Giffen Goods, Gross Complements, and Instability: Examples

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In[1]:= Off[General::spell1, Solve::ifun, Plot::plnr, General::spell]

Introduction

In the traditional textbooks on microeconomics, the explanation of Giffen Goods is one of the "must"s. Furthermore, instability of equilibrium has been attributed to the Giffen Good property (Layards and Walters [1978]). The construction of this good utilizing concrete utility function, however, has not been provided according to my poor knowledge, while it is always explained that the existence of Giffen Good was noticed in the Ireland's famine of the 19th century. Exception is Fukiharu [2008], which asserts in terms of simulation approach utilizing concrete utility function that something peculiar emerges in the financial market with Giffen Good. In this paper, the Giffen Good case is provided by the simulation on the textbook style household's utility maximization with pure trade model. The relation between the Giffen Good and the instability of equilibrium is also examined. The utility function utilized in this simulation is CES (constant elasticity of substitution) type. Indeed, depending on the parameters of CES production function, various results with contrast have been attained on environmental problems, or income distribution (see Fukiharu [2006, 2007]). In this simulation, Mathematica is extensively utilized to provide a Giffen Good.

In[2]:= Clear[ua, ub, a, r, ea, eb, sola, solb, zl1, zl1]
1. Construction of Giffen Good

In an economy with no production, there are two goods. There are two households, A and B, with the initial endowments, $e_A = \{e_A^1, e_A^2\}$, and $e_B = \{e_B^1, e_B^2\}$. It is assumed that
\[ e_A^1 = 1, \quad e_A^2 = 0, \quad e_B^1 = 0, \quad e_B^2 = 1. \] (1)

They have the same type of utility function, which is of the following CES (constant elasticity of substitution) type, while parameters may be different, where $x_i$ is the quantity of the $i$ th good ($i=1, 2$) and others are parameters.
\[ u = u(x_1, x_2) = (a_1 x_1^{-t} + a_2 x_2^{-t})^{-\frac{1}{r}} \] (2)

Each household aims at the utility maximization, subject to the budget constraint, where $p_i$ is the price of the $i$ th good ($i=1, 2$).
\[ \max u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 = p_i e_i, \quad i=A, B \] (3)

It is assumed that
\[ a_1 = \frac{3}{10}, \quad a_2 = \frac{7}{10}, \quad \text{for household A}, \quad a_1 = \frac{7}{10}, \quad a_2 = \frac{3}{10}, \quad \text{for household B}, \]
\[ n = 1, r = 2. \] (4)

In[3]:= ua = (a*x1^(-r) + (1-a)*x2^(-r))^(-1/r);
    ub = ((1-a)*x1^(-r) + a*x2^(-r))^(-1/r);
    a = 3/10; r = 2;
    ea = {ea1, ea2}; ea1 = 1; ea2 = 0;
    eb = {eb1, eb2}; eb1 = 0; eb2 = 1;

- Household A’s Excess Demand Function

Household A’s indifference curves for the utility function specified by (2) has standard shape, convex to the origin, as shown by the following figure.
From the household A’s utility maximization behavior subject to the budget constraint, (3), demand functions for 2 goods, \( d_A[p_1, p_2] \) and \( d_A[p, p_2] \), are derived. *Mathematica* gives rise to 3 solutions as the candidates for them.

The first solution, given as in what follows, is the pair of demand functions, \( d_A[p_1, p_2] \) and \( d_A[p, p_2] \).

Indeed, other solutions give rise to complex solutions.
In[11]:= N[sola /. {p1 \[Rule] 1, p2 \[Rule] 1}]

Out[11]= {{x2 -> 0.570143, x1 -> 0.429857}, {x2 -> 0.764929 - 0.801653 i, x1 -> 0.235071 + 0.801653 i}, {x2 -> 0.764929 + 0.801653 i, x1 -> 0.235071 - 0.801653 i}}

Household A's excess demand function for good 1, \(z_{A1}[p_1]=d_{A1}[p_1, 1]-e_{A1}\), is given as in what follows.

In[12]:= x1a = x1 /. sola[[1]]; 
   x2a = x2 /. sola[[1]]; 
   z1a = x1a - ea1; z2a = x2a - ea2; z2a1 = z2a /. p2 \[Rule] 1; z1al = Simplify[(x1a /. p2 \[Rule] 1)]

Out[14]= \(-1 + \frac{2^{2/3} 21^{1/3} \left(7 p_1^2 - 3 p_1^4 + \sqrt{p_1^4 \left(7 + 3 p_1^2\right)^2}\right)^{1/3} + p_1^2 \left(6 - \frac{22^{2/3} 21^{2/3}}{\sqrt{7 p_1^2 - 3 p_1^4}}\right)^{1/3}}{2 \left(7 + 3 p_1^2\right)}\)

From the household A's point of view, the 1st good is Giffen Good over the whole domain of \(p_1\).

In[15]:= Plot[(x1a /. p2 \[Rule] 1), {p1, 0, 100}, AxesLabel -> {"p1", "x1A"}];

Indeed, in the following graph, the optimal selection of 2 goods when \(p_1=1\), and \(p_2=1\) is depicted as \(A_1=\{0.429857,0.570143\}\), while the optimal selection of 2 goods when \(p_1=3/2\), and \(p_2=1\) is depicted as \(A_2=\{0.496969,0.754546\}\).
Hosehold B’s Excess Demand Function

Household B’s indifference curves for the utility function specified by (2) has standard shape, convex to the origin, as shown by the following figure.
From the household B’s utility maximization behavior subject to the budget constraint, (3), demand functions for 2 goods, \( d_{B1}[p_1, p_2] \) and \( d_{B2}[p_1, p_2] \), are derived. Mathematica gives rise to 3 solutions as the candidates for them.

The first solution, given as in what follows, is the pair of demand functions, \( d_{A1}[p_1, p_2] \) and \( d_{A2}[p_1, p_2] \).

Indeed, other solutions give rise to complex solutions.
Household B’s excess demand function for good 1, $z_{B1}[p_1]=d_{B1}[p_1, 1]-e_{B1}$, is given as in what follows.

> From the household B’s point of view, the 1st good is Non-Giffen Good over the whole domain of $p_1$.

**Market Excess Demand Function and General Equilibrium**

The market excess demand function, $z_1[p_1]$, is defined by $z_1[p_1]=z_{A1}[p_1]+z_{B1}[p_1]$, and the graph of $z_1[p_1]$ is depicted as in what follows.

When price $p_1$ is displayed on the vertical axe as in the traditional textbook style, we have the following figure, in which the Giffen Good property emerges for the price $p_1$ greater than 10.
In[34]:=
list1 = Table[((z1b1 + z1a1) /. p1 -> (1/100) * i, (1/100) * i), {i, 1, 10000}];
fig1 = ListPlot[list1, PlotRange -> {{-0.17, 0.1}, {0, 100}},
  PlotJoined -> True, AxesLabel -> {"z1[p1]", "p1"}, DisplayFunction -> Identity];
fig2 = Show[Graphics[{Text["E", {0.01, 5}]}], DisplayFunction -> Identity];
Show[{fig1, fig2}, DisplayFunction -> $DisplayFunction];

Note that z1[p1] converges to 0 as p1 approaches to \(\infty\).

In[36]:=
Limit[(z1b1 + z1a1), p1 -> Infinity]
Out[36]= 0

Equilibrium price for the 1st good, E, is 1, and it is locally stable.

In[37]:=
Simplify[(z1b1 + z1a1) /. p1 -> 1]
Out[37]= 0

The market excess demand function for the 2nd good, z2[p1, p2], is defined by z2[p1, p2] = zA1[p1, p2] + zB1[p1, p2], and the graph of z2[p1] = z2[p1, 1] is depicted as in what follows. From the graph, it is clear that the 2nd good is Gross Complement to the 1st good on the domain of p1 between 0 and 0.2 with p2 = 1, while the Gross Substitutability prevails for other domain of p1 with p2 = 1.

In[38]:=
Plot[(z2b1 + z2a1), {p1, 0.001, 2}, AxesLabel -> {"p1", "z2[p1]"}];
2. Giffen Good and Local Instability

In the previous section, there existed the unique market equilibrium, E, which is locally stable. Thus, the existence of Giffen Good does not necessarily imply the local instability when the Giffen Good property emerges for only part of the price domain. In this section, the multiple equilibrium case, in which one of equilibria is locally unstable, is constructed from the same CES utility function with different parameters. In this section, assumptions, (1), (2), and (3) are retained. The assumptions on \( a_1 \) and \( a_2 \) in (4) are modified as in what follows.

\[
a_1 = \frac{99}{100}, \quad a_2 = \frac{1}{100}, \text{ for household A}, \quad a_1 = \frac{1}{100}, \quad a_2 = \frac{99}{100}, \text{ for household B},
\]

\( n = 1, \quad t = 2. \)  

\[
(4')
\]

\[
\text{In[39]}:= \text{Clear}[\text{ua}, \text{ub}, a, r, \text{ea}, \text{eb}, \text{solA}, \text{solB}, z1a1, z1b1]
\]

\[
\text{In[40]}:= \text{ua} = (a1 * x1^(-r) + a2 * x2^(-r)) ^ (-1 / r); \quad \text{ub} = (a2 * x1^(-r) + a1 * x2^(-r)) ^ (-1 / r);
\]

\[
a1 = \frac{99}{100}; \quad a2 = \frac{1}{100}; \quad r = 2;
\]

\[
ea = \{\text{ea1}, \text{ea2}\}; \quad \text{ea1} = 1; \quad \text{ea2} = 0;
\]

\[
\text{eb} = \{\text{eb1}, \text{eb2}\}; \quad \text{eb1} = 0; \quad \text{eb2} = 1;
\]

- **Household A’s Excess Demand Function**

The indifference curves of household A for the utility function specified by (1) has usual shape, convex to the origin, as shown by the following figure.

\[
\text{In[44]}:= \text{ContourPlot}[\text{ua}, \{x1, 0.1, 100\}, \{x2, 0.1, 100\}];
\]

From the household A’s utility maximization behavior subject to the budget constraint, (3), demand functions for 2 goods, \( d_{A1}[p_1, p_2] \) and \( d_{A2}[p_1, p_2] \), are derived. Mathematica gives rise to 3 solutions as the candidates for them.

\[
\text{In[45]}:= \text{sola} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{ea1} + p2 * \text{ea2}, \{x1, x2}\}];
\]

\[
\text{In[46]}:= \text{solb} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{eb1} + p2 * \text{eb2}, \{x1, x2}\}];
\]

\[
\text{In[47]}:= \text{solA} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{ea1} + p2 * \text{eb2}, \{x1, x2}\}];
\]

\[
\text{In[48]}:= \text{solB} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{eb1} + p2 * \text{eb2}, \{x1, x2}\}];
\]

\[
\text{In[49]}:= \text{solC} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{ea1} + p2 * \text{ea2}, \{x1, x2}\}];
\]

\[
\text{In[50]}:= \text{solD} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{eb1} + p2 * \text{ea2}, \{x1, x2}\}];
\]

\[
\text{In[51]}:= \text{solE} = \text{Simplify}[
\quad \text{Solve}[\{D[\text{ua}, x1]/D[\text{ua}, x2] = p1/p2, p1 * x1 + p2 * x2 = p1 * \text{eb1} + p2 * \text{ea1}, \{x1, x2}\}];
\]
The first solution, given as in what follows, is the pair of demand functions, \( d_{A1} [p_1, p_2] \) and \( d_{A2} [p_1, p_2] \).

Indeed, other solutions give rise to complex solutions.

Household A's excess demand function for good 1, \( z_{A1} [p_1] = d_{A1} [p_1, 1] - e_{A1} \), is given as in what follows.
In[54]:= check1 = Simplify[x1a /. p2 -> 1]

Out[54]= \[\frac{1}{2} (1 + 99 p1^2)^{1/3} \left(6^{2/3} 11^{1/3} \left(p1^2 - 99 p1^4 + \sqrt{p1^4 (1 + 99 p1^2)^2}\right)^{1/3}\right] + p1^2 \left(198 - \frac{6^{1/3} 11^{2/3}}{p1^2 - 99 p1^4 + \sqrt{p1^4 (1 + 99 p1^2)^2}}\right)^{1/3}\]

### Household B's Excess Demand Function

The indifference curves of the household B for the utility function specified by (1) has usual shape, convex to the origin, as shown by the following figure.

In[55]:= ContourPlot[ub, {x1, 0.1, 100}, {x2, 0.1, 100}];

From the household B’s utility maximization behavior subject to the budget constraint, (3), demand functions for 2 goods, \(d_{B1}[p_1, p_2]\) and \(d_{B2}[p_1, p_2]\), are derived. *Mathematica* gives rise to 3 solutions as the candidates for them.

In[56]:= solb = Simplify[Solve[\(D[ub, x1] / D[ub, x2] = p1 / p2, p1 \cdot x1 + p2 \cdot x2 = p1 \cdot eb1 + p2 \cdot eb2\), \{x1, x2\}];

In[57]:= Length[solb]

Out[57]= 3

The first solution, given as in what follows, is the pair of demand functions, \(d_{B1}[p_1, p_2]\) and \(d_{B2}[p_1, p_2]\). Indeed, other solutions give rise to complex solutions.

In[58]:= N[solb /. \{p1 -> 1/10, p2 -> 1\}]

Out[58]= \{\{x2 \to 0.955501, x1 \to 0.444992\}, \{x2 \to 1.0221 - 0.0422063 i, x1 \to -0.220981 + 0.422063 i\}, \{x2 \to 1.0221 + 0.0422063 i, x1 \to -0.220981 - 0.422063 i\}\}
Household B’s excess demand function for good 1, \( z_{B1}[p_1] = d_{B1}[p_1, 1] - e_{B1} \), is given as in what follows.

\[
\text{Out}[64] = \frac{6^{2/3} 11^{1/3} (99 p_1^2 - p_1^4 + \sqrt{p_1^4 (99 + p_1^2)^2})^{1/3} + p_1^2 \left(2 - \frac{6^{4/3} 11^{2/3}}{99 p_1^2 - p_1^4 + \sqrt{p_1^4 (99 + p_1^2)^2}}\right)}{2 (99 p_1^2 + p_1^4)}
\]

### Market Excess Demand Function and General Equilibria

The market excess demand function, \( z_1[p_1] \), is defined by \( z_1[p_1] = z_{A1}[p_1] + z_{B1}[p_1] \), and the graph of \( z_1[p_1] \) for the domain \([0, 2]\) is depicted as in what follows. There are two equilibria, where smaller one is locally stable and the greater one is locally unstable.

\[
\text{Out}[64] = \frac{6^{2/3} 11^{1/3} (99 p_1^2 - p_1^4 + \sqrt{p_1^4 (99 + p_1^2)^2})^{1/3} + p_1^2 \left(2 - \frac{6^{4/3} 11^{2/3}}{99 p_1^2 - p_1^4 + \sqrt{p_1^4 (99 + p_1^2)^2}}\right)}{2 (99 p_1^2 + p_1^4)}
\]

The graph of \( z_1[p_1] \) for the domain \([2, 50]\) is depicted as in what follows. There is one equilibrium, which is locally stable.

\[
\text{Out}[64] = \frac{6^{2/3} 11^{1/3} (99 p_1^2 - p_1^4 + \sqrt{p_1^4 (99 + p_1^2)^2})^{1/3} + p_1^2 \left(2 - \frac{6^{4/3} 11^{2/3}}{99 p_1^2 - p_1^4 + \sqrt{p_1^4 (99 + p_1^2)^2}}\right)}{2 (99 p_1^2 + p_1^4)}
\]

\( z_1[p_1] \) converges to 0 as \( p_1 \) approaches to \( \infty \).
When price \( p_1 \) is displayed on the vertical axe as in the traditional textbook style, we have the following figure on the price domain \([2, 50]\), in which \( E_1 = 36.7714 \) is a locally stable equilibrium, and the Giffen Good property emerges on the price domain \([2, 4]\).
Conclusions

As shown by Arrow and Hahn [1971], in 2-goods economy the global stability prevails for general equilibria. Thus, starting from arbitrary initial point of prices, the price trajectory on the Walrasian tatonnement converges to an equilibrium. This does not necessarily imply that arbitrary equilibrium is locally stable. In textbooks, the possibility of multiple equilibria and the local instability has been alluded without concrete examples of the economy with 2 households, who maximize utility subject to budget constraint. The aim of this paper is to construct those concrete examples. In doing so, the Constant Elasticity of Substitution (CES) type utility function is utilized. It is known that CES type function produces diverse conclusions depending on the specification of parameters (Fukiharu [2006, 2007]). In section 1 of this paper, it is assumed that there are 2 households, who possess initial endowment of two goods, and there is no production. The households maximize utility subject to budget constraint. The diverse distribution of initial endowment and the specification of CES utility function produces "Giffen goods" property for the one of social excess demand functions, as well as "Gross Complement" property for the other social excess demand function. The price domain on which those properties emerges is restricted on some part of the whole domain. The specification in section 1 does not guarantee the multiple equilibria. In section 1 there is unique equilibrium with local stability. In section 2, the assumption of diversity of utility function is added. This addition guarantees the multiple equilibria. In section 2, there are 3 equilibria, in which local instability emerges for one of them.

References


