frac{1-t^2}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2 (t^2 - 2(\cos\theta_j)t + 1)}} dt,$$

$$\int_{C_2} \frac{i(1+z^2)}{w} dz = \int_r^1 \frac{1+t^2}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2 (t^2 - 2(\cos\theta_j)t + 1)}} dt,$$

$$\int_{C_2} \frac{2z}{w} dz = -2i \int_r^1 \frac{t}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2 (t^2 - 2(\cos\theta_j)t + 1)}} dt,$$

$$\int_{C_6} \frac{1-z^2}{w} dz = -i \int_{-1}^{-a} \frac{1-t^2}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt,$$

$$\int_{C_6} \frac{i(1+z^2)}{w} dz = \int_{-1}^{-a} \frac{1+t^2}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt,$$

$$\int_{C_6} \frac{2z}{w} dz = -2i \int_{-1}^{-a} \frac{t}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt,$$

$$\int_{C_7} \frac{1-z^2}{w} dz = \int_{-a}^{r} \frac{1-t^2}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt,$$

$$\int_{C_7} \frac{i(1+z^2)}{w} dz = i \int_{-a}^{r} \frac{1+t^2}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt,$$

$$\int_{C_7} \frac{2z}{w} dz = 2 \int_{-a}^{r} \frac{t}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt.$$

Next we treat periods along $z = e^{it}$. Setting $x = (z+1/z)/2 = \cos t$, we have

$$\left(\frac{z^2}{w}\right)^2 = \frac{1}{(z+\frac{1}{z}-r-\frac{1}{r})(z+\frac{1}{z}+a+\frac{1}{a})\prod_{j=1}^{2}(z+\frac{1}{z}-2\cos\theta_j)}$$

$$= -\frac{1}{4(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)} \begin{cases} < 0 & \text{on } C_3,\ C_5 \\ > 0 & \text{on } C_4 \end{cases}$$

Taking suitable branches, we find

$$\frac{z^2}{w} = \begin{cases} -\dfrac{i}{2\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}} & \text{on } C_3,\ C_5 \\ \dfrac{1}{2\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}} & \text{on } C_4 \end{cases}$$

By $dz = 2z^2dx/(z^2-1)$, we have

$$\frac{1-z^2}{w}dz = \frac{-2z^2}{w}dx, \quad \frac{i(1+z^2)}{w}dz = \frac{z^2}{w}\left(z+\frac{1}{z}\right)\frac{2i}{z-\frac{1}{z}}dx, \quad \frac{2z}{w}dz = \frac{-2iz^2}{w}\frac{2i}{z-\frac{1}{z}}dx.$$

Hence

$$\int_{C_3} \frac{1-z^2}{w} dz = -i \int_{\cos\theta_1}^{1} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}},$$

$$\int_{C_3} \frac{i(1+z^2)}{w} dz = i \int_{\cos\theta_1}^{1} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}} dx,$$

$$\int_{C_3} \frac{2z}{w} dz = \int_{\cos\theta_1}^{1} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}},$$

$$\int_{C_4} \frac{1-z^2}{w} dz = \int_{\cos\theta_2}^{\cos\theta_1} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}},$$

$$\int_{C_4} \frac{i(1+z^2)}{w} dz = -\int_{\cos\theta_2}^{\cos\theta_1} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}} dx,$$

$$\int_{C_4} \frac{2z}{w} dz = i \int_{\cos\theta_2}^{\cos\theta_1} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}},$$

$$\int_{C_5} \frac{1-z^2}{w} dz = -i \int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}},$$

$$\int_{C_5} \frac{i(1+z^2)}{w} dz = i \int_{-1}^{\cos\theta_2} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}} dx,$$

$$\int_{C_5} \frac{2z}{w} dz = \int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}.$$

From $\displaystyle\int_{(C_2\cup C_3)+j^m(C_4)+j^n(C_5\cup C_6)+j^\ell(C_7)} \frac{1-z^2}{w} dz = 0$, we find $n=1$ and $m \neq \ell$, that is, $(m,\, n,\, \ell) = (1,\, 1,\, 0),\ (0,\, 1,\, 1)$. Moreover, we have

$$\begin{aligned} &\int_{\cos\theta_2}^{\cos\theta_1} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}} \\ &= \int_{-a}^{r} \frac{1-t^2}{\sqrt{(r-t)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\int_{r}^{1} \frac{1-t^2}{\sqrt{(t-r)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}} dt \\ &+ \int_{\cos\theta_1}^{1} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}} \end{aligned} \tag{2.2}$$

$$= \int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}$$
$$+ \int_{-1}^{-a} \frac{1-t^2}{\sqrt{(r-t)(\frac{1}{r}-t)(-t-a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt.$$

If we assume $(m,\, n,\, \ell) = (1,\, 1,\, 0)$, then the imaginary part of the left hand side of

$$\int_{(C_2\cup C_3)+j^m(C_4)+j^n(C_5\cup C_6)+j^\ell(C_7)} \frac{2z}{w} dz = 0$$

leads to a contradiction. Thus we find $(m,\, n,\, \ell) = (0,\, 1,\, 1)$, and so $(C_2 \cup C_3) + C_4 + j(C_5 \cup C_6) + j(C_7) \sim 0$ holds. Furthermore, we have

$$\int_{r}^{1} \frac{1+t^2}{\sqrt{(t-r)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt \tag{2.3}$$
$$= \int_{\cos\theta_2}^{\cos\theta_1} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}}dx$$
$$+ \int_{-1}^{-a} \frac{1+t^2}{\sqrt{(r-t)(\frac{1}{r}-t)(-t-a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$\int_{\cos\theta_1}^{1} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}dx \tag{2.4}$$
$$= \int_{-1}^{\cos\theta_2} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}dx$$
$$+ \int_{-a}^{r} \frac{1+t^2}{\sqrt{(r-t)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$\int_{\cos\theta_2}^{\cos\theta_1} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)(\cos\theta_1-x)(x-\cos\theta_2)}} \tag{2.5}$$
$$= 2\int_{r}^{1} \frac{t}{\sqrt{(t-r)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt$$

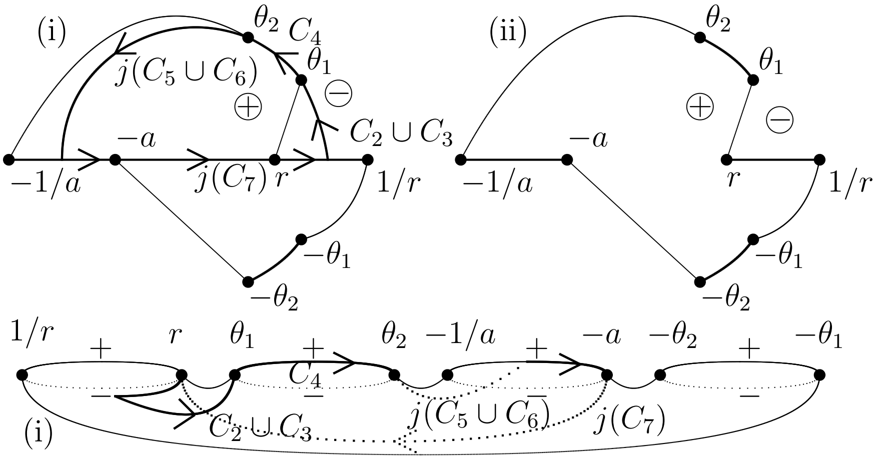


Figure 2.4: C_4, $j(C_5 \cup C_6)$, $j(C_7)$ in M

$$
\begin{aligned}
&-2\int_{-1}^{-a} \frac{t}{\sqrt{(r-t)(\frac{1}{r}-t)(-t-a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,\\
&\int_{\cos\theta_1}^{1} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}} \qquad (2.6)\\
&=\int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}\\
&+2\int_{-a}^{r} \frac{t}{\sqrt{(r-t)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt
\end{aligned}
$$

and Figure 2.4.

We now introduce

$$
\begin{aligned}
C_8 = &\left\{(z,\, w) = \left(t,\, i\left((r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right)^{\frac{1}{2}}\right)\right.\\
&\left.\quad\middle|\; -\tfrac{1}{a} \le t \le -a,\, \sqrt{*} > 0\right\},\\
C_9 = &\{(z,\, w) = (e^{it},\, w(t)) \mid \theta_2 \le -t \le \pi,\, w(-\pi) \in i\mathbb{R}_{>0}\}.
\end{aligned}
$$

Note that C_8 is an extension of C_6, and $C_6 \cap C_9 \neq \varnothing$ yields the Figure 2.5.

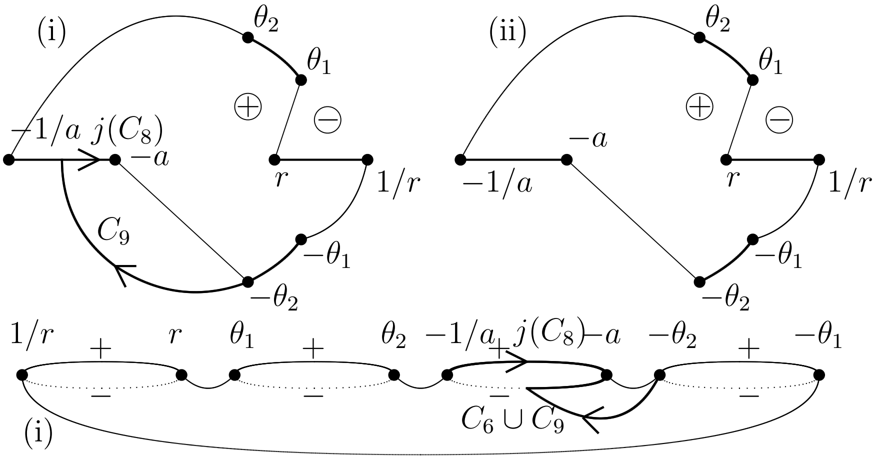


Figure 2.5: $j(C_8)$ and $C_6 \cup C_9$ in M

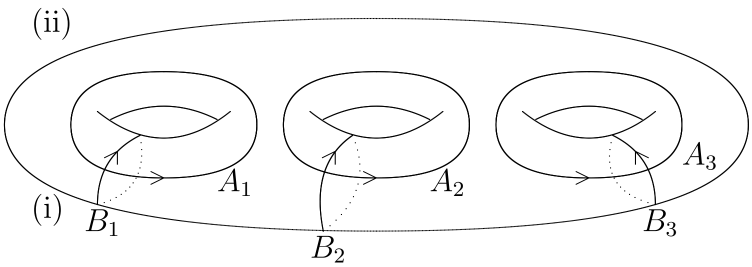


Figure 2.6: A canonical homology basis on M

Therefore, we obtain a canonical homology basis on M as follows (see Figure 2.6).

$$
\begin{aligned}
&A_1 = C_2 \cup C_3 - j(C_2 \cup C_3), \quad A_2 = -C_5 - C_6 + C_8 + j(C_5 + C_6 - C_8), \\
&A_3 = -(C_6 \cup C_9) + j(C_6 \cup C_9), \quad B_1 = -C_1 + j(C_1), \\
&B_2 = -C_4 + j(C_4) + B_1, \quad B_3 = C_8 - j(C_8) + B_2.
\end{aligned} \tag{2.7}
$$

2.2 The period matrix and the Jacobian variety

In this subsection, we shall calculate periods of

$$
{}^t\Phi := \left(\frac{1 - z^2}{w} dz, \ \frac{i\,(1 + z^2)}{w} dz, \ \frac{2z}{w} dz \right)
$$

along the canonical homology basis which is given in (2.7), and obtain its Jacobian variety.

First we have

$$\int_{C_1} \frac{1-z^2}{w}dz = 0,$$

$$\int_{C_1} \frac{i(1+z^2)}{w}dz = 2\int_r^1 \frac{1+t^2}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$\int_{C_1} \frac{2z}{w}dz = -4i\int_r^1 \frac{t}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$\int_{C_8} \frac{1-z^2}{w}dz = 0,$$

$$\int_{C_8} \frac{i(1+z^2)}{w}dz = 2\int_{-1}^{-a} \frac{1+t^2}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$\int_{C_8} \frac{2z}{w}dz = -4i\int_{-1}^{-a} \frac{t}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^2(t^2-2(\cos\theta_j)t+1)}}dt.$$

Next we consider periods along $z = e^{it}$. Setting $x = (z+1/z)/2 = \cos t$ on C_8, we find

$$\frac{z^2}{w} = -\frac{i}{2\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^2(x-\cos\theta_j)}}.$$

From $dz = 2z^2dx/(z^2-1)$, we have

$$\frac{1-z^2}{w}dz = \frac{-2z^2}{w}dx, \quad \frac{i(1+z^2)}{w}dz = \frac{z^2}{w}\left(z+\frac{1}{z}\right)\frac{2i}{z-\frac{1}{z}}dx, \quad \frac{2z}{w}dz = \frac{-2iz^2}{w}\frac{2i}{z-\frac{1}{z}}dx.$$

It follows that

$$\int_{C_9} \frac{1-z^2}{w}dz = -i\int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^2(x-\cos\theta_j)}},$$

$$\int_{C_9} \frac{i(1+z^2)}{w}dz = -i\int_{-1}^{\cos\theta_2} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^2(x-\cos\theta_j)}}dx,$$

$$\int_{C_9} \frac{2z}{w}dz = -\int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^2(x-\cos\theta_j)}}.$$

Therefore, by setting

$$A = 2\int_{-a}^{r} \frac{1-t^2}{\sqrt{(r-t)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$B = 2\int_{r}^{1} \frac{1+t^2}{\sqrt{(t-r)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$C = 2\int_{-1}^{-a} \frac{1+t^2}{\sqrt{(r-t)(\frac{1}{r}-t)(-t-a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$D = 2\int_{\cos\theta_1}^{1} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}},$$

$$E = 2\int_{-1}^{\cos\theta_2} \frac{dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}},$$

$$F = 2\int_{r}^{1} \frac{1-t^2}{\sqrt{(t-r)(\frac{1}{r}-t)(t+a)(t+\frac{1}{a})\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt$$
$$+ 2\int_{\cos\theta_1}^{1} \frac{dx}{\sqrt{(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}},$$

$$G = 2\int_{\cos\theta_1}^{1} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}dx,$$

$$H = 2\int_{-1}^{\cos\theta_2} \frac{x}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)(a+\frac{1}{a}+2x)\prod_{j=1}^{2}(x-\cos\theta_j)}}dx,$$

$$I = 4\int_{r}^{1} \frac{t}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

$$J = 4\int_{-1}^{-a} \frac{t}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(-t-a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)}}dt,$$

the period matrix

$$\Omega := \left(\int_{A_1}\Phi \quad \int_{A_2}\Phi \quad \int_{A_3}\Phi \quad \int_{B_1}\Phi \quad \int_{B_2}\Phi \quad \int_{B_3}\Phi\right)$$

is obtained by

$$\begin{pmatrix} -iF & iF & iF & 0 & -A & -A \\ B+iG & C-iH & -C+iH & -2B & -B-C & -B+C \\ D-iI & -E-iJ & E+iJ & 2iI & i(I+J) & i(I-J) \end{pmatrix}. \tag{2.8}$$

Thus its Jacobian variety is given by $\mathbb{C}^3/\Omega$.

3 Construction of a two-parameter family

Let p_0 be a fixed point on M. In this section, setting

$$f(p) = \Re \int_{p_0}^{p} \Phi$$

on M, we will choose a, r, θ_1, θ_2 such that f defines a conformal minimal immersion in a flat 3-torus.

3.1 A translation of the period condition

For Ω given in (2.8), we have to choose suitable parameters a, r, θ_1, θ_2 such that $\mathrm{rank}_{\mathbb{Q}} \Re(\Omega) = 3$. Straightforward calculation yields

$$\Re(\Omega) = \begin{pmatrix} 0 & 0 & 0 & 0 & -A & -A \\ B & C & -C & -2B & -B-C & -B+C \\ D & -E & E & 0 & 0 & 0 \end{pmatrix}.$$

Multiplying $\Re(\Omega)$ by the following fundamental transformation of matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

from the right, we obtain

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -A & 2A \\ B & C & 0 & -2B & -B-C & 0 \\ D & -E & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

As a consequence, we have

Lemma 3.1. *f defines a conformal minimal immersion of M into a flat 3-torus if and only if*

$$\frac{C}{B},\ \frac{E}{D} \in \mathbb{Q} \tag{3.2}$$

Proof. It is easy to verify that $\operatorname{rank}_{\mathbb{Q}} \Re(\Omega) = 3$ is equivalent to (3.2) by (3.1). □

Let θ_0 and a_0 be two constants satisfying $0 < \theta_0 < \frac{\pi}{2}$ and $0 < a_0 < 1$. Suppose now that $0 < \theta_1 < \theta_2 < \theta_0$ and $a_0 < a < 1$. We will show the existence of a, r, θ_1, θ_2 so that $D = E$ and $\frac{C}{B} \in \mathbb{Q}$ in the next subsections.

3.2 The existence of parameters so that $D - E = 0$

Lemma 3.2. (i) *There exists $r_0 \in (0,\, 1)$ such that $D - E < 0$ for $a \in (a_0,\, 1)$, $r \in (0,\, r_0]$, and $0 < \theta_1 < \theta_2 < \theta_0$.* (ii) *For $a \in (a_0,\, 1)$, there exists $r_1 \in (r_0,\, 1)$ such that $D - E > 0$ for $r \in [r_1,\, 1)$ and $0 < \theta_1 < \theta_2 < \theta_0$.*

Proof. From (2.6), we find

$$\begin{aligned}
D - E &= 4\int_{-a}^{r} \frac{t}{\sqrt{(r-t)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2 - 2(\cos\theta_j)t + 1)}}dt \\
&\underbrace{=}_{x=(t+1/t)/2} -2\int_{\frac{a+\frac{1}{a}}{2}}^{\infty} \frac{dx}{\sqrt{(x^2-1)(r+\frac{1}{r}+2x)(2x-a-\frac{1}{a})(x+\cos\theta_1)(x+\cos\theta_2)}} \\
&\quad + 2\int_{\frac{r+\frac{1}{r}}{2}}^{\infty} \frac{dx}{\sqrt{(x^2-1)(2x-r-\frac{1}{r})(2x+a+\frac{1}{a})(x-\cos\theta_1)(x-\cos\theta_2)}} \\
&\underbrace{=}_{x=\pm\frac{1}{t}} \frac{2}{\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}} \\
&\qquad \times \Bigg\{ -\int_{0}^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&\qquad\quad - \int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \Bigg\}.
\end{aligned}$$

Thus we must check the sign of

$$
-\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt
$$
$$
-\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt.
$$

(i) We will show $D-E<0$ as $r\to 0$ for $a\in(a_0,\,1)$ and $0<\theta_1<\theta_2<\theta_0$. First we have

$$
\begin{aligned}
&-\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&< -\frac{1}{\sqrt{(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{1}{\cos\theta_0})^2}}\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{\frac{2}{a+\frac{1}{a}}-t}}dt.
\end{aligned}
$$

Note that

$$
\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{\frac{2}{a+\frac{1}{a}}-t}}dt = \left(\frac{2}{a+\frac{1}{a}}\right)^{\frac{3}{2}} \mathrm{Beta}\left(2,\frac{1}{2}\right),
$$

where $\mathrm{Beta}(a,\,b) := \displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the Beta function. Next we find

$$
\begin{aligned}
&-\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&< -\frac{1}{\sqrt{\frac{2}{a+\frac{1}{a}}(1-(\frac{2}{r+\frac{1}{r}})^2)(-\frac{2}{r+\frac{1}{r}}+1)^2}}\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{t+\frac{2}{r+\frac{1}{r}}}}dt \\
&= \frac{1}{\sqrt{\frac{2}{a+\frac{1}{a}}(1-(\frac{2}{r+\frac{1}{r}})^2)(-\frac{2}{r+\frac{1}{r}}+1)^2}}\left(\frac{2}{r+\frac{1}{r}}\right)^{\frac{3}{2}} \mathrm{Beta}\left(2,\frac{1}{2}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&-\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&\quad -\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&< -\frac{1}{\sqrt{(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{1}{\cos\theta_0})^2}}\left(\frac{2}{a+\frac{1}{a}}\right)^{\frac{3}{2}}\mathrm{Beta}\left(2,\frac{1}{2}\right) \\
&\quad +\frac{1}{\sqrt{\frac{2}{a+\frac{1}{a}}(1-(\frac{2}{r+\frac{1}{r}})^2)(-\frac{2}{r+\frac{1}{r}}+1)^2}}\left(\frac{2}{r+\frac{1}{r}}\right)^{\frac{3}{2}}\mathrm{Beta}\left(2,\frac{1}{2}\right) \\
&< -\frac{1}{\sqrt{(1+\frac{2}{r+\frac{1}{r}})(1+\frac{1}{\cos\theta_0})^2}}\left(\frac{2}{a_0+\frac{1}{a_0}}\right)^{\frac{3}{2}}\mathrm{Beta}\left(2,\frac{1}{2}\right) \\
&\quad +\frac{1}{\sqrt{\frac{2}{a_0+\frac{1}{a_0}}(1-(\frac{2}{r+\frac{1}{r}})^2)(-\frac{2}{r+\frac{1}{r}}+1)^2}}\left(\frac{2}{r+\frac{1}{r}}\right)^{\frac{3}{2}}\mathrm{Beta}\left(2,\frac{1}{2}\right)
\end{aligned}
$$

holds. By letting r tend to zero, the assertion is proved. We remark that we can choose such r_0 uniformly for $a \in (a_0, 1)$ and $0 < \theta_1 < \theta_2 < \theta_0$.
(ii) For $a \in (a_0, 1)$, we will show $D - E > 0$ as $r \to 1$ for $0 < \theta_1 < \theta_2 < \theta_0$. First we have

$$
\begin{aligned}
&-\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&> -\frac{1}{\sqrt{\frac{2}{r+\frac{1}{r}}(1-(\frac{2}{a+\frac{1}{a}})^2)}}\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{\frac{2}{a+\frac{1}{a}}-t}}dt \\
&= -\frac{1}{\sqrt{\frac{2}{r+\frac{1}{r}}(1-(\frac{2}{a+\frac{1}{a}})^2)}}\left(\frac{2}{a+\frac{1}{a}}\right)^{\frac{3}{2}}\mathrm{Beta}\left(2,\frac{1}{2}\right).
\end{aligned}
$$

Next we find

$$
\begin{aligned}
&-\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&> -\frac{1}{\sqrt{\frac{1}{\cos\theta_1\cos\theta_2}(1+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})}} \int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1+t)(t+\frac{2}{r+\frac{1}{r}})}}dt \\
&> -\frac{1}{\frac{1}{\cos\theta_0}\sqrt{(1+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})}} \int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1+t)(t+\frac{2}{r+\frac{1}{r}})}}dt.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1+t)(t+\frac{2}{r+\frac{1}{r}})}}dt = -\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t+1-1}{\sqrt{(1+t)(t+\frac{2}{r+\frac{1}{r}})}}dt \\
&= -\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \sqrt{\frac{t+1}{t+\frac{2}{r+\frac{1}{r}}}}dt + \int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{1}{\sqrt{(1+t)(t+\frac{2}{r+\frac{1}{r}})}}dt.
\end{aligned}
$$

The first term must be finite as $r \to 1$, and so we treat the second term as follows. Straightforward calculation yields

$$
\begin{aligned}
&\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{1}{\sqrt{(1+t)(t+\frac{2}{r+\frac{1}{r}})}}dt \underbrace{=}_{t=-\frac{2}{r+\frac{1}{r}}\{1-\frac{1}{2}(r+\frac{1}{r}-2)x\}} \int_0^{\frac{2}{r+\frac{1}{r}-2}} \frac{dx}{\sqrt{x(x+1)}} \\
&\underbrace{=}_{\sqrt{x}=\tan\theta} 2\int_0^{\arctan\sqrt{\frac{2}{r+\frac{1}{r}-2}}} \frac{d\theta}{\cos\theta} \underbrace{=}_{s=\sin\theta} \int_0^{\sqrt{\frac{2}{r+\frac{1}{r}}}} \left(\frac{1}{s+1}-\frac{1}{s-1}\right)ds \\
&= \log\left|\frac{\sqrt{\frac{2}{r+\frac{1}{r}}}+1}{\sqrt{\frac{2}{r+\frac{1}{r}}}-1}\right|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&-\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt \\
&\quad -\int_{-\frac{2}{r+\frac{1}{r}}}^{0} \frac{t}{\sqrt{(1-t^2)(t+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-t)(t+\frac{1}{\cos\theta_1})(t+\frac{1}{\cos\theta_2})}}dt
\end{aligned}
$$

$$> -\frac{1}{\sqrt{\frac{2}{r+\frac{1}{r}}(1-(\frac{2}{a+\frac{1}{a}})^2)}}\left(\frac{2}{a+\frac{1}{a}}\right)^{\frac{3}{2}}\mathrm{Beta}\left(2,\frac{1}{2}\right)+\frac{1}{\frac{1}{\cos\theta_0}\sqrt{(1+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})}}$$

$$\times\left(-\int_{-\frac{2}{r+\frac{1}{r}}}^{0}\sqrt{\frac{t+1}{t+\frac{2}{r+\frac{1}{r}}}}dt+\log\left|\frac{\sqrt{\frac{2}{r+\frac{1}{r}}}+1}{\sqrt{\frac{2}{r+\frac{1}{r}}}-1}\right|\right)$$

holds, and the result follows immediately. In this case, we must choose such r_1 for $a \in (a_0, 1)$. $\square$

For $a \in (a_0, 1)$, combining Lemma 3.2 and the intermediate value theorem implies the existence of r, θ_1, θ_2 so that $D - E = 0$.

Lemma 3.3. *Using the notation as in Lemma* 3.2*, we have* $\frac{\partial}{\partial r}(D-E) > 0$ *on* $\{(\theta_1, \theta_2, r) \mid r_0 < r < r_1,\ 0 < \theta_1 < \theta_2 < \theta_0,\ D - E = 0\}$.

Proof. For simplicity, we set

$$\varphi_1(x) = \left((1-x^2)\left(r+\frac{1}{r}-2x\right)\left(a+\frac{1}{a}+2x\right)\prod_{j=1}^{2}(x-\cos\theta_j)\right)^{\frac{1}{2}}.$$

First we find

$$\frac{\partial}{\partial r}\left(\frac{1}{\varphi_1(x)}\right) = -\frac{1}{2\varphi_1(x)^3}\frac{\partial}{\partial r}\left\{(1-x^2)\left(r+\tfrac{1}{r}-2x\right)\left(a+\tfrac{1}{a}+2x\right)\textstyle\prod_{j=1}^{2}(x-\cos\theta_j)\right\}$$

$$= -\frac{1}{2\varphi_1(x)^3}\left(1-\frac{1}{r^2}\right)(1-x^2)\left(a+\frac{1}{a}+2x\right)\prod_{j=1}^{2}(x-\cos\theta_j)$$

$$= -\frac{1}{2\varphi_1(x)^3}\left(1-\frac{1}{r^2}\right)\frac{\varphi_1(x)^2}{r+\frac{1}{r}-2x} = \left(\frac{1}{r^2}-1\right)\frac{1}{2(r+\frac{1}{r}-2x)\varphi_1(x)}.$$

Next, the integral interval of D ($\cos\theta_1 \le x \le 1$) yields

$$\left(\frac{1}{r+\frac{1}{r}-2\cos\theta_2}\le\right)\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\le\frac{1}{r+\frac{1}{r}-2x}\le\frac{1}{r+\frac{1}{r}-2}.$$

Also, that of E ($-1 \le x \le \cos\theta_2$) implies

$$-\frac{1}{r+\frac{1}{r}-2\cos\theta_2}\le-\frac{1}{r+\frac{1}{r}-2x}\le-\frac{1}{r+\frac{1}{r}+2}.$$

Hence we have

$$
\begin{aligned}
&\frac{\partial}{\partial r}(D-E) = 2\,\frac{\partial}{\partial r}\left(\int_{\cos\theta_1}^{1}\frac{dx}{\varphi_1(x)} - \int_{-1}^{\cos\theta_2}\frac{dx}{\varphi_1(x)}\right)\\
&= 2\left(\int_{\cos\theta_1}^{1}\frac{\partial}{\partial r}\left(\frac{1}{\varphi_1(x)}\right)dx - \int_{-1}^{\cos\theta_2}\frac{\partial}{\partial r}\left(\frac{1}{\varphi_1(x)}\right)dx\right)\\
&= \left(\frac{1}{r^2}-1\right)\left(\int_{\cos\theta_1}^{1}\frac{1}{(r+\frac{1}{r}-2x)\varphi_1(x)}dx - \int_{-1}^{\cos\theta_2}\frac{1}{(r+\frac{1}{r}-2x)\varphi_1(x)}dx\right)\\
&> \left(\frac{1}{r^2}-1\right)\left(\int_{\cos\theta_1}^{1}\frac{1}{(r+\frac{1}{r}-2\cos\theta_2)\varphi_1(x)}dx - \int_{-1}^{\cos\theta_2}\frac{1}{(r+\frac{1}{r}-2\cos\theta_2)\varphi_1(x)}dx\right)\\
&= \frac{1}{2(r+\frac{1}{r}-2\cos\theta_2)}\left(\frac{1}{r^2}-1\right)(D-E).
\end{aligned}
$$

It follows that $\frac{\partial}{\partial r}(D-E) > 0$ at every point where $D-E=0$, which proves the lemma. □

The next proposition is an immediate consequence of Lemma 3.3 and the implicit function theorem.

Proposition 3.1. $\{(\theta_1,\,\theta_2,\,r)\,|\,r_0 < r < r_1,\,0 < \theta_1 < \theta_2 < \theta_0,\,D-E=0\}$ *is a graph on* $\{(\theta_1,\,\theta_2)\,|\,0<\theta_1<\theta_2<\theta_0\}$.

By Lemma 3.3, $D-E$ is monotone increasing for r on $\{(\theta_1,\,\theta_2,\,r)\,|\,r_0 < r < r_1,\,0<\theta_1<\theta_2<\theta_0,\,D-E=0\}$. Combining this and Proposition 3.1 yields

Theorem 3.1. *There exists a unique component of* $\{(\theta_1,\,\theta_2,\,r)\,|\,r_0<r<r_1,\,0<\theta_1<\theta_2<\theta_0,\,D-E=0\}$ *which is a graph on* $\{(\theta_1,\,\theta_2)\,|\,0<\theta_1<\theta_2<\theta_0\}$ *and divides the open triangle prism* $\{(\theta_1,\,\theta_2,\,r)\,|\,r_0<r<r_1,\,0<\theta_1<\theta_2<\theta_0\}$ *into two components.* (*See Figure* 3.1.)

3.3 The existence of parameters so that $\frac{C}{B} \in \mathbb{Q}$

Lemma 3.4. *Use the notation as in* § 3.2. *There exist* $q \in \mathbb{Q}$, θ_* *and* $\theta_\sharp$ *satisfying* $0 < \theta_* < \theta_\sharp < \theta_0$ *such that* (i) $\frac{C}{B} - q > 0$ *for* $a \in (a_0,\,1)$, $r \in (r_0,\,1)$, *and* $\theta_\sharp < \theta_1 < \theta_2 < \theta_0$, (ii) $\frac{C}{B} - q < 0$ *for* $a \in (a_0,\,1)$, $r \in (r_0,\,1)$, *and* $0 < \theta_1 < \theta_2 < \theta_*$.

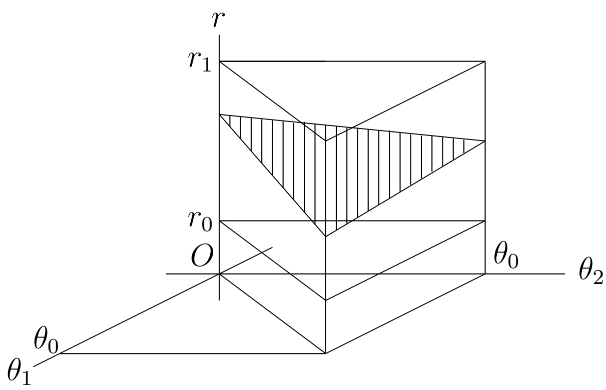


Figure 3.1: $\{(\theta_1,\, \theta_2,\, r) \mid D - E = 0\}$ (the shaded part)

Proof. First, B and C can be written as

$$B \underbrace{=}_{x=(t+1/t)/2} 2\int_1^{\frac{r+\frac{1}{r}}{2}} \frac{x}{\sqrt{(x^2-1)(r+\frac{1}{r}-2x)(2x+a+\frac{1}{a})(x-\cos\theta_1)(x-\cos\theta_2)}}dx$$

$$= \frac{2}{\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}}$$

$$\times \int_{\frac{2}{r+\frac{1}{r}}}^{1} \frac{dx}{\sqrt{(1-x^2)(x-\frac{2}{r+\frac{1}{r}})(x+\frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_1}-x)(\frac{1}{\cos\theta_2}-x)}},$$

$$= \frac{2}{\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}}$$

$$\times \int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{(1-x^2)(-x-\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-x)(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}},$$

$$C \underbrace{=}_{x=(t+1/t)/2} 2\int_1^{\frac{a+\frac{1}{a}}{2}} \frac{x}{\sqrt{(x^2-1)(r+\frac{1}{r}+2x)(a+\frac{1}{a}-2x)(x+\cos\theta_1)(x+\cos\theta_2)}}dx$$

$$= \frac{2}{\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}}$$

$$\times \int_{\frac{2}{a+\frac{1}{a}}}^{1} \frac{dx}{\sqrt{(1-x^2)(x+\frac{2}{r+\frac{1}{r}})(x-\frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}.$$

It follows that

$$\frac{C}{B} = \frac{\int_{\frac{2}{a+\frac{1}{a}}}^{1} \frac{dx}{\sqrt{(1-x^2)(x+\frac{2}{r+\frac{1}{r}})(x-\frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}}{\int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{(1-x^2)(-x-\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-x)(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}}.$$

We now show that there exist $\theta_\sharp \in (0,\, \theta_0)$ and a constant $K > 0$ so that $\frac{C}{B} > K$ for $a \in (a_0,\, 1)$, $r \in (r_0,\, 1)$, and $\theta_\sharp < \theta_1 < \theta_2 < \theta_0$.

To start with, we have

$$\begin{aligned}
&\int_{\frac{2}{a+\frac{1}{a}}}^{1} \frac{dx}{\sqrt{(1-x^2)(x+\frac{2}{r+\frac{1}{r}})(x-\frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}\\
&> \frac{1}{\sqrt{2(1+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}+1)(\frac{1}{\cos\theta_2}+1)}} \int_{\frac{2}{a+\frac{1}{a}}}^{1} \frac{dx}{\sqrt{(1-x)(x-\frac{2}{a+\frac{1}{a}})}}\\
&> \frac{1}{(\frac{1}{\cos\theta_0}+1)\sqrt{2(1+\frac{2}{r+\frac{1}{r}})}} \operatorname{Beta}\left(\frac{1}{2},\, \frac{1}{2}\right),\\
&\int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{(1-x^2)(-x-\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-x)(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}\\
&< \frac{1}{\sqrt{(1+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}-1)(\frac{1}{\cos\theta_2}-1)}} \int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{(1+x)(-x-\frac{2}{r+\frac{1}{r}})}}\\
&= \frac{1}{\sqrt{(1+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}-1)(\frac{1}{\cos\theta_2}-1)}} \operatorname{Beta}\left(\frac{1}{2},\, \frac{1}{2}\right).
\end{aligned}$$

Thus we find

$$\frac{C}{B} > \frac{\sqrt{(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}-1)(\frac{1}{\cos\theta_2}-1)}}{\sqrt{2}(\frac{1}{\cos\theta_0}+1)}.$$

Hence there exists $\theta_\sharp \in (0,\, \theta_0)$ such that

$$\frac{C}{B} > \frac{(\frac{1}{\cos\theta_\sharp}-1)\sqrt{\frac{2}{a_0+\frac{1}{a_0}}+\frac{2}{r_0+\frac{1}{r_0}}}}{\sqrt{2}(\frac{1}{\cos\theta_0}+1)}\ (>0)$$

for $a \in (a_0, 1)$, $r \in (r_0, 1)$, and $\theta_\sharp < \theta_1 < \theta_2 < \theta_0$. So, choosing

$$K = \frac{(\frac{1}{\cos\theta_\sharp} - 1)\sqrt{\frac{2}{a_0+\frac{1}{a_0}} + \frac{2}{r_0+\frac{1}{r_0}}}}{\sqrt{2}(\frac{1}{\cos\theta_0} + 1)}$$

implies the claim.

We next show that there exists $\theta_* \in (0, \theta_\sharp)$ such that $\frac{C}{B} < \frac{K}{2}$ for $a \in (a_0, 1)$, $r \in (r_0, 1)$, and $0 < \theta_1 < \theta_2 < \theta_*$. First, we have

$$\int_{\frac{2}{a+\frac{1}{a}}}^{1} \frac{dx}{\sqrt{(1-x^2)(x+\frac{2}{r+\frac{1}{r}})(x-\frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}$$

$$< \frac{1}{\sqrt{(1+\frac{2}{a+\frac{1}{a}})(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}+\frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_2}+\frac{2}{a+\frac{1}{a}})}} \int_{\frac{2}{a+\frac{1}{a}}}^{1} \frac{dx}{\sqrt{(1-x)(x-\frac{2}{a+\frac{1}{a}})}}$$

$$< \frac{1}{\sqrt{(1+\frac{2}{a+\frac{1}{a}})^3(\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}})}} \,\mathrm{Beta}\left(\frac{1}{2}, \frac{1}{2}\right),$$

$$\int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{(1-x^2)(-x-\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-x)(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}$$

$$> \frac{1}{\sqrt{2(1+\frac{2}{a+\frac{1}{a}})}} \int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{-(1+x)(x+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}.$$

Note that

$$\int_{-1}^{-\frac{2}{r+\frac{1}{r}}} \frac{dx}{\sqrt{-(1+x)(x+\frac{2}{r+\frac{1}{r}})(\frac{1}{\cos\theta_1}+x)(\frac{1}{\cos\theta_2}+x)}}$$

$$\underbrace{=}_{t^2=\frac{-\frac{2}{r+\frac{1}{r}}-x}{1+x}} 2\int_0^\infty \frac{dt}{\sqrt{(\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}}+(\frac{1}{\cos\theta_1}-1)\,t^2)(\frac{1}{\cos\theta_2}-\frac{2}{r+\frac{1}{r}}+(\frac{1}{\cos\theta_2}-1)\,t^2)}}$$

$$> 2\int_0^\infty \frac{dt}{\frac{1}{\cos\theta_2}-\frac{2}{r+\frac{1}{r}}+(\frac{1}{\cos\theta_2}-1)\,t^2}$$

$$\underbrace{=}_{\left(\frac{\frac{1}{\cos\theta_2}-1}{\frac{1}{\cos\theta_2}-\frac{2}{r+\frac{1}{r}}}\right)^{\frac{1}{2}} t=\tan\theta} \frac{\pi}{\sqrt{(\frac{1}{\cos\theta_2}-1)(\frac{1}{\cos\theta_2}-\frac{2}{r+\frac{1}{r}})}}.$$

As a consequence, we find

$$
\begin{aligned}
\frac{C}{B} &< \frac{\sqrt{2(\frac{1}{\cos\theta_2}-1)(\frac{1}{\cos\theta_2}-\frac{2}{r+\frac{1}{r}})}}{\pi(1+\frac{2}{a+\frac{1}{a}})\sqrt{\frac{2}{a+\frac{1}{a}}+\frac{2}{r+\frac{1}{r}}}}\,\mathrm{Beta}\left(\frac{1}{2},\,\frac{1}{2}\right) \\
&< \frac{\sqrt{2(\frac{1}{\cos\theta_2}-1)(\frac{1}{\cos\theta_2}-\frac{2}{r_0+\frac{1}{r_0}})}}{\pi(1+\frac{2}{a_0+\frac{1}{a_0}})\sqrt{\frac{2}{a_0+\frac{1}{a_0}}+\frac{2}{r_0+\frac{1}{r_0}}}}\,\mathrm{Beta}\left(\frac{1}{2},\,\frac{1}{2}\right).
\end{aligned}
$$

Thus, there exists $\theta_* \in (0,\,\theta_\sharp)$ so that $\frac{C}{B} < \frac{K}{2}$ for $a \in (a_0,\,1)$, $r \in (r_0,\,1)$, and $0 < \theta_1 < \theta_2 < \theta_*$.

Therefore, by the density of $\{q \in \mathbb{Q} \mid \frac{K}{2} < q < K\}$ in $(\frac{K}{2},\,K)$, we obtain the desired result. $\square$

For $a \in (a_0,\,1)$, combining Lemma 3.4 and the intermediate value theorem yields the existence of parameters $r \in (r_0,\,1)$ and $\theta_* < \theta_1 < \theta_2 < \theta_\sharp$ so that $\frac{C}{B} = q \in \mathbb{Q}$. We remark that, by the arguments in the proof of Lemma 3.4, we can choose q, θ_* and $\theta_\sharp$ which are independent of $a \in (a_0,\,1)$ and $r \in (r_0,\,1)$.

Lemma 3.5. *Using the notation as above, we have* $\frac{\partial}{\partial\theta_1}(qB - C) < 0$ *on* $\{(\theta_1,\,\theta_2,\,r) \mid r_0 < r < 1,\ \theta_* < \theta_1 < \theta_2 < \theta_\sharp,\ qB - C = 0\}$.

Proof. For simplicity, we set

$$
\varphi_2(t) = \left((t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right)^{\frac{1}{2}}.
$$

First we find

$$
\begin{aligned}
&\frac{\partial}{\partial\theta_1}\left(\frac{1}{\varphi_2(t)}\right) = -\frac{1}{2\varphi_2(t)^3}\frac{\partial}{\partial\theta_1}\left\{(t-r)\left(\tfrac{1}{r}-t\right)(t+a)\left(t+\tfrac{1}{a}\right)\textstyle\prod_{j=1}^{2}(t^2-2(\cos\theta_j)t+1)\right\} \\
&= -\frac{1}{2\varphi_2(t)^3}(t-r)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)(t^2-2(\cos\theta_2)t+1)\times 2\,t\sin\theta_1 \\
&= -\frac{1}{2\varphi_2(t)^3}\frac{2\,t\sin\theta_1\varphi_2(t)^2}{t^2-2(\cos\theta_1)t+1} = -\frac{t\sin\theta_1}{(t^2-2(\cos\theta_1)t+1)\varphi_2(t)}.
\end{aligned}
$$

Thus we have

$$
\frac{\partial}{\partial\theta_1}(qB-C) = 2\,\frac{\partial}{\partial\theta_1}\left(q\int_r^1 \frac{1+t^2}{\varphi_2(t)}dt - \int_{-1}^{-a}\frac{1+t^2}{\varphi_2(t)}dt\right)
$$

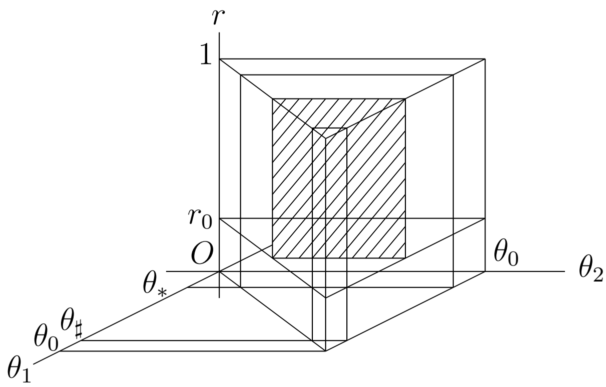


Figure 3.2: $\{(\theta_1, \theta_2, r) \mid \frac{C}{B} = q\}$ (the shaded part)

$$= 2\left(q\int_r^1 \frac{\partial}{\partial\theta_1}\left(\frac{1+t^2}{\varphi_2(t)}\right)dt - \int_{-1}^{-a}\frac{\partial}{\partial\theta_1}\left(\frac{1+t^2}{\varphi_2(t)}\right)dt\right)$$
$$= 2\left(-q\int_r^1 \frac{(1+t^2)t\sin\theta_1}{(t^2-2(\cos\theta_1)t+1)\varphi_2(t)}dt + \int_{-1}^{-a}\frac{(1+t^2)t\sin\theta_1}{(t^2-2(\cos\theta_1)t+1)\varphi_2(t)}dt\right) < 0,$$

which proves the lemma. □

We now introduce the map $\pi : \{(\theta_1, \theta_2, r) \mid r_0 < r < 1, \theta_* < \theta_1 < \theta_2 < \theta_\sharp, qB - C = 0\} \ni (\theta_1, \theta_2, r) \mapsto (\theta_2, r) \in \{(\theta_2, r) \mid r_0 < r < 1, \theta_* < \theta_2 < \theta_\sharp\}$. The next proposition follows from the implicit function theorem and Lemma 3.5.

Proposition 3.2. $\{(\theta_1, \theta_2, r) \mid r_0 < r < 1, \theta_* < \theta_1 < \theta_2 < \theta_\sharp, qB - C = 0\}$ *is a graph on* $\pi(\{(\theta_1, \theta_2, r) \mid r_0 < r < 1, \theta_* < \theta_1 < \theta_2 < \theta_\sharp, qB - C = 0\})$.

Lemma 3.5 implies that $qB - C$ is monotone decreasing for θ_1. By this fact and Proposition 3.2, we obtain the following.

Theorem 3.2. *There exists a unique component* Y *of* $\{(\theta_1, \theta_2, r) \mid r_0 < r < 1, \theta_* < \theta_1 < \theta_2 < \theta_\sharp, qB - C = 0\}$ *so that* (i) Y *is a graph on* $\pi(Y)$. (ii) Y *separates the open prism* $\{(\theta_1, \theta_2, r) \mid r_0 < r < 1, \theta_* < \theta_1 < \theta_2 < \theta_\sharp\}$ *into two components.* (*See Figure* 3.2.)

3.4 The existence of a two-parameter family

For $a \in (a_0, 1)$, by Lemma 3.3 and the implicit function theorem, r can be represented as a smooth function of the parameters θ_1, θ_2, set $r = \alpha(\theta_1, \theta_2)$.

Also, θ_1 can be rewritten as $\theta_1 = \beta(\theta_2, r)$ by Lemma 3.5. Thus we have $\theta_1 - \beta(\theta_2, \alpha(\theta_1, \theta_2)) = 0$. Now, setting

$$F(\theta_1, \theta_2) = \theta_1 - \beta(\theta_2, \alpha(\theta_1, \theta_2)),$$

we will apply the implicit function theorem for the locus $F(\theta_1, \theta_2) = 0$ in this subsection.

First we find

$$\frac{\partial F}{\partial \theta_1} = 1 - \frac{\partial \beta}{\partial r} \cdot \frac{\partial \alpha}{\partial \theta_1}.$$

By the implicit function theorem for the loci $D - E = 0$ and $qB - C = 0$, we remark that

$$\frac{\partial \alpha}{\partial \theta_1} = -\frac{\frac{\partial}{\partial \theta_1}(D-E)}{\frac{\partial}{\partial r}(D-E)}, \quad \frac{\partial \beta}{\partial r} = -\frac{\frac{\partial}{\partial r}(qB-C)}{\frac{\partial}{\partial \theta_1}(qB-C)}.$$

From the proof of Lemma 3.3 and Lemma 3.5, we have

$$\frac{\partial}{\partial r}(D-E) = \frac{1-r^2}{r^2}\left\{\int_{\cos\theta_1}^{1} \frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)} - \int_{-1}^{\cos\theta_2} \frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}\right\},$$

$$\frac{\partial}{\partial \theta_1}(qB-C) = 2\sin\theta_1\left\{-q\int_{r}^{1} \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} + \int_{-1}^{-a} \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)}\right\}.$$

By (2.6) and setting

$$\varphi_3(t) = \left((r-t)\left(\frac{1}{r}-t\right)(t+a)\left(t+\frac{1}{a}\right)\prod_{j=1}^{2}(t^2+1-2\cos\theta_j\,t)\right)^{\frac{1}{2}},$$

we find

$$\frac{\partial}{\partial \theta_1}(D-E) = -4\sin\theta_1 \int_{-a}^{r} \frac{t^2}{(t^2+1-2\cos\theta_1\,t)\varphi_3(t)}dt.$$

Setting $q = \frac{n}{m}\,(<1)$ implies

$$qB - C = \frac{1}{m}\{n(B-C) - (m-n)C\}.$$

Straightforward calculation yields

$$\frac{\partial}{\partial r}C = -\frac{1-r^2}{r^2}\int_{-1}^{-a} \frac{(1+t^2)\,t\,dt}{(r-t)(\frac{1}{r}-t)\varphi_2(t)},$$

and it follows from (2.3) that

$$\frac{\partial}{\partial r}(B-C)=\frac{1-r^2}{r^2}\int_{\cos\theta_2}^{\cos\theta_1}\frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)},$$

where

$$\varphi_4(x)=\left((1-x^2)\left(r+\frac{1}{r}-2x\right)\left(a+\frac{1}{a}+2x\right)(\cos\theta_1-x)(x-\cos\theta_2)\right)^{\frac{1}{2}}.$$

Hence we have

$$\begin{aligned}
&\frac{\partial}{\partial r}(qB-C)=\frac{1}{m}\left\{n\frac{\partial}{\partial r}(B-C)-(m-n)\frac{\partial}{\partial r}C\right\}\\
&=\frac{1-r^2}{r^2}\left\{q\int_{\cos\theta_2}^{\cos\theta_1}\frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)}+(1-q)\int_{-1}^{-a}\frac{(1+t^2)\,t\,dt}{(r-t)(\frac{1}{r}-t)\varphi_2(t)}\right\}.
\end{aligned}$$

The following is an immediate consequence of the above arguments.

Proposition 3.3.

$$\frac{\partial}{\partial r}(D-E)\cdot\left\{-\frac{\partial}{\partial\theta_1}(qB-C)\right\}+\frac{\partial}{\partial\theta_1}(D-E)\cdot\frac{\partial}{\partial r}(qB-C)\neq 0$$

is equivalent to

$$\begin{aligned}
&\left\{\int_{\cos\theta_1}^{1}\frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}-\int_{-1}^{\cos\theta_2}\frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}\right\}\\
&\quad\times\left\{q\int_{r}^{1}\frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)}-\int_{-1}^{-a}\frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)}\right\}\\
&-2\int_{-a}^{r}\frac{t^2}{(t^2+1-2\cos\theta_1\,t)\varphi_3(t)}dt\\
&\quad\times\left\{q\int_{\cos\theta_2}^{\cos\theta_1}\frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)}+(1-q)\int_{-1}^{-a}\frac{(1+t^2)\,t\,dt}{(r-t)(\frac{1}{r}-t)\varphi_2(t)}\right\}\neq 0.
\end{aligned}$$

If the hypothesis of Proposition 3.3 is true, then θ_1 can be represented as a smooth function of θ_2 by the implicit function theorem. (See Figure 3.3.) In fact, for a, there exists $\theta(a)\in(\theta_*,\,\theta_\sharp)$ such that (i) θ_1 is a smooth function of $\theta_2\in(\theta(a),\,\theta_0)$, (ii) $\theta_1\to\theta(a)$ as $\theta_2\to\theta(a)$.

The next theorem guarantees the existence of a satisfying the hypothesis of Proposition 3.3.

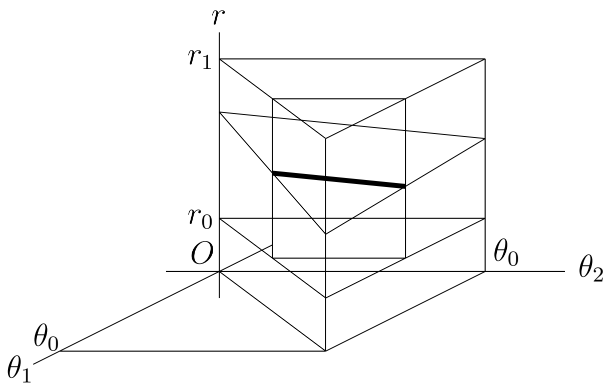


Figure 3.3: $\{(\theta_1,\, \theta_2,\, r) \mid \frac{E}{D} = 1,\, \frac{C}{B} = q\}$ (the thick curve)

Theorem 3.3. *Suppose that* $D - E = 0$ *and* $\frac{C}{B} = q$*. There exists* $a_1 \in (a_0,\, 1)$ *such that*

$$\begin{aligned}
&\left\{ \int_{\cos\theta_1}^{1} \frac{dx}{(r + \frac{1}{r} - 2x)\varphi_1(x)} - \int_{-1}^{\cos\theta_2} \frac{dx}{(r + \frac{1}{r} - 2x)\varphi_1(x)} \right\} \\
&\quad \times \left\{ q \int_{r}^{1} \frac{(1+t^2)\, t\, dt}{(t^2 + 1 - 2\cos\theta_1\, t)\varphi_2(t)} - \int_{-1}^{-a} \frac{(1+t^2)\, t\, dt}{(t^2 + 1 - 2\cos\theta_1\, t)\varphi_2(t)} \right\} \\
&- 2 \int_{-a}^{r} \frac{t^2}{(t^2 + 1 - 2\cos\theta_1\, t)\varphi_3(t)} dt \\
&\quad \times \left\{ q \int_{\cos\theta_2}^{\cos\theta_1} \frac{x\, dx}{(r + \frac{1}{r} - 2x)\varphi_4(x)} + (1-q) \int_{-1}^{-a} \frac{(1+t^2)\, t\, dt}{(r - t)(\frac{1}{r} - t)\varphi_2(t)} \right\} > 0
\end{aligned}$$

for $a \in (a_1,\, 1)$ *and* $\theta_* < \theta_1 < \theta_2 < \theta_\sharp$.

Theorem 3.3 follows immediately from the following three lemmas.

Lemma 3.6. *Suppose that* $D = E$*, then* $r \to 1$ *as* $a \to 1$.

Proof. Combining (2.6) and $D = E$ implies

$$\begin{aligned}
&\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{t}{\sqrt{(1-t^2)(t + \frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}} - t)(t + \frac{1}{\cos\theta_1})(t + \frac{1}{\cos\theta_2})}} dt \\
&= \int_0^{\frac{2}{r+\frac{1}{r}}} \frac{t}{\sqrt{(1-t^2)(\frac{2}{r+\frac{1}{r}} - t)(t + \frac{2}{a+\frac{1}{a}})(\frac{1}{\cos\theta_1} - t)(\frac{1}{\cos\theta_2} - t)}} dt.
\end{aligned}$$

By letting a tend to 1, the left hand side goes to infinity. If $r \to 1$ does not hold as $a \to 1$, then the right hand side must be finite. Thus we would obtain a contradiction and the assertion is proved. $\square$

Lemma 3.7.

$$
\begin{aligned}
&q\int_r^1 \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} - \int_{-1}^{-a} \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} \\
&> q\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)} + (1-q)\int_{-1}^{-a} \frac{(1+t^2)\,t\,dt}{(r-t)(\frac{1}{r}-t)\varphi_2(t)}
\end{aligned}
$$

holds.

Proof. We first have

$$
\begin{aligned}
\int_r^1 \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} &= \int_r^1 \frac{(1+t^2)\,dt}{(t+\frac{1}{t}-2\cos\theta_1)\varphi_2(t)} \\
&> \frac{1}{r+\frac{1}{r}-2\cos\theta_1}\int_r^1 \frac{(1+t^2)\,dt}{\varphi_2(t)}, \\
\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)} &< \frac{1}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{\varphi_4(x)}.
\end{aligned}
$$

Hence we find

$$
\begin{aligned}
&q\int_r^1 \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} - \int_{-1}^{-a} \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} \\
&> q\int_r^1 \frac{(1+t^2)\,t\,dt}{(t^2+1-2\cos\theta_1\,t)\varphi_2(t)} > \frac{q}{r+\frac{1}{r}-2\cos\theta_1}\int_r^1 \frac{(1+t^2)\,dt}{\varphi_2(t)} \\
&\underbrace{=}_{(2.3)} \frac{q}{r+\frac{1}{r}-2\cos\theta_1}\left(\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{\varphi_4(x)} + \int_{-1}^{-a} \frac{(1+t^2)\,dt}{\varphi_2(t)}\right) \\
&> \frac{q}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{\varphi_4(x)} > q\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)} \\
&> q\int_{\cos\theta_2}^{\cos\theta_1} \frac{x\,dx}{(r+\frac{1}{r}-2x)\varphi_4(x)} + (1-q)\int_{-1}^{-a} \frac{(1+t^2)\,t\,dt}{(r-t)(\frac{1}{r}-t)\varphi_2(t)}.
\end{aligned}
$$

$\square$

Lemma 3.8. *There exists $a_1 \in (a_0,\, 1)$ such that*

$$\int_{\cos\theta_1}^{1} \frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)} - \int_{-1}^{\cos\theta_2} \frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}$$
$$> 2\int_{-a}^{r} \frac{t^2}{(t^2+1-2\cos\theta_1\, t)\varphi_3(t)}dt$$

for $a \in (a_1,\, 1)$ and $\theta_* < \theta_1 < \theta_2 < \theta_\sharp$.

Proof. First we have

$$2\int_{-a}^{r} \frac{t^2}{(t^2+1-2\cos\theta_1\, t)\varphi_3(t)}dt$$
$$= 2\int_{-a}^{0} \frac{t^2}{(t^2+1-2\cos\theta_1\, t)\varphi_3(t)}dt + 2\int_{0}^{r} \frac{t^2}{(t^2+1-2\cos\theta_1\, t)\varphi_3(t)}dt$$
$$\underbrace{=}_{x=\pm\frac{1}{2}(\frac{1}{t}+t)} \frac{1}{2}\int_{\frac{a+\frac{1}{a}}{2}}^{\infty} \frac{dx}{(x+\cos\theta_1)\sqrt{(x^2-1)(2x+r+\frac{1}{r})(2x-a-\frac{1}{a})\prod(x+\cos\theta_j)}}$$
$$+\frac{1}{2}\int_{\frac{r+\frac{1}{r}}{2}}^{\infty} \frac{dx}{(x-\cos\theta_1)\sqrt{(x^2-1)(2x-r-\frac{1}{r})(2x+a+\frac{1}{a})\prod(x-\cos\theta_j)}}$$
$$= \frac{1}{2\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}}\times$$
$$\left(\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{s^2\, ds}{(1+\cos\theta_1 s)\sqrt{(1-s^2)(s+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-s)\prod(s+\frac{1}{\cos\theta_j})}}\right.$$
$$\left.+\int_0^{\frac{2}{r+\frac{1}{r}}} \frac{s^2\, ds}{(1-\cos\theta_1 s)\sqrt{(1-s^2)(\frac{2}{r+\frac{1}{r}}-s)(s+\frac{2}{a+\frac{1}{a}})\prod(\frac{1}{\cos\theta_j}-s)}}\right)$$
$$< \frac{1}{2\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}}\times$$
$$\left(\frac{2}{a+\frac{1}{a}+2\cos\theta_1}\int_0^{\frac{2}{a+\frac{1}{a}}} \frac{s\, ds}{\sqrt{(1-s^2)(s+\frac{2}{r+\frac{1}{r}})(\frac{2}{a+\frac{1}{a}}-s)\prod(s+\frac{1}{\cos\theta_j})}}\right.$$
$$\left.+\frac{2}{r+\frac{1}{r}-2\cos\theta_1}\int_0^{\frac{2}{r+\frac{1}{r}}} \frac{s\, ds}{\sqrt{(1-s^2)(\frac{2}{r+\frac{1}{r}}-s)(s+\frac{2}{a+\frac{1}{a}})\prod(\frac{1}{\cos\theta_j}-s)}}\right)$$

$$\underbrace{=}_{D-E=0} \frac{1}{\sqrt{(r+\frac{1}{r})(a+\frac{1}{a})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)$$

$$\times\int_0^{\frac{2}{r+\frac{1}{r}}}\frac{s\,ds}{\sqrt{(1-s^2)(\frac{2}{r+\frac{1}{r}}-s)(s+\frac{2}{a+\frac{1}{a}})\prod(\frac{1}{\cos\theta_j}-s)}}$$

$$< \frac{1}{\sqrt{2(r+\frac{1}{r})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)$$

$$\times\int_0^{\frac{2}{r+\frac{1}{r}}}\frac{s\,ds}{\sqrt{(1-s)(\frac{2}{r+\frac{1}{r}}-s)(\frac{1}{\cos\theta_1}-s)}}$$

$$\underbrace{=}_{t=\sqrt{\frac{\frac{2}{r+\frac{1}{r}}-s}{1-s}}} \frac{\sqrt{2}}{\sqrt{(r+\frac{1}{r})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)$$

$$\times\int_0^{\sqrt{\frac{2}{r+\frac{1}{r}}}}\frac{\frac{2}{r+\frac{1}{r}}-t^2}{(1-t^2)\{\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}}-(\frac{1}{\cos\theta_1}-1)\,t^2\}}\,dt$$

$$< \frac{\sqrt{2}}{\sqrt{(r+\frac{1}{r})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)$$

$$\times\int_0^{\sqrt{\frac{2}{r+\frac{1}{r}}}}\frac{dt}{\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}}-(\frac{1}{\cos\theta_1}-1)\,t^2}$$

$$= \frac{1}{\sqrt{2(r+\frac{1}{r})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)$$

$$\times\frac{1}{\sqrt{\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}}}}\int_0^{\sqrt{\frac{2}{r+\frac{1}{r}}}}\left(\frac{1}{\sqrt{\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}}}+\sqrt{\frac{1}{\cos\theta_1}-1}\,t}\right.$$

$$\left.+\frac{1}{\sqrt{\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}}}-\sqrt{\frac{1}{\cos\theta_1}-1}\,t}\right)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2(r+\frac{1}{r})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)\\
&\quad\times\frac{1}{\sqrt{(\frac{1}{\cos\theta_1}-1)(\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}})}}\log\frac{\sqrt{r+\frac{1}{r}-2\cos\theta_1}+\sqrt{2(1-\cos\theta_1)}}{\sqrt{r+\frac{1}{r}-2\cos\theta_1}-\sqrt{2(1-\cos\theta_1)}}\\
&= \frac{\sqrt{2}}{\sqrt{(r+\frac{1}{r})\cos\theta_1\cos\theta_2}}\left(\frac{1}{a+\frac{1}{a}+2\cos\theta_1}+\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\right)\\
&\quad\times\frac{1}{\sqrt{(\frac{1}{\cos\theta_1}-1)(\frac{1}{\cos\theta_1}-\frac{2}{r+\frac{1}{r}})}}\log\frac{\sqrt{r+\frac{1}{r}-2\cos\theta_1}+\sqrt{2(1-\cos\theta_1)}}{r+\frac{1}{r}-2}. \qquad (3.3)
\end{aligned}
$$

Next we note that

$$
\begin{aligned}
&\int_{\cos\theta_1}^{1}\frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}-\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_1}^{1}\frac{dx}{\varphi_1(x)}\\
&=\frac{2}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_1}^{1}\frac{(x-\cos\theta_1)\,dx}{(r+\frac{1}{r}-2x)\varphi_1(x)},\\
&\int_{-1}^{\cos\theta_2}\frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}-\frac{1}{r+\frac{1}{r}-2\cos\theta_2}\int_{-1}^{\cos\theta_2}\frac{dx}{\varphi_1(x)}\\
&=\frac{2}{r+\frac{1}{r}-2\cos\theta_2}\int_{-1}^{\cos\theta_2}\frac{(x-\cos\theta_2)\,dx}{(r+\frac{1}{r}-2x)\varphi_1(x)},
\end{aligned}
$$

and hence we find

$$
\begin{aligned}
&\int_{\cos\theta_1}^{1}\frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}-\int_{-1}^{\cos\theta_2}\frac{dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}\\
&=\frac{1}{r+\frac{1}{r}-2\cos\theta_1}\underbrace{\int_{\cos\theta_1}^{1}\frac{dx}{\varphi_1(x)}}_{=\frac{D}{2}}-\frac{1}{r+\frac{1}{r}-2\cos\theta_2}\underbrace{\int_{-1}^{\cos\theta_2}\frac{dx}{\varphi_1(x)}}_{=\frac{E}{2}}\\
&\quad+\frac{2}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_1}^{1}\frac{(x-\cos\theta_1)\,dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}\\
&\quad-\frac{2}{r+\frac{1}{r}-2\cos\theta_2}\int_{-1}^{\cos\theta_2}\frac{(x-\cos\theta_2)\,dx}{(r+\frac{1}{r}-2x)\varphi_1(x)}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{r+\frac{1}{r}-2\cos\theta_1} - \frac{1}{r+\frac{1}{r}-2\cos\theta_2}\right)\frac{D}{2} \\
&\quad + \frac{2}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_1}^{1} \frac{(x-\cos\theta_1)\,dx}{(r+\frac{1}{r}-2x)\varphi_1(x)} \\
&\quad + \frac{2}{r+\frac{1}{r}-2\cos\theta_2}\int_{-1}^{\cos\theta_2} \frac{(\cos\theta_2-x)\,dx}{(r+\frac{1}{r}-2x)\varphi_1(x)} \\
&= \frac{\cos\theta_1-\cos\theta_2}{(r+\frac{1}{r}-2\cos\theta_1)(r+\frac{1}{r}-2\cos\theta_2)}\,D \\
&\quad + \frac{2}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_1}^{1} \frac{\sqrt{x-\cos\theta_1}\,dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)^3(2x+a+\frac{1}{a})(x-\cos\theta_2)}} \\
&\quad + \frac{2}{r+\frac{1}{r}-2\cos\theta_2}\int_{-1}^{\cos\theta_2} \frac{\sqrt{\cos\theta_2-x}\,dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)^3(2x+a+\frac{1}{a})(\cos\theta_1-x)}} \\
&> \frac{2}{r+\frac{1}{r}-2\cos\theta_1}\int_{\cos\theta_1}^{1} \frac{\sqrt{x-\cos\theta_1}\,dx}{\sqrt{(1-x^2)(r+\frac{1}{r}-2x)^3(2x+a+\frac{1}{a})(x-\cos\theta_2)}} \\
&> \frac{2}{(r+\frac{1}{r}-2\cos\theta_1)\sqrt{2(2+a+\frac{1}{a})}}\int_{\cos\theta_1}^{1} \frac{\sqrt{x-\cos\theta_1}\,dx}{\sqrt{(1-x)(r+\frac{1}{r}-2x)^3(x-\cos\theta_2)}} \\
&> \frac{2}{(r+\frac{1}{r}-2\cos\theta_1)\sqrt{2(2+a+\frac{1}{a})(1-\cos\theta_2)}}\int_{\cos\theta_1}^{1} \frac{\sqrt{x-\cos\theta_1}\,dx}{\sqrt{(1-x)(r+\frac{1}{r}-2x)^3}} \\
&\underbrace{=}_{x=1-\frac{r+\frac{1}{r}-2}{2}\tan^2\theta} \frac{2\sqrt{1-\cos\theta_1}}{(r+\frac{1}{r}-2)(r+\frac{1}{r}-2\cos\theta_1)\sqrt{(2+a+\frac{1}{a})(1-\cos\theta_2)}} \\
&\qquad\qquad \times \int_0^{\arctan\sqrt{\frac{2(1-\cos\theta_1)}{r+\frac{1}{r}-2}}} \sqrt{1-\frac{r+\frac{1}{r}-2\cos\theta_1}{2(1-\cos\theta_1)}\sin^2\theta}\,d\theta \\
&= \frac{2\sqrt{1-\cos\theta_1}}{(r+\frac{1}{r}-2)(r+\frac{1}{r}-2\cos\theta_1)\sqrt{(2+a+\frac{1}{a})(1-\cos\theta_2)}} \\
&\qquad\qquad \times E\left(\sqrt{\frac{r+\frac{1}{r}-2\cos\theta_1}{2(1-\cos\theta_1)}},\ \arctan\sqrt{\frac{2(1-\cos\theta_1)}{r+\frac{1}{r}-2}}\right),
\end{aligned}
\tag{3.4}
$$

where $E(k, \varphi) = \displaystyle\int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi$ is the elliptic integral of the second kind.

Letting a tend to 1, comparing (3.3) and (3.4), and recalling Lemma 3.6, we conclude the existence of $a_1 \in (a_0, 1)$ as desired. □

Finally, we prove the following lemma to show the Main Theorem in the introduction.

Lemma 3.9. *Under the condition $D - E = 0$, r is monotone increasing for a.*

Proof. Recall that $\frac{\partial}{\partial r}(D - E) > 0$ on $\{(\theta_1, \theta_2, r) \,|\, r_0 < r < r_1,\ 0 < \theta_1 < \theta_2 < \theta_0,\ D - E = 0\}$. By the implicit function theorem, $r = r(a, \theta_1, \theta_2)$ and

$$\frac{\partial}{\partial a}(D - E) + \frac{\partial}{\partial r}(D - E) \times \frac{\partial r}{\partial a} = 0. \tag{3.5}$$

For simplicity, we set

$$\varphi_1(x) = \left((1 - x^2)\left(r + \frac{1}{r} - 2x\right)\left(a + \frac{1}{a} + 2x\right)\prod_{j=1}^{2}(x - \cos\theta_j)\right)^{\frac{1}{2}}.$$

First we find

$$\begin{aligned}
\frac{\partial}{\partial a}\left(\frac{1}{\varphi_1(x)}\right) &= -\frac{1}{2\varphi_1(x)^3}\frac{\partial}{\partial a}\left\{(1 - x^2)\left(r + \tfrac{1}{r} - 2x\right)\left(a + \tfrac{1}{a} + 2x\right)\textstyle\prod_{j=1}^{2}(x - \cos\theta_j)\right\} \\
&= -\frac{1}{2\varphi_1(x)^3}\left(1 - \frac{1}{a^2}\right)(1 - x^2)\left(r + \frac{1}{r} - 2x\right)\prod_{j=1}^{2}(x - \cos\theta_j) \\
&= -\frac{1}{2\varphi_1(x)^3}\left(1 - \frac{1}{a^2}\right)\frac{\varphi_1(x)^2}{a + \frac{1}{a} + 2x} = \left(\frac{1}{a^2} - 1\right)\frac{1}{2(a + \frac{1}{a} + 2x)\varphi_1(x)}.
\end{aligned}$$

Next, the integral interval of D ($\cos\theta_1 \le x \le 1$) yields

$$\frac{1}{a + \frac{1}{a} + 2} \le \frac{1}{a + \frac{1}{a} + 2x} \le \left(\frac{1}{a + \frac{1}{a} + 2\cos\theta_1} \le\right) \frac{1}{a + \frac{1}{a} + 2\cos\theta_2}.$$

Also, that of E ($-1 \le x \le \cos\theta_2$) implies

$$-\frac{1}{a + \frac{1}{a} - 2} \le -\frac{1}{a + \frac{1}{a} + 2x} \le -\frac{1}{a + \frac{1}{a} + 2\cos\theta_2}.$$

Hence we have

$$\frac{\partial}{\partial a}(D-E)=2\,\frac{\partial}{\partial a}\left(\int_{\cos\theta_1}^{1}\frac{dx}{\varphi_1(x)}-\int_{-1}^{\cos\theta_2}\frac{dx}{\varphi_1(x)}\right)$$
$$=2\left(\int_{\cos\theta_1}^{1}\frac{\partial}{\partial a}\left(\frac{1}{\varphi_1(x)}\right)dx-\int_{-1}^{\cos\theta_2}\frac{\partial}{\partial a}\left(\frac{1}{\varphi_1(x)}\right)dx\right)$$
$$=\left(\frac{1}{a^2}-1\right)\left(\int_{\cos\theta_1}^{1}\frac{1}{(a+\frac{1}{a}+2x)\varphi_1(x)}dx-\int_{-1}^{\cos\theta_2}\frac{1}{(a+\frac{1}{a}+2x)\varphi_1(x)}dx\right)$$
$$<\left(\frac{1}{a^2}-1\right)\left(\int_{\cos\theta_1}^{1}\frac{1}{(a+\frac{1}{a}+2\cos\theta_2)\varphi_1(x)}dx-\int_{-1}^{\cos\theta_2}\frac{1}{(a+\frac{1}{a}+2\cos\theta_2)\varphi_1(x)}dx\right)$$
$$=\frac{1}{2(a+\frac{1}{a}+2\cos\theta_2)}\left(\frac{1}{a^2}-1\right)(D-E).$$

It follows that $\frac{\partial}{\partial a}(D-E)<0$ at every point where $D-E=0$. From (3.5), we have $\frac{\partial r}{\partial a}>0$ which proves the lemma. $\square$

By Lemma 3.9 and the existence of the interval $(a_1,\,1)$ in Theorem 3.3, there exist two-parameters a and θ_2 such that $E/D=1$ and $C/B\in\mathbb{Q}$. Therefore, we conclude that f defines a two-parameter family parametrized by a and θ_2 of conformal minimal immersions in flat 3-tori, and the proof of the Main Theorem is completed. We remark that we can take θ_2 sufficiently near to the locus $\theta_1=\theta_2$.

4 Gallery

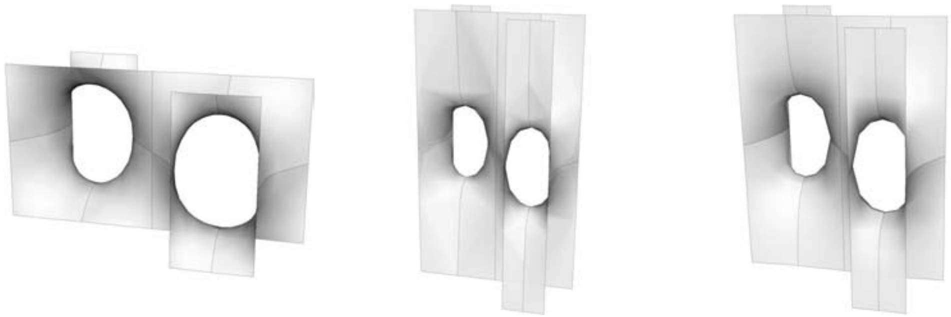
In this section we exhibit some graphics of surfaces we constructed in the previous section.

We used the FindRoot command in Mathematica to find solutions for

$$D-E=0,\qquad qB-C=0,$$

where q is some rational number such that $0<q<1$. Throughout this section we fix $q=1/2$.

Here we only draw just one period of the surfaces. The vertical boundaries of each surface are indeed vertical straight lines. So each surface can be extended analytically by rotation of angle π with respect to these straight lines, and we see that each surface has self intersections after the analytic



$a = 0.5$,	$a = 0.999$,	$a \to 1$,
$r \approx 0.376650$,	$r \approx 0.963693$,	$r \to 1$,
$\theta_1 \approx 0.821065$,	$\theta_1 \approx 1.004779$,	$\theta_1 \approx 0.968716$,
$\theta_2 \approx 1.534785$.	$\theta_2 \approx 1.476486$.	$\theta_2 \approx 1.519815$.

Figure 4.1: A family of surfaces. As $a \to 1$, we see $r \to 1$ as well, and the triply periodic minimal surface converges to a genus 1 doubly periodic minimal surface with 4 parallel ends.

extensions for appropriate times, because the image of the projection of these vertical lines to x_1x_2-plane is vertices of some hexagon.

Figure 4.1 shows a family of surfaces such that θ_2 is not close to θ_1. As $a \to 1$, we see $r \to 1$ by Lemma 3.6. As a limit, we have a genus 1 doubly periodic minimal surface with 4 parallel ends.

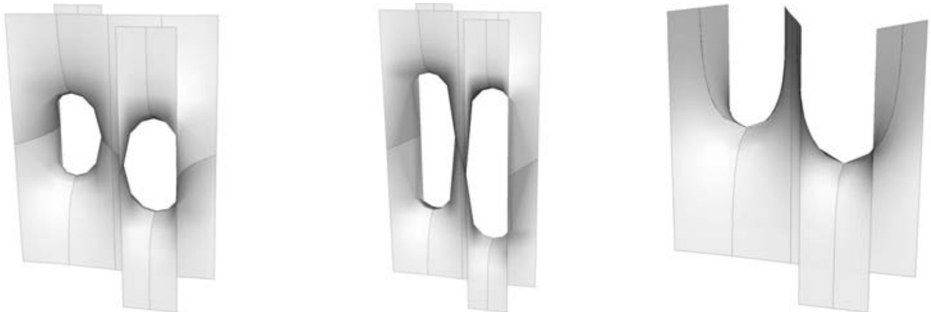
Moreover, if we take the limit $\theta_2 \to \theta_1$ for this doubly periodic minimal surface, we have a genus 0 doubly periodic minimal surface with 4 non-parallel ends. See Figure 4.2.

Figure 4.3 shows a family of surfaces such that θ_1 and θ_2 becomes closer each other. As a limit $\theta_2 \to \theta_1$, we have a genus 1 doubly periodic minimal surface with 4 non-parallel ends.

Moreover, if we take the limit $a \to 1$ for this doubly periodic minimal surface, we see $r \to 1$ as well and have a genus 0 doubly periodic minimal surface with 4 non-parallel ends. See Figure 4.4.

References

[1] C. Arezzo and G. P. Pirola, *On the existence of periodic minimal surfaces*, J. Algebraic Geom. **8** (1999), 765–785.



$\theta_1 \approx 0.968716$, $\theta_2 \approx 1.519815$. | $\theta_1 = 1.2$, $\theta_2 \approx 1.262261$. | $\theta_1 \approx 1.230959$, $\theta_2 \to \theta_1$.

Figure 4.2: A family of genus 1 doubly periodic minimal surfaces with 4 parallel ends given by $a = r = 1$. As $\theta_2 \to \theta_1$, the genus 1 doubly periodic minimal surface converges to a genus 0 doubly periodic minimal surface with 4 non-parallel ends.

$r \approx 0.376650$, $\theta_1 \approx 0.821065$, $\theta_2 \approx 1.534785$. | $r \approx 0.370514$, $\theta_1 \approx 1.167345$, $\theta_2 \approx 1.167355$. | $r \approx 0.370515$, $\theta_1 \approx 1.167350$, $\theta_2 \to \theta_1$.

Figure 4.3: Surfaces with $a = 0.5$. As $\theta_2 \to \theta_1$, the triply periodic minimal surface converges to a genus 1 doubly periodic minimal surface with 4 non-parallel ends. The surface in the left hand side is the same as the surface in the left hand side in Figure 4.1 with translation by a half period in the x_3-direction.

$a = 0.5$, $r \approx 0.370515$. $a = 0.999$, $r \approx 0.963129$. $a \to 1$ $r \to 1$.

Figure 4.4: A family of genus 1 doubly periodic minimal surfaces with 4 non-parallel ends given by $\theta_1 = \theta_2 \approx 1.167350$. As $a \to 1$, we see $r \to 1$ as well, and the genus 1 doubly periodic minimal surface converges to a genus 0 doubly periodic minimal surface with 4 non-parallel ends. The surface on the right hand side is the same as the surface on the right hand side in Figure 4.2.

[2] H. M. Farkas and I. Kra, *Riemann Surfaces*, Second edition, G.T.M. Springer (1991).

[3] S. Fujimori and M. Weber, *Triply periodic minimal surfaces bounded by vertical symmetry planes*, Manuscripta math. **129** (2009), 29–53.

[4] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley Classics Library (1994).

[5] H. Karcher, *The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions*, Manuscripta math. **64** (1989), 291–357.

[6] S. Lidin and S. Larsson, *Bonnet Transformation of Infinite Periodic Minimal Surfaces with Hexagonal Symmetry*, J. Chem. Soc. Farday trans **86**(5) (1990), 769–775.

[7] W. H. Meeks III, *The theory of triply periodic minimal surfaces*, Indiana Univ Math Journal **39** No3 (1990), 877–935.

[8] W. H. Meeks III and H. Rosenberg, *The geometry of periodic minimal surfaces*, Comment. Math. Helvetici **68** (1993), 538–578.

[9] T. Nagano and B. Smyth, *Periodic minimal surfaces*, Comment. Math. Helvetici **53** (1978), 29–55.

[10] T. Nagano and B. Smyth, *Periodic minimal surfaces and Weyl groups*, Acta. Math. **145** (1980), 1–27.

[11] G. E. Schröder-Turk, A. Fogden, and S. T. Hyde, *Bicontinuous geometries and molecular self-assembly: comparison of local curvature and global packing variations in genus-three cubic, tetragonal and rhombohedral surfaces*, Eur. Phys. J. B. **54** (2006), 509–524.

Norio Ejiri
Department of Mathematics, Meijo University
Tempaku, Nagoya 468-8502, Japan.
ejiri@meijo-u.ac.jp

Shoichi Fujimori
Department of Mathematics, Okayama University
Tsushimanaka, Okayama 700-8530, Japan.
fujimori@math.okayama-u.ac.jp

Toshihiro Shoda
Faculty of Education, Saga University
1 Honjo-cho, Saga-city, Saga, 840-8502, Japan.
tshoda@cc.saga-u.ac.jp