Asset Market Equilibrium: A Simulation

Toshitaka Fukiharu
(Faculty of Economics, Hiroshima University)

December, 2006

Introduction

The aim of this paper is to examine the asset price variation when the asset market structure changes. When the bubble emerges in an asset market, there exists the modification of preceding circumstances in many cases: e.g. in Netherland's Tulip bubble. In the bubble economy, the investors appear to be risk-lovers. Indeed, it is a standard exercise that assuming the existence of one risky asset and one riskless asset, when the investor's income increases, he or she reduces the ratio of investment on risky asset, raising the one on the riskless asset (Layard and Walters [1978, Chapter 13]). This exercise is solved under the assumption of a quadratic utility function, a special type of risk-averters assumption. At first glance, it appears that under the assumption of the society with risk averters, the bubble of asset price cannot be explained in terms of this exercise. The exercise, however, does not imply the shrinkage of absolute demand for the risky asset. There might be the increase of absolute demand for the risky asset. Unfortunately, this exercise does not explain how the asset price changes.

This paper examines how the asset price is determined in the asset market, and how it changes through the modification of market structure, utilizing Lucas [1978] model, which incorporates the asset price. The main motivation in Lucas [1978] was the existence of equilibrium asset price, while the one in this paper is the comparative statics: the variation of asset price, especially when the dividend from the asset becomes riskier. This paper examines if the asset price rises in the case of increased uncertainty when the investors are risk-averters, utilizing simulation approach.
1. Certain Asset Dividend Case

Lucas [1978] constructed a household’s asset purchasing plan over indefinite period, examining the existence of equilibrium in the asset market. Since the purpose of the present paper is to examine the asset price variation under the modified circumstance, we simplify the Lucas model as much as possible. In this section, we assume that there is the aggregate household, planning the optimum asset possession and goods consumption over two periods, under pure exchange model. There are commodity and asset. There is no production, so that the model may be well understood if we suppose that the asset is the foreign asset, and the household’s consumption from the dividend is provided in the foreign trade. In this section one unit of the asset is assumed to yield the certain divident, $y$. The assumption of certainty of divident allows the investor to consume the same $y$ for the two period. In the first period, the aggregate investor possesses the asset on the amount of $z_0=1$. When $p$ is the asset price, in terms of consumption good, the aggregate investor possess the income on the amount of $(y+p)z_0$ in the first period. The aggregate investor plans the optimum consumption over the two period: $c_1$ and $c_2$, and the purchase of the asset, $z_1$, believing that the purchase of the asset on the amount of $z_1$ guarantees the consumption on the second period, $c_2=yz_1$, while the asset is bequested in the second period. Assuming $u(c)$ to be the utility function, the aggregate investor’s behavior is expressed by the following maximization.

$$\max u(c_1)+\beta u(c_2) \text{ s.t. } c_1+pz_1 \leq (y+p)z_0, c_2 \leq yz_1$$  \hspace{1cm} (1-1)

From this maximization, we have demand functions, $c_1(p)$, $c_2(p)$, and $z_1(p)$. The certain pure exchange stock market equilibrium is defined by $p^*$, which satisfies

$$z_1(p^*)=1, c_1(p^*)=y, \text{ and } c_2(p^*)=y$$ \hspace{1cm} (2-1)

We have the following result (Fukiharu [1991]). This result is nothing but the fundamental theorem on finance; the asset price is the present value of the stream of dividends.

**Proposition 1**: Suppose that $u'(c)>0$ and $u''(c)<0$ ($\forall c>0$). (3-1)

(i) $z_1(p)<1$ holds when $\beta y<p$.
(ii) $z_1(p)>1$ holds when $\beta y>p$.
(iii) Necessary and sufficient conditions for (2-1) to hold is $p^*=\beta y$.

From Proposition 1 it follows that $p^*=\beta y$ is the stable pure exchange stock market. Assumption (3-1) appears to be a standard one. In what follows, a simulation is conducted by specifying utility function.

In[1]:= Off[Solve::incnst, Solve::tdep, Solve::ifun, General::spell1]

1.1: Power Function I–Certainty Case

One of the typical example which satisfies (3-1) is the power function:

$$u(c)=c^\gamma \text{  } 0<\gamma<1.$$ \hspace{1cm} (4)

As shown in what follows, (iii) in Proposition 1 is derived.
As shown in what follows, (i) and (ii) in Proposition 1 are derived.

One of the other typical examples which satisfies (3-1) is the exponential function:

\[ u(c) = 1 - e^{-\mu c} \quad \mu > 0. \]  

(5)

As shown in what follows, (iii) in Proposition 1 is derived when \( \mu = 1 \).

1.2: Exponential Function I–Certainty Case

As shown in what follows, (i) and (ii) in Proposition 1 are derived.
This figure reveals an important property of the exponential utility function: when the asset price, \( p \), is large, a further increase of \( p \) raises the demand for the asset: the asset as a *Giffen* good, or strong *income* effect.

## 1.3: CES Function.

In (1-1) and (3-1), the separability of utility function is assumed. The following CES type function does not satisfy the separability, where \( n \) is the degree of homogeneity and \( \tau \) is the elasticity of substitution.

\[
U(c_1, c_2) = (c_1^{-\tau} + \beta c_2^{-\tau})^{-\frac{1}{\tau}}
\]

(6)

In what follows, it is examined if the conclusions hold when the separability is not assumed.

### 1.3.1: When \( \tau = -1/2 \)

When \( \tau = -1/2 \) and \( n = 1 \), the solution is \( p^* = by \), as shown in what follows.

\[
Out[18] = \{p \rightarrow by\}
\]

As shown in what follows, (i) and (ii) in *Proposition 1* are derived.
1.3.2: When $r=1/2$.

When $r=1/2$ and $n=1$, the computation is somewhat complicated. However, after deriving the excess demand function for the asset, we can easily check that the excess demand equals zero when $p^*=\beta y$, by selecting arbitrary specification on $\beta$ and $y$.

\[
\begin{align*}
\text{In[20]} & := \quad t = 1/2; \\
& \quad u = (c1^(-t) + b * c2^(-t))^(-t); \\
& \quad s1 = \text{Solve}[(D[u, c1] / D[u, c2] = 1 / (p / y), c1 + (p / y) * c2 = (y + p) z0), \{c1, c2\}][[1]] / . z0 \rightarrow 1; \\
& \quad z10 = \text{Simplify}[(\text{PowerExpand}[c2 / . s1]) / y]; \\
& \quad \text{PowerExpand}[(\text{Simplify}[(\text{PowerExpand}[z10 - 1 /. p \rightarrow y * b])] / . \{b \rightarrow 9/10, y \rightarrow 1}\right)
\end{align*}
\]

\[\text{Out[24]} = 0\]

As shown in what follows, (i) and (ii) in Proposition 1 are derived.

\[
\begin{align*}
\text{In[25]} & := \quad \text{Plot}[z10 - 1 /. \{b \rightarrow 9/10, y \rightarrow 1\}, \{p, 0.1, 10\}, \text{AxesLabel} \rightarrow \{"p", "z1-l"}] ;
\end{align*}
\]
1.4: Quadratic Function

As pointed out in Introduction of this paper, the quadratic utility function has been utilized for convenience, since it allows the expression of expected utility as a function of mean and variance. This function, however, satisfies only one of the two conditions: \( u''(c) < 0 \). For example,

\[
u(c) = -c^2 + 3c + 1 \tag{7}
\]

has \( u'(c) < 0 \) when \( c > 3/2 \). Although this does not prevent the maximizing behavior of the investors, the asset price is not determined when \( y = 3/2 \).

In[26]:= \( v = 1 + 3 \times x - x^2 \); Plot[v, \{x, 0, 3\}];

\[
0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3
\]

\[
0 \quad 0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3
\]

In order to show this, suppose that \( y \geq 3/2 \). Note that the indifference curve of the two period utility in (1-1) has the global maximum point at \( c_1 = 3/2 \) and \( c_2 = 3/2 \), or saturation point, independently of \( \beta \). The following figure shows the contour of the two period utility, or the indifference curves, when \( \beta = 9/10 \).

In[27]:= \( u = (v / . x \rightarrow c1) + \beta \times (v / . x \rightarrow c2) \); ContourPlot[(u / . \( \beta \rightarrow 9/10 \)), \{c1, 0, 3\}, \{c2, 0, 3\}];

\[
0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3
\]

\[
0 \quad 0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3
\]

Meanwhile, the boundary of the budget set in (1-1);
\[ c_2 = \frac{-y}{p} (c_1 - \bar{y}) + \bar{y} \quad (8) \]

passes through \{ \bar{y}, \bar{y} \}. Thus, when \( y = 3/2 \), the investor has the greatest utility when selecting \{ \bar{y}, \bar{y} \}. In order to achieve this, \( z_1 = 1 \) is selected for whatever \( p \) prevails. When \( y > 3/2 \), the investor has the greatest utility when selecting \{3/2, 3/2\}. In order to achieve this, \( z_1 = z_0 \) is selected for whatever \( p \) prevails. Thus, \( p \) continues to decline from the initial value: the instability. Only when \( y < 3/2 \) Proposition 1 holds. Indeed, the following result obtains: \( p^* = \beta \bar{y} \).

In what follows, it is shown that the certain asset market equilibrium exists so long as \( y < 3/2 \) holds, however \( y \) is close to 3/2. In order to show that, suppose that \( y = 299/200 \), and \( \beta = 9/10 \). The attained utility level, \( u_e \), is computed as in what follows.

In[28]:= sol1 = Solve[[D[u, c1]/D[u, c2] = 1/(p/y), c1 + (p/y) * c2 = (y + p) z0], \{c1, c2\}][[1]]; z10 = Simplify[(c2 /. sol1)/y] /. z0 -> 1; pe = PowerExpand[Solve[z10 = 1, p]][[1]]

Out[28]= \{p -> b y\}

In what follows, it is shown that the certain asset market equilibrium exists so long as \( y < 3/2 \) holds, however \( y \) is close to 3/2. In order to show that, suppose that \( y = 299/200 \), and \( \beta = 9/10 \). The attained utility level, \( u_e \), is computed as in what follows.

In[30]:= ue = Simplify[u /. sol1] /. pe /. b -> 9/10 /. z0 -> 1 /. y -> 299/200

Out[30]= \[2469981\]

In the following figure, the indifference curve corresponding to \( u_e \) is depicted as the solid circle, while the budget line corresponding to (8), where \( y = 299/200 \), and \( \beta = 9/10 \), is depicted as the dashed line. Thus, when \( p^* = \beta \bar{y} \), \{ \bar{y}, \bar{y} \} is the utility maximizing consumption point.

In[31]:= u0 = u /. b -> 9/10;
sol2 = Solve[u0 = ue, c2];
sol3 = Solve[-899981 + 1200000 c1 - 400000 c1^2 = 0, c1];
figq1 = Plot[c2 /. sol2[[1]],
\{c1, cl /. sol3[[1]], cl /. sol3[[2]]\}, DisplayFunction -> Identity];
figq2 = Plot[c2 /. sol2[[2]], \{c1, cl /. sol3[[1]], cl /. sol3[[2]]\},
DisplayFunction -> Identity];
figq3 = Plot[(y - cl)/b + y /. \{y -> 299/200, b -> 9/10\}, \{c1, 1.49, 1.508\},
PlotStyle -> Dashing[\{0.01, 0.01\}], DisplayFunction -> Identity];
Show\{figq1, figq2, figq3\}, PlotRange -> All, AxesLabel -> \{"c1", "c2"\},
DisplayFunction -> $DisplayFunction];

This equilibrium asset price is stable, as shown by the following figure.
This figure reveals an important property of the exponential utility function: when the asset price, \( p \), is large, a further increase of \( p \) raises the demand for the asset: the asset as a Giffen good, or strong income effect.
2. Uncertain Asset Dividend Case: Uncertain Future Period Case

In this section, the uncertainty of asset dividend is introduced. Suppose that through the modified economic circumstances, the dividend in the second period becomes uncertain. When the expected dividend in the second period is greater than the certain dividend in the first period, the asset price in the modified situation may well be higher than the one in the original circumstance. When the former is exactly as the latter, however, the asset price may be expected to be lower than the one in the original circumstance, due to the risk-averse assumption. The element in this problem is the *income effect*, which is clear in (1-1). The rise of asset price raises the income of the aggregate household.

This problem is formulated as in what follows. In the first period, the aggregate investor possesses the asset on the amount of \( z_0 = 1 \). When \( p \) is the asset price, the aggregate investor possess the income on the amount of \( y \hat{y} + p \) on the first period. The aggregate investor plans the optimum consumption over the two period: \( c_1, c_{21}, \text{ and } c_{22} \), and the purchase of the asset, \( z_1 \), believing that the purchase of the asset on the amount of \( z_1 \) yields the two different probable consumption in the second period, \( c_{21} = \hat{y} z_1 \) with probability \( \lambda_1 \), and \( c_{22} = \hat{y} z_1 \) with probability \( \lambda_2 \), with \( \lambda_1 + \lambda_2 = 1 \), while the asset is bequested in the second period. Assuming \( u(c) \) to be the utility function, the aggregate investor's behavior is expressed by the following maximization.

\[
\max u(c_1) + \beta \{ \lambda_1 u(c_{21}) + \lambda_2 u(c_{22}) \} \quad \text{s.t.} \quad c_1 + p z_1 \leq (y \hat{y} + p) z_0, \quad z_0 = 1, \quad c_{21} \leq \hat{y} z_1, \quad c_{22} \leq \hat{y} z_1. \tag{1-2}
\]

From this maximization, we have demand functions, \( c_1(p), c_{21}(p), c_{22}(p) \), and \( z_1(p) \). The *uncertain pure exchange* asset market equilibrium is defined by \( p^{**} \), which satisfies

\[
z_1(p^{**}) = 1, \quad c_1(p^{**}) = \hat{y}, \quad \text{and} \quad c_{21}(p^{**}) = \hat{y} z_1, \quad c_{22}(p^{**}) = \hat{y} z_1. \tag{2-2}
\]

It is examined if the following holds.

\[p^{**} \leq p^{*}. \tag{9}\]

We have the following result (Fukiharu [1991]).

**Proposition 2** (Fukiharu [1991]): Suppose that in addition to (3-1), the following holds.

\[R(c) = -u''(c) c / u'(c) \leq 1 \quad \text{and} \quad R'(c) \geq 0 \quad (\forall c > 0). \tag{3-2}\]

Then, there exists the *uncertain pure exchange* asset market equilibrium, \( p^{**} \), which satisfies (9).

In the proof of Proposition 2, the following Lemma is crucial.

**Lemma** (Fukiharu [1991]): Suppose that (3-1) and (3-2) are satisfied. Then, the following inequality holds.

\[\lambda_1 u'(\hat{y} z_1) + \lambda_2 u'(\hat{y} z_1) \leq u'(\hat{y}) \hat{y}.\]

In Proposition 2, \( R(c) = -u''(c) c / u'(c) \) is called the *relative risk-aversion*. The power function, defined in (4) satisfies (3-1) and (3-2), since \( R(c) = 1 - \gamma \).
2.1: Power Function II–Future Uncertainty Case

In this subsection, assuming the power function, defined in (4), we actually derive $p^{**}$ when $g = 1/2$.

In[39]:= Clear[u];
   a = 1/2;
   u = c1^a + b (r1*c21^a + r2*c22^a); {c21 -> y1*z1, c22 -> y2*z1};
   d1 = Simplify[PowerExpand[D[u, c1] / D[u, z1]]];
   sol1 = Simplify[Solve[{d1 - 1/p, c1 + p*z1 == (y + p)*z0}, {c1, z1}]] /. z0 -> 1;
   sol2 = Solve[{(z1 /. sol1) == 1, p}][2]]

Out[44]=

We have $p^{**} = b\bar{y} \sqrt{(\lambda_1 \sqrt{\bar{y}^2} + \lambda_2 \sqrt{\bar{y}^2})^2} / \bar{y}$. It is easy to show that $\bar{y} - (\lambda_1 \sqrt{\bar{y}^2} + \lambda_2 \sqrt{\bar{y}^2})^2 \geq 0$ always holds for $\lambda_1$ and $\lambda_2$, where $\lambda_1 + \lambda_2 = 1$.

In[45]:= Simplify[PowerExpand[r1*y1 + r2*y2 - (r1*y1^/(1/2) + r2*y2^/(1/2))^2 /. r2 -> 1 - r1]]

Out[45]=

When $\lambda_1 = \lambda_2 = 1/2$, $y_1 = 0$, $y_2 = 2$, and $\bar{y} = 1$, $b = 9/10$, the excess demand function is exhibited as in what follows: stable equilibrium.

In[46]:= ed = z1 - 1 /. sol1 /. r2 -> 1 - r1 /. r1 -> 1/2 /. y -> 1 /. y1 -> 0 /. y2 -> 2 /. b -> 9/10;
   Plot[ed, {p, 0.1, 10}, AxesLabel -> {"p", "z1-1"}]

2.2: Exponential Function II–Future Uncertainty Case

When the exponential function in (5) is assumed for the utility function, $R(c) = \mu c$, so that (3-2) is not satisfied. Note that when $c$ is small, (3-2) is satisfied, while it is not the case when $c$ is not small. In this subsection, we examine what would happen when the exponential function in (5) is assumed for the utility function.

In[48]:= Clear[u]; u =
   1 - E^(-c1) + b (r1 * (1 - E^(-c21)) + r2 * (1 - E^(-c22))) /. {c21 -> y1*z1, c22 -> y2*z1};

In what follows, we examine two cases, depending on the value of $\bar{y}$. We start with the case, in which $\bar{y}$ is small.
\section*{2.2.1: When $\bar{y}$ Is Small}

Suppose that
\[
\bar{y}_1 = 0, \quad \bar{y}_2 = 2, \quad \lambda_1 = \lambda_2 = 1/2, \quad \text{and} \quad \bar{y} = 1, \quad b = 9/10.
\]

(10-1)

When (10-1) is assumed, $p^{**}$ is computed as in what follows.

```plaintext
In[49]:= d1 = (D[u, c1] / D[u, z1]) = 1 / p, c1 + p * z1 = (y + p) * z0) /.
   {y -> 1, y1 -> 0, y2 -> 2, r1 -> 1/2, r2 -> 1/2, z0 -> 1, b -> 9/10};
   sol1 = Solve[d1, {c1, z1}][[1]];    
   sol2 = Solve[{z1 /. sol1 == 1, p}][[1]]
Out[51]= \{p \rightarrow \frac{9}{10} \}
```

Note that $p^{**} < 0.9 = b \bar{y} = p^*$.  

```plaintext
In[52]:= N[\%]  
Out[52]= \{p \rightarrow 0.331091\}
```

The excess demand function is depicted as in what follows: $p^{**}$ is the stable equilibrium.

```plaintext
In[53]:= ed = z1 - 1 / . sol1 / . (b -> 9/10);    
   Plot[ed, (p, 0.001, 1), PlotRange -> All, AxesLabel -> {"p", "z1 -1"}];
```

\section*{2.2.2: When $\bar{y}$ Is Large}

Suppose that
\[
\bar{y}_1 = 4, \quad \bar{y}_2 = 6, \quad \lambda_1 = \lambda_2 = 1/2, \quad \text{and} \quad \bar{y} = 5, \quad b = 9/10.
\]

(10-2)

When (10-2) is assumed, $p^{**}$ is computed as in what follows. First of all, the simultaneous equations, which derives $c_1(p)$ and $z_1(p)$ are as follows.

```plaintext
In[55]:= d2 = (D[u, c1] / D[u, z1]) = 1 / p, c1 + p * z1 = (y + p) * z0) /.
   {y -> 5, y1 -> 4, y2 -> 6, r1 -> 1/2, r2 -> 1/2, z0 -> 1, b -> 9/10}
Out[55]= \{10 e^{-c1} \left(3 e^{-6 z1} + 2 e^{-4 z1}\right) = \frac{1}{p} c1 + p z1 = 5 + p\}
```
Mathematica cannot solve this simultaneous equation directly, so that $c_1$ is derived as a function of $z_1$ and $p$ from the second equation. Substituting this $c_1$ into the first equation, we derive $p^{**}$ by setting $z_1=1$.

In[56]:= \[d3 = \text{Solve}[d2[[2]], c1][[1]];
\]
\[d4 = \text{Solve}[d2[[1]] / . d3 / . z1 \rightarrow 1, p][[1]]\]

Out[57]= \[
\{p \rightarrow \frac{9 (3 + 2 e^2)}{10 e}\}\]

Note that $p^{**} > 4.5 = \bar{b}$ by $\bar{b} = p^*$.

In[58]:= \[N[\%]\]
Out[58]= \[
\{p \rightarrow 5.88618\}\]

In what follows, it is examined if this solution is indeed the maximization of the expected utility function when (10-2) is assumed. The expected utility, $u_0$, when (10-2) is assumed, is derived as follows.

In[59]:= \[u0 = u \cdot (y1 \rightarrow 4, y2 \rightarrow 6, z1 \rightarrow 1/2, z2 \rightarrow 1/2, b \rightarrow 9/10)\]

Out[59]= \[
1 - e^{-c1} + \frac{9}{10} \left( \frac{1}{2} (1 - e^{-6 z1}) + \frac{1}{2} (1 - e^{-4 z1}) \right)\]

Under $p^{**}$, the optimum consumption, $c_1^{**}$, is derived as in what follows.

In[60]:= \[d5 = d3 / . d4 / . z1 \rightarrow 1\]
Out[60]= \[
\{c1 \rightarrow 5\}\]

Substituting $z_1^{**} = 1$ and $c_1^{**} = 5$ into the utility function, the achieved utility, $u_{00}$, is derived as in what follows.

In[61]:= \[N[u00 = u0 / . d5 / . z1 \rightarrow 1]\]
Out[61]= \[
1.8839\]

Thus, the indifference curve with utility level $u_{00}$ under the budget constraint on the second period, is depicted as the solid curve, while the budget line of the first period is depicted as the dashed curve, in the following figure.
It is indeed convex to the origin, and we may safely conclude that \(c_1**, z_1**\) is indeed the utility maximization and \(p**\) is the uncertain pure exchange asset market equilibrium. Finally, it is examined if the equilibrium is stable. Utilizing the Newton method, the excess demand function for the asset is depicted as in what follows. The uncertain pure exchange asset market equilibrium, \(p**\), is stable.

\[
\text{In}[65]:= \quad \text{data1} = \text{Table}[\{z_1, \text{FindRoot}[d2[[1]] /. d3 /. p \to i, \{z_1, 1\}\}, \{i, 1, 100\}];
\]

This figure reveals an important property of the exponential utility function: when the asset price, \(p\), is large, a further increase of \(p\) raises the demand for the asset: the asset as a Giffen good, or strong income effect.

2.3: Quadratic Function II–Future Uncertainty Case

In this subsection, we assume the quadratic utility function, (7). Note that (3-2) is not satisfied for the quadratic utility function. \(R(c) = -u''(c)c/u'(c)\) is computed as in what follows.

\[
\text{In}[67]:= \quad \text{Clear}[u, v]; v[x_] := 1 + 3x - x^2; -D[v[c], c, c] * c / D[v[c], c]
\]

In this subsection, assuming the quadratic function, defined in (7), we actually derive \(p**\).

\[
\text{In}[68]:= \quad u = v[c1] + b \left(r1*v[c21] + r2*v[c22]\right) /. \{c21 \to y1*z1, c22 \to y2*z1\};
\]

In the previous section, it was shown that so long as \(y < 3/2\) is satisfied, there exists certain asset market equilibrium. It is shown in this section, even if \(y < 3/2\) is satisfied, there is a possibility that exists no uncertain asset market equilibrium.

To show this, suppose that parameters are specified by the following.

\[
\bar{y_1}=3/2-11/100, \quad \bar{y_2}=3/2+10/100, \quad \lambda_1=\lambda_2=1/2, \text{ and } \bar{y}=299/200, b=9/10. \quad (10-3)
\]

When (10-3) is specified, \(p**\) must be negative. In other words, there exists no uncertain asset equilibrium.
In this non-existence case, some might expect the bubble: the expansion of asset price. This expectation is not supported. To show this, suppose that when there is no uncertainty, the equilibrium asset price, $p^* = \beta \bar{y}$, prevails. Now the uncertainty emerges where $\lambda_1 \bar{y}_1 + \lambda_2 \bar{y}_2 = \bar{y}$, where $\lambda_1 + \lambda_2 = 1$. In this uncertain circumstance, the excess demand is negative around the previous equilibrium price, $p^* = \beta \bar{y}$. Thus, asset price continuously declines, not rises. In other words, this analysis might be used for the collapse of the bubble. As the certain dividend increases in the bubble economy, the asset price rises. Approaching the satiation point of consumption, however, when the uncertainty emerges regarding the divident, the bubble might collapse even if the expected divident is invariable.

Then, suppose that the following holds.

$$\bar{y}_1 \leq \bar{y} \leq \bar{y}_2 < \frac{3}{2}. \quad (11)$$

Under (11), $p^{**}$ is always positive. Subsstituting $\lambda_1 \bar{y}_1 + \lambda_2 \bar{y}_2 = \bar{y}$, $p^{**} - p^*$ is expressed as in what follows.

$$
2b \left( -r1 y1^2 + r1^2 y1^2 + 2 r1 r2 y1 y2 - r2 y2^2 + r2^2 y2^2 \right) \\
-3 + 2 r1 y1 + 2 r2 y2
$$

The numerator of $p^{**} - p^*$ is negative as shown in what follows. Therefore, so long as (11) is satisfied, $p^{**} \leq p^*$ holds.

When $\lambda_1 = \lambda_2 = 1/2$, $y_1 = 0$, $y_2 = 1$, and $y = 1/2$, $b = 9/10$, the excess demand function is exhibited as in what follows: stable equilibrium.
In[76]:= ed = z1 - 1 /. sol1 /. r2 -> 1 - r1 /. r1 -> 1/2 /. y1 -> 1/2 /. y2 -> 1/0 /. y2 -> 1/. b -> 9/10;
Plot[ed, {p, 0.1, 10}, AxesLabel -> {"p", "z1-1"}];

This figure reveals an important property of the exponential utility function: when the asset price, \( p \), is large, a further increase of \( p \) raises the demand for the asset: the asset as a Giffen good, or strong income effect.

3. Uncertain Asset Dividend Case: Uncertain Present and Future Periods Case

Finally, suppose that through the modified economic circumstances, the dividend in the first period also becomes uncertain, as well as in the second period. In the original circumstance, the dividend is certain with \( \bar{y} \) in both periods. In this section, it is assumed that in both periods, the dividends are \( \bar{y}_1 \) with probability \( \lambda_1 \) and \( \bar{y}_2 \) with probability \( \lambda_2 \), where \( \bar{y} = \lambda_1 \bar{y}_1 + \lambda_2 \bar{y}_2 \). Thus, the aggregate investor plans the optimum consumption over the two period: \( c_{11}, c_{12}, c_{211}, c_{212}, c_{221}, \) and \( c_{222} \), and the purchase of the asset, \( z_1 \) and \( z_2 \) in the first period, believing that the purchase of the asset on the amount of \( z_1 \) yields the two different probable consumption in the second period, \( c_{211} = \bar{y}_1 z_1 \) with probability \( \lambda_1 \), and \( c_{212} = \bar{y}_2 z_1 \) with probability \( \lambda_2 \), and the purchase of the asset on the amount of \( z_2 \) yields the two different probable consumption in the second period, \( c_{221} = \bar{y}_1 z_2 \) with probability \( \lambda_1 \), and \( c_{222} = \bar{y}_2 z_2 \) with probability \( \lambda_2 \), with \( \lambda_1 \bar{y}_1 + \lambda_2 \bar{y}_2 = \bar{y} \), where \( \lambda_1 + \lambda_2 = 1 \), while the asset is bequested in the second period. Assuming \( u(c) \) to be the utility function, the aggregate investor's behavior is expressed by the following maximization.

\[
\text{max } \lambda_1 u(c_{11}) + \lambda_2 u(c_{12}) + \beta [\lambda_1 u(c_{211}) + \lambda_2 u(c_{212}) + \lambda_1 u(c_{221}) + \lambda_2 u(c_{222})] \\
\text{s.t. } c_{11} + p_1 z_1 \leq (\bar{y}_1^* + p) z_0, \ c_{12} + p_2 z_2 \leq (\bar{y}_2^* + p) z_0, \ z_0 = 1, \ c_{211} \leq \bar{y}_1 z_1, \ c_{212} \leq \bar{y}_2 z_1, \\
c_{221} \leq \bar{y}_1 z_2, \ c_{222} \leq \bar{y}_2 z_2.
\]

(1-3)

From this maximization, we have demand functions, \( c_{11} (p_1, p_2, c_{12} (p_1, p_2, c_{211} (p_1, p_2, c_{212} (p_1, p_2, c_{221} (p_1, p_2, c_{222} (p_1, p_2, z_1 (p_1, p_2), and z_2 (p_1, p_2). The uncertain pure exchange asset market equilibrium is defined by \( \{p_1, p_2 \} \) which satisfies

\[
z_1 (p_1, p_2, p_2) = 1, \ z_2 (p_1, p_2, p_2) = 1, \ c_{11} (p_1, p_2, p_2) = \bar{y}_1, \ c_{12} (p_1, p_2, p_2) = \bar{y}_2, \ c_{211} (p_1, p_2, p_2) = \bar{y}_1, \\
c_{212} (p_1, p_2, p_2) = \bar{y}_2, \ c_{221} (p_1, p_2, p_2) = \bar{y}_1, \ c_{222} (p_1, p_2, p_2) = \bar{y}_2.
\]

(2-3)

Proposition 3: Suppose that (3-1) and (3-2) are satisfied, where \( \bar{y}_1 \leq \bar{y} \leq \bar{y}_2 \) holds. Then, the following holds:

\[
p_1 \leq p^* = \beta \bar{y} \leq p_2^*. \]

(12)

This conclusion implies that not all the uncertain asset prices cannot exceed the certain asset price.
To prove Proposition 3, we actually derive \( p_1 **, p_2 ** \). First, \( p_1 ** \) is derived as in what follows.

\[
\text{In}[78]:= \text{Clear}[u, v];
\]
\[
u = r_1 * v[c[11]] + r_2 * v[c[12]] + b (r_1^2 * v[c[21]] + r_1 * r_2 * v[c[21]] + r_2 * r_1 * v[c[21]] + r_2 * r_1 * v[c[22]] + r_2 * r_1 * v[c[22]]);\]
\[
\text{Out}[78]= \frac{\text{Clear}[u, v];}{u = r_1 * v[c[11]] + r_2 * v[c[12]] + b (r_1^2 * v[c[21]] + r_1 * r_2 * v[c[21]] + r_2 * r_1 * v[c[21]] + r_2 * r_1 * v[c[22]] + r_2 * r_1 * v[c[22]])}\]

\[
\text{In}[86]:= d_6 = \text{Solve}[d_2[[4]], c[12]][[1]];\]
\[
d_7 = d_2[[2]];\]
\[
d_8 = \text{Solve}[d_7/.(z_2 \rightarrow 1), p_1][[1]]\]
\[
\text{Out}[88]= \{p_2 \rightarrow \frac{b r_1 y_1 v'[y_1] + b r_2 y_2 v'[y_2]}{v'[y_2]}\}
\]

By Lemma, \( p_1 ** \leq \beta y = p^* \) holds. Meanwhile, \( p_2 ** \) is derived as in what follows.

\[
\text{In}[86]:= d_6 = \text{Solve}[d_2[[4]], c[12]][[1]];\]
\[
d_7 = d_2[[2]];\]
\[
d_8 = \text{Solve}[d_7/.(z_2 \rightarrow 1), p_1][[1]]\]
\[
\text{Out}[88]= \{p_2 \rightarrow \frac{b r_1 y_1 v'[y_1] + b r_2 y_2 v'[y_2]}{v'[y_2]}\}
\]

Since \( u'(c)<0 \) is assumed, \( y_1 \leq y_2 \) implies \( 1 \leq u'(y_1)/u'(y_2) \), so that \( p^* = \beta y \leq p_2 ** \).

In what follows, we examine the concrete cases.

### 3.1: Power Function III—Present and Future Uncertainty Case

As pointed out in Section 2, the power function, (4), satisfies (3-1) and (3-2). we actually compute \( p_1 ** \) and \( p_2 ** \), by assuming (4).

\[
\text{In}[89]:= \text{Clear}[a]; u = r_1 * c[11]^a + r_2 * c[12]^a + b (r_1^2 * c[21]^a + r_1 * r_2 * c[21]^a + r_2 * r_1 * c[21]^a + r_2 * r_1 * c[22]^a + r_2 * r_1 * c[22]^a);\]
\[
\text{Out}[92]= \{p_1 \rightarrow b y_1^{1-a} (r_1 y_1^a + r_2 y_2^a)\}
\]

\[
\text{In}[93]:= \text{Factor}[y_1^{1-a} (r_1 y_1^a + r_2 y_2^a)]\]
\[
\text{Out}[93]= r_2 y_1^{1-a} (-y_1^a y_2 + y_1 y_2^a)\]

Since \( -y_1^a y_2 + y_1 y_2^a = y_1^a y_2^a (-y_1^{1-a} + y_1^{1-a}) < 0 \), it follows that \( p_1 ** \leq \beta y = p^* \). Meanwhile, \( p_2 ** \) is computed as in what follows.
In[96]:= \text{d33 = Solve[d2[[4]], c12][[1]];}
   \text{d34 = d2[[2]] /.
   \text{Solve[d34 /. z2 \rightarrow 1, p2][[1]]}
   Out[96]= \{p2 \rightarrow b \ y1^{1-a} (r1 \ y1^a + r2 \ y2^a)\}

In[97]:= \text{Factor[y2^{1-a} (r1 \ y1^a + r2 \ y2^a) - \ (r1 \ y1 + r2 \ y2)]}
   Out[97]= -r1 \ y2^a (-y^a \ y2 + y1 \ y2^a)

Since \(-y1^a \ y2 + y1 \ y2^a = y1^a \ y2^a (\ -y2^{1-a} + y1^{1-a})<0\), it follows that \(p^* \beta \bar{y} \leq p_2^*\). Thus, \(p_1^{**} \leq p^* \beta \bar{y} \leq p_2^{**}\) holds.

### 3.2: Exponential Function III–Present and Future Uncertainty Case

It is examined in this subsection what would happen to Proposition 3 when the utility function assumption is replaced by the exponential function, (5). Regarding Proposition 2, the exponential function does not guarantee (9). It is shown that (5) guarantees (12) in Proposition 3.

To prove Proposition 3, assuming the exponential function, defined in (5), we actually derive \(\{p_1^{**}, p_2^{**}\}\). First, \(p_1^{**}\) is derived as in what follows.

In[98]:= \text{u = r1 \ (1 - E^{(-c11)}) + r2 \ (1 - E^{(-c12)}) + b \ (r1^2 \ (1 - E^{(-c211)}) + r1 \ r2 \ (1 - E^{(-c212)}) + r2 \ r1 \ (1 - E^{(-c221)}) + r2^2 \ (1 - E^{(-c222)}) \} / .
   \text{\{c211 \rightarrow y1 \ z1, c212 \rightarrow y2 \ z1, c221 \rightarrow y1 \ z2, c222 \rightarrow y2 \ z2\};}
   \text{d11 = PowerExpand[Simplify[PowerExpand[D[u, c11] / D[u, z1]]]];}
   \text{d12 = PowerExpand[Simplify[PowerExpand[D[u, c12] / D[u, z2]]]];}
   \text{d2 = (d11 = 1 / p1, d12 = 1 / p2,}
   \text{c11 \ p1 \ z1 = (y1 + p1) \ z0, c12 \ p2 \ z2 = (y2 + p2) \ z0) / . z0 \rightarrow 1;}
   \text{d3 = Solve[d2[[3]], c11][[1]]};
   \text{d4 = d2[[1]] / . d3;}
   \text{d5 = Solve[d4 / . \{z1 \rightarrow 1, p1\}][[1]]}
   Out[104]=
   \{p1 \rightarrow b \ e^{-y2} \ (e^{y2} \ r1 \ y1 + e^{y1} \ r2 \ y2)\}

Since \(y_1 < y_2\), \(p_1^{**} \leq \beta \bar{y} = p^*\) holds. On the other hand, \(p_2^{**}\) is derived as in what follows.

In[105]:= \text{d6 = Solve[d2[[4]], c12][[1]];}
   \text{d7 = d2[[2]] / . d6;}
   \text{d8 = Solve[d7 /. \{z2 \rightarrow 1, p2\}][[1]]}
   Out[107]=
   \{p2 \rightarrow b \ e^{-y1} \ (e^{y2} \ r1 \ y1 + e^{y1} \ r2 \ y2)\}

Since \(y_1 < y_2\), \(p^* = \beta \bar{y} \leq p_2^{**}\) holds. Thus, (12) in Proposition 3 holds. In Section 2, where the uncertainty emerges only in the second period, it was shown that the income effect may exceed the effect of risk-averse, resulting in the example in which the asset price rises in spite of the emergence of uncertainty. In this section with further introduction of uncertainty, it was shown that the income effect cannot exceed the effect of risk-aversion: at least one asset price is smaller than the certain asset price. Note, however, that expected asset price in the uncertain world is greater than the asset price in the certain world, as shown by the following.
$3.3$: Quadratic Function III–Present and Future Uncertainty Case

It is examined in this subsection what would happen to Proposition 3 when the utility function assumption is replaced by the quadratic function, (7). Regarding Proposition 2, the quadratic function does not guarantee the existence of asset price. The existence of asset price is examined in this subsection whether the uncertainty increases. To do this, first, $p_1^{**}$ is derived as in what follows.

\[
\begin{align*}
v[x_] &:= 1 + 3x - x^2; \\
u &= r_1 + v[c11] + r_2 + v[c12] + b (r_1 x) + r_2 (r_1 z) + 2 br_1 y_1 + r_2 z_2 / . \{c211 \to y_1 + z_1, c212 \to y_2 + z_1, c221 \to y_1 + z_2, c222 \to y_2 + z_2\}; \\
d11 &= \text{PowerExpand}[\text{Simplify}[\text{PowerExpand}[\text{D}[u, c11] / \text{D}[u, z1]]]]; \\
d12 &= \text{PowerExpand}[\text{Simplify}[\text{PowerExpand}[\text{D}[u, c12] / \text{D}[u, z2]]]]; \\
d2 &= \text{Solve}[\text{d2}[[3]], c11][[1]]; \\
d3 &= \text{Solve}[\text{d2}[[1]] /. \text{d3}; \\
d5 &= \text{Solve}[\text{d4} /. \{z1 \to 1\}, \{\text{z0} \to 1\}]; \\
\text{Out}[116] &= \{p1 \to -\frac{3 b r_1 y_1 + 2 b r_1 y_1^2 - 3 b r_2 y_2 + 2 b r_2 y_2^2}{-3 + 2 y_1}\}
\end{align*}
\]

When (10-3) is specified, $p_1^{**}$ must be negative, as shown in what follows. In other words, we have the same conclusion as in Section 2: there exists no uncertain asset market equilibrium under (10-3). Trivially, when (11) is assumed the existence is guaranteed.

\[
\begin{align*} 
\text{Out}[117] &= \{p1 \to -\frac{639}{22000}\}
\end{align*}
\]

Next, $p_2^{**}$ is derived as in what follows.

\[
\begin{align*} 
\text{Out}[120] &= \{p2 \to -\frac{3 b r_1 y_1 + 2 b r_1 y_1^2 - 3 b r_2 y_2 + 2 b r_2 y_2^2}{-3 + 2 y_2}\}
\end{align*}
\]

Trivially, when (11) is assumed, (12) in Proposition 3 holds.
4. Conclusion

This paper examined the variation of asset price when the uncertainty emerges regarding the prospect for dividend receipt, assuming that the investors are risk-aversers. It is expected that in this modified situation the demand for asset declines, leading to the decline of asset price. When the investors plan to maximize the utility level under fixed income, this expectation might be supported. In fact, the income increases when the asset price rises: the argument of reservation demand, such as in the one of the labor supply and saving. Due to the income effect the expectation might prove to be wrong. In this paper, following Lucas' formulation, we examined if there is any utility function which provides the case in which the asset price rises in spite of the emergence of uncertainty regarding the prospect for dividend receipt. Constructing a risk-aversers' two period maximization problem, it was shown that if the uncertainty is introduced for only one period, the exponential utility function provides the case mentioned above. It was shown, however, that when the uncertainty is introduced for two periods, the exponential utility function does not provide the case. Thus, when the uncertainty is not so strong, the income effect may raise the asset price in spite of the increased uncertainty. It was shown in this case that the asset is a Giffen good when the asset price is already high. Historically, when the bubble economy took place, the market-structural change preceded it. This paper pointed out a theoretical possibility that even if the investors are risk-aversers the bubble economy may take place. In examining this problem, it was also found that quadratic utility function may explain the collapse of bubble economy.

References

Fukiharu, T. [1991], "Stock Prices in General Equilibrium Theory", Discussion Paper 9005 (Kobe University, Faculty of Economics)


