Evolutionarily Stable Strategy: Globally Stable or Locally Stable?

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Introduction

Nash [1949] established the existence of equilibrium for the Nash non-cooperative game. His result was applied to the existence proof of general equilibrium theory by Arrow and Debreu [1954]. In general equilibrium theory, the analyses of uniqueness and stability were conducted to examine the structure of equilibria. In the game theory the concept of evolutionarily stable strategy (ESS) was provided by Maynard Smith and Price [1973], to examine the structure of equilibria. ESS was utilized to explain the activity of animals and human beings, by constructing the strategy sets for the two-player-Nash-game as "hawks (hawkish strategy)" and "doves (dovish strategy)". ESS is one of the standard topics for the graduate level textbooks for the game theory.

In such a standard textbook, as Osborne and Rubinstein [1994, Section 3.4], evolutionary equilibrium is examined for "the members of a single population of organisms, such as animals and human beings, interacting with each other pairwise" (pp.48-9). The definition of ESS appears to correspond with "local" stability in general equilibrium theory: robustness of ESS against "small scale" invasions of mutants. In examining the exercise in their book, however, ESS in Hawk-Dove game satisfies "global" stability: robustness of ESS against "large scale" invasions of mutants. Dichotomy of ESS between global stability and local stability enriches the game theory in examining how ESS is robust to the invasion of mutants.

In this paper, Hawk-Dove game and other games constructed from the motion pictures are examined from the viewpoint of global and local stabilities. Following Osborne and Rubinstein [1994], this paper begins with the consideration of Hawk-Dove game, proceeding to the one of Sir Thomas More's game, constructed from the motion picture, A man for all seasons (1966), and then to the one of Confess-Hide game, from the motion picture, Gentleman's agreement (1947).

<Mathematica Programm for Computing Nash Solutions>

In[35]:= Off[Solve::svars, General::stop, General::spell1]
I. Hawk-Dove Game

Evolutionary equilibrium is considered for the game with symmetry, since in this game players are assumed to be identical. We consider the case in which identical animals (or human beings) play a game in front of a prey. Their strategies consist of "Hawk" and "Dove". When both selected "Dove" they split the prey in half. When one of them selects "Hawk", while the other selecting "Dove", the former gains all of the prey, while the latter gains nothing. When both players select "Hawk", the gain is the same for them, less than half, while it may well be minus, since the fight (or war) breaks out between them. Suppose that the gains for them when they select "Hawk" is -1 and the payoff matrices for player A and player B, are as in what follows.

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

where \( Z^T \) stands for the transpose matrix of a matrix \( Z \).

1-1A. Special case A

\[ c = 3; \]
\[ \text{In[36] := hd = \{\{1/2, 1/2\}, \{0, 1\}\}, \{\{1, 0\}, \{(1 - c) / 2, (1 - c) / 2\}\}\}; } \]
\[ \text{hd0 = hd // MatrixForm; } \]
\[ \text{hdq = Table[hd[[1, i]], \{i, 1, 2\}]; } \]
\[ \text{hdq0 = hdp // MatrixForm; } \]
\[ \text{Out[41] = } \begin{pmatrix} 1/2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1/2 & 1 \\ 0 & -1 \end{pmatrix} \]

Utilizing Mathematica program for computing Nash non-cooperative solutions, provided in Varian [1993], we know that there are three types of solutions: two pure strategy solutions and one mixed strategy solution.

\[ \text{In[42] := Nash[hd]} \]
\[ \text{Out[42] = } \{\{\{0, 1\}, \{1, 0\}\}, \\{\{2/3, 1/3\}, \{2/3, 1/3\}\}, \{\{1, 0\}, \{0, 1\}\}\} \]

Nash non-cooperative solution for the two players, A and B, \( s_A^* \) and \( s_B^* \), is defined as in what follows, on the strategy sets, \( S_A \) and \( S_B \), where \( S_A = \{p_1, 1-p_1 \} | 0 \leq p_1 \leq 1 \} \), \( S_B = \{q_1, 1-q_1 \} | 0 \leq q_1 \leq 1 \} \).

\[ s_A^* P_A - s_B^* \geq s_A P_A s_B^* \quad \forall s_A \in S_A \quad (1) \]
\[ s_A^* P_A - s_B^* \geq s_A^* P_A s_B^* \quad \forall s_B \in S_B \quad (2) \]

The mixed strategy equilibrium, \( s_A^* = \{2/3, 1/3\} \) and \( s_B^* = \{2/3, 1/3\} \), can be found by computing \( s_B \) which satisfies \( \partial(s_A P_A s_B) / \partial p_1 = 0 \) and by computing \( s_A \) which satisfies \( \partial(s_A P_A s_B) / \partial q_1 = 0 \), since 2/3 is the value of \( p_1 \) for the interior maximization of (1) given \( s_B^* \) and 2/3 is the value of \( q_1 \) for the interior maximization of (2) given \( s_A^* \).

\[ \text{In[43] := sol1 = Solve[D[\{p1, 1-p1\}.hdp.\{q1, 1-q1\}, p1] = 0, q1][[1]]; } \]
\[ \text{bq = (q1, 1-q1) / . sol1} \]
\[ \text{Out[44] = } \{\{2/3, 1/3\}\} \]
In[45]:= sol2 = Solve[D[p1, 1 - p1] . hdq. {q1, 1 - q1}, q1] == 0, p1][[1]]; 
   bp = (p1, 1 - p1) /. sol2
Out[46]= {\(\frac{2}{3}, \frac{1}{3}\)}

Evolutionary equilibrium, especially, evolutionarily stable strategy (ESS), is defined by the strategies, \(s_A^*\) and \(s_B^*\), which satisfies

\[
\begin{align*}
    s_A^* &= s_B^* \quad (3) \\
    (1-\epsilon)(s_A^* . P_A . s_A^*) + \epsilon(s_A^* . P_A . s_A^*) + \epsilon(s_A^* . P_A . s_A^*) &< (1-\epsilon)(s_A^* . P_A . s_A^*) + \epsilon(s_A^* . P_A . s_A^*) \quad \text{for any small } \epsilon > 0 \text{ and } \\
    \forall s_A \in S_A \text{ and } s_A \neq s_A^* \quad (4)
\end{align*}
\]

The condition (4) is the one which prevents the small invasion of mutants, destroying \(s_A^* = s_B^*\). Suppose that a mutant emerges and tries to destroy \(s_A^* = s_B^*\). To succeed in the destruction, there must be a fraction \(\epsilon > 0\) of population who act as mutants. In other words, to prevent the destruction, for any \(\epsilon > 0\), the expected utility of the act of leaving \(s_A^* = s_B^*\) must be smaller than the expected utility of the act of staying at \(s_A^* = s_B^*\). On the one hand, when player \(A\) acts as a mutant selecting \(s_A\), he (or she) encounters player \(B\) who acts as mutant selecting \(s_A\) with probability \(\epsilon\), while he (or she) encounters player \(B\) who acts as a non-mutant selecting \(s_A^*\) with probability \(1-\epsilon\). The left hand side of (4) is the expected utility of player \(A\) when he (or she) acts as a mutant selecting \(s_A\). On the other hand, when player \(A\) acts as a non-mutant selecting \(s_A^* = s_B^*\), he (or she) encounters player \(B\) who acts as a mutant selecting \(s_A\) with probability \(\epsilon\), while he (or she) encounters player \(B\) who acts as a non-mutant selecting \(s_A^* = s_B^*\) with probability \(1-\epsilon\). The right hand side of (4) is the expected utility of player \(A\) when he (or she) acts as a non-mutant selecting \(s_A^* = s_B^*\).

This definition corresponds with local stability in general equilibrium theory. In this paper, when ESS satisfies (3) and (4) it is called Local ESS: LESS. When ESS satisfies (3) and the following (4') it is called Global ESS: GESS.

\[
\begin{align*}
    (1-\epsilon)(s_A^* . P_A . s_A^*) + \epsilon(s_A^* . P_A . s_A^*) &< (1-\epsilon)(s_A^* . P_A . s_A^*) + \epsilon(s_A^* . P_A . s_A^*) \quad \forall \epsilon \in [0,1) \text{ and } \\
    \forall s_A \in S_A \text{ and } s_A \neq s_A^* \quad (4')
\end{align*}
\]

In this specified parameter, the candidate for ESS is \(s_A^* = \{2/3, 1/3\}\) and \(s_B^* = \{2/3, 1/3\}\) from (3). We show that it is the case and GESS. In (4), the first element, \(s_A^* . P_A . s_A^*\) is computed as follows.

In[47]:= check1 = Simplify[(p1, 1 - p1) . hdq. {q1, 1 - q1} /. sol1]
Out[47]= \(\frac{1}{3}\)

In (4), the second element, \(s_A^* . P_A . s_A^*\) is computed as follows.

In[48]:= check2 = Simplify[(p1, 1 - p1) . hdq. (p1, 1 - p1)]
Out[48]= \(-1 + 3p1 - \frac{3p1^2}{2}\)

In (4), the third element, \(s_A^* . P_A . s_A^*\) is computed as follows, which is equivalent to the first element.

In[49]:= check3 = Simplify[(p1, 1 - p1) . hdq. {q1, 1 - q1} /. sol1 /. sol2]
Out[49]= \(\frac{1}{3}\)

In (4), the fourth element, \(s_A^* . P_A . s_A^*\) is computed as follows.
The difference between the fourth element and second element is computed as follows.

\[ \text{Out}[51] = 2 - 2p_1 + \frac{3p_1^2}{2} \]

This function is minimized at \( p_1 = \frac{2}{3} \). Thus, (4') is satisfied and the pair of \( s_A^* = \{2/3, 1/3\} \) and \( s_B^* = \{2/3, 1/3\} \) is GESS.

1-1B. Special case B

Suppose that the payoff matrices for them, \( P_A \) for player A and \( P_B \) for player B, are given as in what follows.

\[ \text{Out}[50] = \{\{0, 1\}, \{0, 1\}\} \]

Utilizing Mathematica programm for computing Nash non-cooperative solutions, we know that there is only one pure strategy solution: \( \{\text{"Dove}_A", \text{"Dove}_B"\} \).
In (4), the second element, \( s_A.P_As_A \) is computed as follows.

\[
\text{check1} = \text{Simplify}[(p_1, 1 - p_1).\text{hdp.(q}_1, 1 - q_1) /. q_1 \to 0]
\]

Out[61]= \( \frac{1 - p_1}{4} \)

In (4), the third element, \( s_A^*P_As_A^* \) is computed as follows, which is greater than the first element except for \( p_1 = 0 \).

\[
\text{check2} = \text{Simplify}[(p_1, 1 - p_1).\text{hdp.(q}_1, 1 - q_1) /. (p_1 \to 0, q_1 \to 0)]
\]

Out[62]= \( \frac{1}{4} + 2p_1 - p_1^2 \)

In (4), the fourth element, \( s_A^*P_As_A^* \) is computed as follows.

\[
\text{check3} = \text{Simplify}[(p_1, 1 - p_1).\text{hdp.(q}_1, 1 - q_1) /. (p_1 \to 0, q_1 \to 0)]
\]

Out[63]= \( \frac{1}{4} P_1 \)

The difference between the fourth element and the second is computed as follows, which is positive except for \( p_1 = 0 \).

\[
\text{check4} = (p_1, 1 - p_1).\text{hdp.(q}_1, 1 - q_1) /. (p_1 \to 0, q_1 \to p_1)
\]

Out[64]= \( \frac{1 - p_1}{4} + p_1 \)

Thus, \((4')\) is satisfied and the pair of \( s_A^* = \{0, 1\} \) and \( s_B^* = \{0, 1\} \) is GESS.

1-2A. General case A: \( c>1 \)

In general, the payoff matrices for them, \( P_A \) for player A and \( P_B \) for player B, are given as in what follows. Here, the parameter, \( c \), might be more than 1. In special case 1A, \( c = 3 \) is assumed, while in special case 1B, \( c = 1/2 \) is assumed.

\[
\text{Clear[c]}
\]

\[
\text{hd} = \{(\{1/2, 1/2\}, \{0, 1\}), (\{1, 0\}, \{(1 - c) / 2, (1 - c) / 2\})\};
\]

\[
\text{hdp = Table[hd[[1, i, 1]], \{i, 1, 2\}], Table[hd[[2, i, 1]], \{i, 1, 2\}]]};
\]

\[
\text{hdp0 = hdp // MatrixForm;}
\]

\[
\text{hdpq = Table[hd[[1, i, 2]], \{i, 1, 2\}], Table[hd[[2, i, 2]], \{i, 1, 2\}]]};
\]

\[
\text{hdpq0 = hdpq // MatrixForm; (hdp0, hdpq0)}
\]

Out[71]= \[ \begin{pmatrix}
\frac{1}{2} & 0 \\
1 & \frac{1}{2}
\end{pmatrix}, \begin{pmatrix}
\frac{1}{2} & 1 \\
0 & \frac{1}{2}
\end{pmatrix} \]

If any, the mixed strategy equilibrium, \( s_A^* = \{p_1^*, 1 - p_1^*\}, 0 < p_1^* < 1 \), and \( s_B^* = \{q_1^*, 1 - q_1^*\}, 0 < q_1^* < 1 \), can be found by computing \( s_B \) which satisfies \( \partial(s_A, P_A, s_B) / \partial p_1 = 0 \) and by computing \( s_A \) which satisfies \( \partial(s_A, P_A, s_B) / \partial q_1 = 0 \), since \( 0 < p_1^* < 1 \) is the value of \( p_1 \) for the interior maximization of \((1)\) given \( s_B^* \) and \( 0 < q_1^* < 1 \) is the value of \( q_1 \) for the interior maximization of \((2)\) given \( s_A^* \). If any, such \( s_B^* \) is derived as in what follows. Thus, \( c>1 \) must be satisfied when the mixed strategy equilibrium exists.
In[72]:= \[\text{sol1} = \text{Solve}\left[ \frac{D}{D p1} (p1 - p1 - \mu hdp. (q1 - q1) < 0, q1 - q1 < 0, p1) \right] \]

Bq = \text{Simplify}[\text{sol1}] 

Out[73]= \[\{-1 + \frac{c}{c}, \frac{1}{c}\} \] 

Likewise, if any, such \( s_A^* \) is derived as in what follows. Thus, \( c > 1 \) must be satisfied when the mixed strategy equilibrium exists.

In[74]:= \[\text{sol2} = \text{Solve}\left[ \frac{D}{D q1} (q1 - q1 - \mu dq. (p1 - p1) < 0, q1 - q1 < 0, p1) \right] \]

Bp = \text{Simplify}[\text{sol2}] 

Out[75]= \[\{-1 + \frac{c}{c}, \frac{1}{c}\} \] 

Indeed, on the one hand, if \( c > 1 \), \( s_A^* = \{1 - 1/c, 1/c\} \) and \( s_B^* = \{1 - 1/c, 1/c\} \) are the pair of mixed strategy equilibrium. On the other hand, when \( c > 1 \), the pure strategy equilibria are, \{0,1\}, \{1,0\} and \{1,0\}, \{0,1\}. In this way, when \( c > 1 \), the candidate for ESS is solely the mixed strategy, \( s_A^* = \{1 - 1/c, 1/c\} \) and \( s_B^* = \{1 - 1/c, 1/c\} \). We show that this mixed strategy is GESS. In (4), the first element, \( s_A.P_A.s_A^* \) is computed as follows.

In[76]:= \[\text{check1} = \text{Simplify}\left[ (p1 - p1 - \mu hdp. (q1 - q1) < 0, q1 - q1 < 0, p1) / \text{sol1} \right] \]

Out[76]= \(-1 + \frac{c}{2c} \) 

In (4), the second element, \( s_A.P_A.s_A \) is computed as follows.

In[77]:= \[\text{check2} = \text{Simplify}\left[ (p1 - p1 - \mu hdp. (p1 - p1) < 0, p1 - p1 < 0, p1) \right] \]

Out[77]= \(\frac{1}{2} (1 - c (-1 + p1)^2) \) 

In (4), the third element, \( s_A^*.P_A.s_A^* \) is computed as follows, which is equivalent to the first element.

In[78]:= \[\text{check3} = \text{Simplify}\left[ (p1 - p1 - \mu hdp. (q1 - q1) < 0, q1 - q1 < 0, p1) / \text{sol1} / \text{sol2} \right] \]

Out[78]= \(-1 + \frac{c}{2c} \) 

In (4), the fourth element, \( s_A^*.P_A.s_A \) is computed as follows.

In[79]:= \[\text{check4} = \text{Simplify}\left[ (p1 - p1 - \mu hdp. (q1 - q1) < 0, q1 - q1 < 0, p1) / \text{sol2} \right] / q1 \rightarrow p1 \]

Out[79]= \(-\frac{1}{2} + \frac{1}{2c} + p1 \) 

The difference between the fourth element and the second element is computed as in what follows. This function is minimized at \( 0 < p1^* = 1 - 1/c < 1 \), and (4') is satisfied for \( s_A^* = \{1 - 1/c, 1/c\} \) and \( s_B^* = \{1 - 1/c, 1/c\} \).

In[80]:= \[\text{check5} = \text{Factor}[\text{Expand}[\text{check4} - \text{check2}]] \]

Out[80]= \(\frac{(1 - c + c p1)^2}{2c} \) 

Thus, the pair of \( s_A^* = \{1 - 1/c, 1/c\} \) and \( s_B^* = \{1 - 1/c, 1/c\} \) is the unique GESS.
1-2B. General case B: c<1

As shown in general case A, c>1 must be satisfied when the mixed strategy equilibrium exists. Indeed, if c<1, \(s_A^*={0, 1}\) and \(s_B^*={0, 1}\) are the unique pair of pure strategy equilibrium. In this way, when c<1, the candidate for ESS is solely the pure strategy, \(s_A^*={0, 1}\) and \(s_B^*={0, 1}\). We show that this pure strategy is GESS. In (4), the first element, \(s_A.P_A.s_A^*\) is computed as follows.

\[
\text{In[81]:= check1 = Simplify[(p1, 1-p1).hdp.(q1, 1-q1) /. {q1 -> 0}]}\\
\text{Out[81]=} \frac{1}{2} (-1 + c) (-1 + p1)
\]

In (4), the second element, \(s_A.P_A.s_A^*\) is computed as follows.

\[
\text{In[82]:= check2 = Simplify[(p1, 1-p1).hdp.(p1, 1-p1)]}\\
\text{Out[82]=} \frac{1}{2} (1 - c (-1 + p1)^2)
\]

In (4), the third element, \(s_A^*.P_A.s_A^*\) is computed as follows, which is greater than the first element except for \(p_1=0\).

\[
\text{In[83]:= check3 = Simplify[(p1, 1-p1).hdp.(q1, 1-q1) /. {p1 -> 0, q1 -> 0}]}\\
\text{Out[83]=} \frac{1 - c}{2}
\]

In (4), the fourth element, \(s_A^*.P_A.s_A^*\) is computed as follows.

\[
\text{In[84]:= check4 = Simplify[(p1, 1-p1).hdp.(q1, 1-q1) /. {p1 -> 0, q1 -> p1}]}\\
\text{Out[84]=} \frac{1}{2} (1 + c (-1 + p1) + p1)
\]

The difference between the fourth element and the first element is computed as in what follows. When c<1, this function is always positive except for \(p_1=0\), and (4') is satisfied. Thus, the pair of pure strategy, \(s_A^*={0, 1}\) and \(s_B^*={0, 1}\) is GESS.

\[
\text{In[85]:= Simplify[check4 - check2]}\\
\text{Out[85]=} \frac{1}{2} (1 + c (-1 + p1)) p1
\]
2. Sir Thomas More's Game

The author Sir Thomas More of *Utopia* [1516] is an important player in the Game theory. In the first place, in his book, he advocated the abolition of death penalty. In this argument, he proposes the payment to the prisoners who inform the crimes in prisons such as jailbreaking. This argument may be a forerunner of "prisoner's dilemma". In the second place, his remarks in the motion picture *A man for all seasons* (1966) provides a game on ESS.

This motion picture describes his opposition to the King Henry VIII’s marriage on account of illegality against law. It also describes his obedience to law even when he may face a danger if he is obedient to law. In the motion picture, respecting Sir Thomas, a young teacher, Richard Rich is seeking employment at Sir Thomas’ house. Sir Thomas does not believe in him, because Rich is a political aspirant or an opportunist. When Rich leaves Sir Thomas house after he rejects Rich’s hope, his family, including daughter's fiancee, Roper, asks Sir Thomas to arrest Rich, fearing Rich’s revenge against him. Sir Thomas denies this asking since Rich has not conducted crimes against him yet, saying "Cut a great road through the law to get after the Devil? ... And when the last law was down, and the Devil turned round on you - where would you hide, Roper, the laws all being flat?"

From Sir Thomas’ remarks we can construct a two-player game. Two players are identical. Their strategies are "Obedience (to the laws)" and "Disobedience (to the laws)".

2-1A. Special case A

Suppose that both players, A and B, obtain the same satisfaction from the legal system, 100. When one of the two players break a law, while the other remains obedient, the former player obtains 102, while the latter player obtains 98. When both players disobey the laws, the legal system breaks down, and their satisfaction is -99. The payoff matrices for them, $P_A$ for player A and $P_B$ for player B, are as in what follows.

```math
\begin{pmatrix}
100 & 98 \\
102 & -99
\end{pmatrix} = 
\begin{pmatrix}
100 & 102 \\
98 & -99
\end{pmatrix}
```

Utilizing *Mathematica* programm for computing Nash non-cooperative solutions, we know that there are three types of solutions: two pure strategy solutions and one mixed strategy solution.

```math
\begin{align*}
\text{In}[86]:=& \text{Nash}[	ext{tm}]
\end{align*}
```

As in section 1, the mixed strategy equilibrium, $s_A^* = \{197/199, 2/199\}$ and $s_B^* = \{197/199, 2/199\}$, can be found by computing $s_B$ which satisfies $\partial(s_A, P_A, s_B) / \partial p_1 = 0$ and by computing $s_A$ which satisfies $\partial(s_A, P_A, s_B) / \partial q_1 = 0$, since 197/199 is the value of $p_1$ for the interior maximization of (1) given $s_B^*$ and 2197/199 is the value of $q_1$ for the interior maximization of (2) given $s_A^*$.
In[93]:= sol1 = Solve[D[p1, 1 - p1] tmp. {q1, 1 - q1}, p1] == 0, q1][[1]];
bq = (q1, 1 - q1) /. sol1
Out[94]= \{197/199, 2/199\}

In[95]:= sol2 = Solve[D[p1, 1 - p1] tmp. {q1, 1 - q1}, q1] == 0, p1][[1]];
bp = (p1, 1 - p1) /. sol2
Out[96]= \{197/199, 2/199\}

In this specified parameter, the candidate for ESS is \(s_A^* = \{197/199, 2/199\}\) and \(s_B^* = \{197/199, 2/199\}\) from (3). We show that it is the case. Indeed it is GESS. In (4), the first element, \(s_A . P_A . s_A^*\) is computed as follows.

In[97]:= check1 = Simplify[p1, 1 - p1] tmp. {q1, 1 - q1} /. sol1
Out[97]= 19896/199

In (4), the second element, \(s_A . P_A . s_A^*\) is computed as follows.

In[98]:= check2 = Simplify[p1, 1 - p1] tmp. {p1, 1 - p1}
Out[98]= -99 + 398 p1 - 199 p1^2

In (4), the third element, \(s_A^* . P_A . s_A^*\) is computed as follows, which is equivalent to the first element.

In[99]:= check3 = Simplify[p1, 1 - p1] tmp. {q1, 1 - q1} /. sol1 /. sol2
Out[99]= 19896/199

In (4), the fourth element, \(s_A^* . P_A . s_A\) is computed as follows.

In[100]:= check4 = Simplify[p1, 1 - p1] tmp. {q1, 1 - q1} /. sol2 /. q1 -> p1
Out[100]= 19108/199 + 4 p1

The difference between the fourth element and second element is computed as follows.

In[101]:= check5 = check4 - check2
Out[101]= 38809/199 - 394 p1 + 199 p1^2

This function is minimized at \(p_1 = 197/199\). Thus, (4’) is satisfied. Thus, the pair of \(s_A^* = \{197/199, 2/199\}\) and \(s_B^* = \{197/199, 2/199\}\) is GESS.
2-1B. Special case B

In this subsection, a modification is made on the payoff matrix. When both players disobey the laws, the legal system breaks down, and their satisfaction is 99. For other cases no modifications are made. The payoff matrices for them, \( P_A \) for player A and \( P_B \) for player B, are as in what follows.

\[
\begin{bmatrix}
100 & 102 \\
98 & 99
\end{bmatrix}
\]

Utilizing Mathematica programm for computing Nash non-cooperative solutions, we know that there is only one pure strategy solution.

\[
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]

In this specified parameter, the candidate for ESS is \( s_A^\ast = \{0, 1\} \) and \( s_B^\ast = \{0, 1\} \) from (3). We show that it is the case and indeed it is GESS. In (4'), the first element, \( s_A.P_A.s_A^\ast \) is computed as follows.

\[
99 - p_1
\]
In (4'), the second element, $s_A.P_A.s_A$ is computed as follows.

```
In[112]:= check2 = Simplify[(p1, 1 - p1).tmp.(p1, 1 - p1)]
Out[112]= 99 + 2 p1 - p1^2
```

In (4), the third element, $s_A*.P_A.s_A*$ is computed as follows, which is greater than the first element except for $p_1 = 0$.

```
In[113]:= check3 = Simplify[(p1, 1 - p1).tmp.(q1, 1 - q1) /. {p1 -> 0, q1 -> 0}]
Out[113]= 99
```

In (4), the fourth element, $s_A*.P_A.s_A$ is computed as follows, which is greater than the second element except for $p_1 = 0$.

```
In[114]:= check4 = Simplify[(p1, 1 - p1).tmp.(q1, 1 - q1) /. {p1 -> 0, q1 -> p1}]
Out[114]= 3 (33 + p1)
```

From the above examination, it was shown that the right hand side is clearly greater than the left hand side except for $p_1 = 0$. This conclusion is confirmed by the direct computation. Thus, the pair of $s_A*={0,1}$ and $s_B*={0,1}$ was shown to be the unique GESS.

2-2A. General case A: $a<c$

In this subsection, we consider the general case for the Sir Thomas More game. The payoff matrices is given by the following one where all the parameters are positive. Special case A is specified by $f=100$, $a=2$, $b=2$, and $c=199$, while in special case B is specified by $f=100$, $a=2$, $b=2$, and $c=1$. The general case A is the one in which $a<c$. When this assumption is satisfied, the mixed strategy solution is the unique ESS.

```
In[115]:= Clear[c]

In[116]:= tm = {{{f, f}, {f - a, f + b}}, {{f + b, f - a}, {f - c, f - c}}};
tmp = {Table[tm[[1, i, 1]], {i, 1, 2}], Table[tm[[2, i, 1]], {i, 1, 2}]};
tmp0 = tmp // MatrixForm;
tmq = {Table[tm[[1, i, 2]], {i, 1, 2}], Table[tm[[2, i, 2]], {i, 1, 2}]};
tmq0 = tmq // MatrixForm;
{tmp0, tmq0}
Out[121]=
{\{
\{f, -a + f\}, \{f + b, -c + f\}\}, \{f, b + f\}}
If any, the mixed strategy equilibrium, \( s_A^* = \{p_1^*, 1-p_1^*\}, 0<p_1^*<1 \) and \( s_B^* = \{q_1^*, 1-q_1^*\}, 0<q_1^*<1 \), can be found by computing \( s_B \) which satisfies \( \partial(s_A, P_A, s_B) / \partial p_1 = 0 \) and by computing \( s_A \) which satisfies \( \partial(s_A, P_A, s_B) / \partial q_1 = 0 \), since \( 0<p_1^*<1 \) is the value of \( p_1 \) for the *interior* maximization of (1) given \( s_B^* \) and \( 0<q_1^*<1 \) is the value of \( q_1 \) for the *interior* maximization of (2) given \( s_A^* \). If any, such \( s_B^* \) is derived as in what follows. Thus, \( a<c \) must be satisfied when the mixed strategy equilibrium exists. Indeed, if \( a<c \) is assumed then \( 0<p_1^*=(a-c)/(a-b-c)<1 \) holds, and \( s_A^* = \{p_1^*, 1-p_1^*\}, 0<p_1^*<1 \) and \( s_B^* = \{q_1^*, 1-q_1^*\}, 0<q_1^*<1 \) are the pair of mixed strategy equilibrium.

\[
\text{In}[122] := \text{soll} = \text{Solve}[[p_1, 1-p_1].\text{tmp.}\{q_1, 1-q_1\}, p_1] = 0, q_1] [[1]]; \\
\text{bq} = (q_1, 1-q_1) / \text{. soll}
\]

\[
\text{In}[123] := \text{sol2} = \text{Solve}[[p_1, 1-p_1].\text{tmp.}\{q_1, 1-q_1\}, q_1] = 0, p_1] [[1]]; \\
\text{bp} = (p_1, 1-p_1) / \text{. sol2}
\]

Indeed, on the one hand, if \( a<c \) is assumed then \( 0<p_1^*=(a-c)/(a-b-c)<1 \) holds, and \( s_A^* = \{p_1^*, 1-p_1^*\}, 0<p_1^*<1 \) and \( s_B^* = \{p_1^*, 1-p_1^*\}, 0<p_1^*<1 \) are the pair of mixed strategy equilibrium. On the other hand, when \( a<c \), the pure strategy equilibria are \{0,1\} and \{1,0\}. In this way, when \( a<c \), the candidate for ESS is solely the mixed strategy, \( s_A^* \) and \( s_B^* \), above defined. We show that this mixed strategy is ESS, and indeed GEES. In (4'), the first element, \( s_A.P_A.s_A^* \) is computed as follows.

\[
\text{In}[126] := \text{check1} = \text{Simplify}[[p_1, 1-p_1].\text{tmp.}\{q_1, 1-q_1\} / \text{. soll}]
\]

\[
\text{Out}[126] = \frac{-(b+c) f + a (b + f)}{a - b - c}
\]

In (4'), the second element, \( s_A.P_A.s_A^* \) is computed as follows.

\[
\text{In}[127] := \text{check2} = \text{Simplify}[[p_1, 1-p_1].\text{tmp.}\{p_1, 1-p_1\}]
\]

\[
\text{Out}[127] = f - c (-1 + p_1)^2 + (a - b) (-1 + p_1) p_1
\]

In (4'), the third element, \( s_A^*.P_A.s_A^* \) is computed as follows, which is equivalent to the first element.

\[
\text{In}[128] := \text{check3} = \text{Simplify}[[p_1, 1-p_1].\text{tmp.}\{q_1, 1-q_1\} / \text{. soll} / \text{. sol2}]
\]

\[
\text{Out}[128] = \frac{-(b+c) f + a (b + f)}{a - b - c}
\]

In (4'), the fourth element, \( s_A^*.P_A.s_A^* \) is computed as follows.
The difference between the fourth element and the second element is computed as in what follows. This function is minimized at \( 0 < p_1 = \frac{(a-c)}{(a-b-c)} < 1 \), and (4') is satisfied for \( s_A^* = \{ p_1^*, 1-p_1^* \} \) and \( s_B^* = \{ p_1^*, 1-p_1^* \} \).

Thus, the mixed strategy, \( s_A^* = \{ p_1^*, 1-p_1^* \} \) and \( s_B^* = \{ p_1^*, 1-p_1^* \} \) for \( 0 < p_1 = \frac{(a-c)}{(a-b-c)} < 1 \), is the unique GESS.

2-2B. General case A: \( c < a \)

If \( a > c \), there exist no mixed strategy equilibria. Only \{Disobedience\_A, Disobedience\_B\} (or, \{0,1\},\{0,1\}) is the pure strategy equilibrium. We show that \( s_A^* = \{0, 1\} \) and \( s_B^* = \{0, 1\} \) are the pair of ESS strategy. Indeed, it is GESS. In (4'), the first element, \( s_A^\cdot P_A\cdot s_A^* \) is computed as follows.

In (4'), the second element, \( s_A^\cdot P_A\cdot s_A^* \) is computed as follows.

In (4'), the third element, \( s_A^\cdot P_A\cdot s_A^* \) is computed as follows, which is greater than the first element except for \( p_1 = 0 \).

In (4'), the fourth element, \( s_A^\cdot P_A\cdot s_A^* \) is computed as follows.
\textbf{In[135] :=}
\begin{verbatim}
check4 = Simplify[(p1, 1 - p1).tmp.(q1, 1 - q1) /. {p1 -> 0, q1 -> 1}]
\end{verbatim}
\textbf{Out[135] =}
\begin{displaymath}
f + c (-1 + p1) + b p1
\end{displaymath}

The difference between the fourth element and the second element is computed as in what follows, which is positive except for \(p_1 = 0\).

\textbf{In[136] :=}
\begin{verbatim}
Factor[Simplify[check4 - check2]]
\end{verbatim}
\textbf{Out[136] =}
\begin{displaymath}
-p1 (-a + c + a p1 - b p1 - c p1)
\end{displaymath}

In this way, when \(a > c\), \(s_A^* = \{0, 1\}\) and \(s_B^* = \{0, 1\}\) are the pair of GESS strategy.

When \(c\) is small compared with when \(a\), both players tend to select "Disobedience" strategy. As \(c\) rises, there is high possibility of the selection of "Obedience" strategy in mixed strategy, although "Obedience" strategy is not selected as the pure strategy.

\section*{3. Confess-Hide Game}

In the motion picture, \textit{Gentleman's Agreement} (1947), racial discrimination against Jewish people, is described. A journalist of a "progressive" magazine investigates the antisemitism in New York City, by checking the people's reaction from his remark that he is a Jewish, although he is not. In his research, he meets secretary, named Elaine Wales. Believing that he is a Jewish, she confesses that she is too a Jewish and her previous name was Estelle Wilovsky, sounding Jewish. After being rejected many times in her effort to get a job, she sends two same CVs with different names, one with Elaine Wales and the other with Estelle Wilovsky to the present "progressive" magazine. While the magazine rejected the CV with the name Estelle Wilovsky, she was employed through the interview on the CV with the name Elaine Wales.

From this motion picture, we can construct a game, named "Confess-Hide Game" played by Jewish people. Suppose that the two Jewish players aim at getting jobs. Their strategies are the same: "Retain (Jewish name)" and "Change (Jewish name)", or we may use "Confess" and "Hide".

\subsection*{3-1A. Special case A}

Suppose that when player A and player B select the strategy, "Hide", their satisfaction from job opportunity are the same 100. When one of the two players selects the strategy "Confess", while the other selects the strategy "Hide", the former player has lower job opportunity and thus lower satisfaction, 50, and the latter player has higher job opportunity and thus higher satisfaction, 150. When both players select the strategy, "Confess", they have the same employment opportunity and their satisfaction rise to 160, due to the improved self-esteem through "coming out". It is assumed that the job needs some skill to perform. Then, the payoff matrices for them, \(P_A\) for player A and \(P_B\) for player B, are as in what follows.

\textbf{In[137] :=}
\begin{verbatim}
c = -60;
\end{verbatim}
In[138]:= jc = {{(100, 100), (150, 50), (100 - c, 100 - c)};
   jcp = Table[jc[[1, i, 1]], {i, 1, 2}], Table[jc[[2, i, 1]], {i, 1, 2}}];
   jcp0 = jcp // MatrixForm;
   jcq = Table[jc[[1, i, 2]], {i, 1, 2]], Table[jc[[2, i, 2]], {i, 1, 2}}];
   jcq0 = jcq // MatrixForm;
   {jcp0, jcq0}

Out[143]=

Utilizing Mathematica program for computing Nash non-cooperative solutions, we know that there are three types of solutions: two pure strategy solutions and one mixed strategy solution.

In[144]:= Nash[jc]

Out[144]=

In this section all of the three Nash solutions are the candidates for the ESS. We begin with the examination of the mixed strategy solution.

**mixed strategy**

As in section 1, the mixed strategy equilibrium, \( s_A^* = \{1/6, 5/6\} \) and \( s_B^* = \{1/6, 5/6\} \), can be found by computing \( s_B \) which satisfies \( \partial(s_A . P_A . s_B) / \partial p_1 = 0 \) and by computing \( s_A \) which satisfies \( \partial(s_A . P_A . s_B) / \partial q_1 = 0 \), since 1/6 is the value of \( p_1 \) for the interior maximization of (1) given \( s_B^* \) and 5/6 is the value of \( q_1 \) for the interior maximization of (2) given \( s_A^* \).

In[145]:=
   sol1 = Solve[D[[p1, 1 - p1] . jcp . (q1, 1 - q1), p1] = 0, q1][[1]];  
   bq = (q1, 1 - q1) /. sol1

Out[146]=

In[147]:=
   sol2 = Solve[D[[p1, 1 - p1] . jcq . (q1, 1 - q1), q1] = 0, p1][[1]];  
   bp = (p1, 1 - p1) /. sol2

Out[148]=

In this specified parameter, one of the candidates for ESS is \( s_A^* = \{1/6, 5/6\} \) and \( s_B^* = \{1/6, 5/6\} \) from (3). We show that it is not the case. In (4), the first element, \( s_A . P_A . s_A^* \) is computed as follows.

In[149]:=
   check1 = Simplify[(p1, 1 - p1) . jcp . (q1, 1 - q1) /. sol1]

Out[149]=

\[
\frac{425}{3}
\]
In (4), the second element, $s_A.P_A.s_A$ is computed as follows.

```
In[150]:= 
check2 = Simplify[(p1, 1 - p1).jcp.{p1, 1 - p1}]
```

```
Out[150]=
20 (8 - 6 p1 + 3 p1²)
```

In (4), the third element, $s_A^* P_A s_A^*$ is computed as follows, which is equivalent to the first element.

```
In[151]:= 
check3 = Simplify[(p1, 1 - p1).jcp.{q1, 1 - q1} /. sol1 /. sol2]
```

```
Out[151]=
425/3
```

In (4), the fourth element, $s_A^* P_A s_A$ is computed as follows.

```
In[152]:= 
check4 = Simplify[(p1, 1 - p1).jcp.{q1, 1 - q1} /. sol2 /. q1 -> p1]
```

```
Out[152]=
475/3 - 100 p1
```

The difference between the fourth element and second element is computed as follows.

```
In[153]:= 
check5 = check4 - check2
```

```
Out[153]=
475/3 - 100 p1 - 20 (8 - 6 p1 + 3 p1²)
```

This function is maximized, not minimized, at $p_1=1/6$. Thus, (4) is not satisfied.

```
In[154]:= 
Plot[check5, {p1, 0, 1}];
```

```
In[155]:= 
check5 /. sol2
```

```
Out[155]=
0
```

Thus, the pair of $s_A^*={1/6,5/6}$ and $s_B^*={1/6,5/6}$ is not ESS.
The pure strategy equilibrium, \( s_A^* = \{0, 1\} \) and \( s_B^* = \{0, 1\} \), is also a candidate for ESS. We show that it is the case. It is LESS. In (4), the first element, \( s_A^* \cdot P_A \cdot s_A^* \) is computed as follows.

\[
\text{check1} = \text{Simplify}\{(p_1, 1-p_1).jcp.(q_1, 1-q_1) / \{q_1 \to 0\}\}
\]

\[
\text{Out[156]} = -10 (-16 + p_1)
\]

In (4), the second element, \( s_A^* \cdot P_A \cdot s_A^* \) is computed as follows.

\[
\text{check2} = \text{Simplify}\{(p_1, 1-p_1).jcp.(p_1, 1-p_1)\}
\]

\[
\text{Out[157]} = 20 (8 - 6 p_1 + 3 p_1^2)
\]

In (4), the third element, \( s_A^* \cdot P_A \cdot s_A^* \) is computed as follows, which is greater than the first element except for \( s_A^* = \{0, 1\} \).

\[
\text{check3} = \text{Simplify}\{(p_1, 1-p_1).jcp.(q_1, 1-q_1) / \{p_1 \to 0, q_1 \to 0\}\}
\]

\[
\text{Out[158]} = 160
\]

In (4), the fourth element, \( s_A^* \cdot P_A \cdot s_A^* \) is computed as follows.

\[
\text{check4} = \text{Simplify}\{(p_1, 1-p_1).jcp.(q_1, 1-q_1) / \{p_1 \to 0, q_1 \to p_1\}\}
\]

\[
\text{Out[159]} = 160 - 110 p_1
\]

The difference between the fourth element and second element is computed as follows.

\[
\text{Simplify[check4 - check2]}
\]

\[
\text{Out[160]} = 10 (1 - 6 p_1) p_1
\]

This difference is not non-negative. However, the difference between the right hand side and the left hand side is computed as in what follows

\[
\text{Simplify[e * check4 + (1-e) * check3 - e * check2 - (1-e) * check1]}
\]

\[
\text{Out[161]} = 10 p_1 (1 - 6 e p_1)
\]

For small \( e > 0 \), this function is positive except for \( s_A^* = \{0, 1\} \). Thus, the pure strategy equilibrium, \( s_A^* = \{0, 1\} \) and \( s_B^* = \{0, 1\} \), is LESS. Note that, for large \( e > 0 \) (4) is not satisfied. In other words, if large portion of population become mutants, then \( s_A^* = \{0, 1\} \) and \( s_B^* = \{0, 1\} \) might be unstable, moving to a different ESS, if any.
pure strategy solution 2: Hide

The pure strategy equilibrium, $s_A^* = \{1,0\}$ and $s_B^* = \{1,0\}$, is also a candidate for ESS. We show that it is the case. In (4), the first element, $s_A^*.P_A.s_A^*$ is computed as follows.

```
In[162]:= check1 = Simplify[(p1, 1 - p1).jcp.{q1, 1 - q1} /. {q1 -> 1})
Out[162]= 50 (1 + p1)
```

In (4), the second element, $s_A^*.P_A.s_A$ is computed as follows.

```
In[163]:= check2 = Simplify[(p1, 1 - p1).jcp.{p1, 1 - p1}]
Out[163]= 20 (8 - 6 p1 + 3 p1^2)
```

In (4), the third element, $s_A^*.P_A.s_A^*$ is computed as follows, which is greater than the first element except for $p_1 = 1$.

```
In[164]:= check3 = Simplify[(p1, 1 - p1).jcp.{q1, 1 - q1} /. {p1 -> 1, q1 -> 1}]
Out[164]= 100
```

In (4), the fourth element, $s_A^*.P_A.s_A$ is computed as follows.

```
In[165]:= check4 = Simplify[(p1, 1 - p1).jcp.{q1, 1 - q1} /. {p1 -> 1, q1 -> p1}]
Out[165]= -50 (-3 + p1)
```

The difference between the fourth element and second element is computed as follows.

```
In[166]:= Simplify[check4 - check2]
Out[166]= -10 (1 - 7 p1 + 6 p1^2)
```

This difference is not non-negative. However, the difference between the right hand side and the left hand side is computed as in what follows.

```
In[167]:= Simplify[e*check4 + (1 - e)*check3 - e*check2 - (1 - e)*check1]
Out[167]= -10 (5 + 6 e (-1 + p1)) (-1 + p1)
```

For small $\epsilon > 0$, this function is positive except for $s_A^* = \{1,0\}$. Thus, the pure strategy equilibrium, $s_A^* = \{1,0\}$ and $s_B^* = \{1,0\}$, is LESS. Note that, for large $\epsilon > 0$ (4) is not satisfied. In other words, if large portion of population become mutants, then $s_A^* = \{1,0\}$ and $s_B^* = \{1,0\}$ might be unstable, moving to a different ESS, $s_A^* = \{0,1\}$ and $s_B^* = \{0,1\}$. 
3-1B: Special case B

Modification is made on the payoff when both players select the strategy "Confess". They have the same employment opportunity, however, and their satisfaction falls to 60, due to the escape of employment opportunity to others than these two players. It is assumed that the job is highly substitutable. Then, the payoff matrices for them, $P_A$ for player A and $P_B$ for player B, are as in what follows.

```mathematica
In[168]:= c = 40;
In[169]:= jc = {{(100, 100), (150, 50)}, {(50, 150), (100 - c, 100 - c)}};
   jcp = {Table[jc[[1, 1]], {i, 1, 2}], Table[jc[[2, 1]], {i, 1, 2]}];
   jcp0 = jcp // MatrixForm;
   jcq = {Table[jc[[1, 2]], {i, 1, 2}], Table[jc[[2, 2]], {i, 1, 2]]};
   jcq0 = jcq // MatrixForm;
{jc0, jcq0}
Out[174]= 
\{\begin{pmatrix} 100 & 150 \\ 50 & 60 \end{pmatrix}, \begin{pmatrix} 100 & 50 \\ 150 & 60 \end{pmatrix}\}
```

Utilizing Mathematica programm for computing Nash non-cooperative solutions, we know that there is unique pure strategy solution. It is shown that it is GESS.

```mathematica
In[175]:= Nash[jc]
Out[175]= {{\{1, 0\}, \{1, 0\}}}
```

In (4'), the first element, $s_A \cdot P_A \cdot s_A^*$ is computed as follows.

```mathematica
In[176]:= check1 = Simplify[{p1, 1 - p1}.jcp.{q1, 1 - q1} /. {q1 -> 1}]
Out[176]= 50 (1 + p1)
```

In (4'), the second element, $s_A \cdot P_A \cdot s_A$ is computed as follows.

```mathematica
In[177]:= check2 = Simplify[{p1, 1 - p1}.jcp.{p1, 1 - p1}]
Out[177]= 60 + 80 p1 - 40 p1^2
```

In (4'), the third element, $s_A^* \cdot P_A \cdot s_A^*$ is computed as follows, which is greater than the first element except for $p_1=1$.

```mathematica
In[178]:= check3 = Simplify[{p1, 1 - p1}.jcp.{q1, 1 - q1} /. {p1 -> 1, q1 -> 1}]
Out[178]= 100
```

In (4'), the fourth element, $s_A^* \cdot P_A \cdot s_A$ is computed as follows.
In[179]:= check4 = Simplify[{p1, 1 - p1}.jcP.{q1, 1 - q1} /. {p1 -> 1, q1 -> p1}]
Out[179]= -50 (-3 + p1)

The difference between the fourth element and the second is computed as follows. This function is minimized at $p_1 = 1$, and (4') is satisfied.

In[180]:= check5 = Simplify[check4 - check2]
Out[180]= 10 (9 - 13 p1 + 4 p1^2)

In[181]:= Plot[check5, {p1, 0, 1};

Thus, the pure strategy equilibrium, $x_A^* = \{1,0\}$ and $x_B^* = \{1,0\}$, is GESS.

3-2A: General case A: $a + c < 0$

In this subsection, we consider the general case for the Confess-Hide game. The payoff matrices is given by the following one where all the parameters are positive. Special case A is specified by $f = 100$, $a = 2$, $b = 2$, and $c = 199$, while in special case B is specified by $f = 100$, $a = 50$, $b = 50$, and $c = -60$. The general case A is the one in which $a + c < 0$. When this assumption is satisfied, there exists a mixed strategy solution. We show that it is not ESS.

In[182]:= Clear[c]

In[183]:= jc = {{(f, f)}, {(f + a, f - b)}, {{f - b, f + a}, {f - c, f - c}}};
jcP = {Table[jc[[1, i]], {i, 1, 2}], Table[jc[[2, i]], {i, 1, 2}]};
jcQ = {Table[jc[[1, i]], {i, 1, 2]], Table[jc[[2, i]], {i, 1, 2}]};
jcQ0 = jcQ // MatrixForm;
jcP0 = jcP // MatrixForm;

Out[188]=
\[
\begin{pmatrix}
  f \\
  -b + f
\end{pmatrix}
\begin{pmatrix}
a + f \\
-c + f
\end{pmatrix}
\]

If any, the mixed strategy equilibrium, $x_A^* = \{p_1^*, 1 - p_1^*\}$, $0 < p_1^* < 1$, and $x_B^* = \{q_1^*, 1 - q_1^*\}$, $0 < q_1^* < 1$, can be found by computing $x_B$ which satisfies $\partial (s_A.P_A.x_B) / \partial p_1 = 0$ and by computing $s_A$ which satisfies $\partial (s_A.P_A.x_B) / \partial q_1 = 0$, since
0<p_1<1 is the value of $p_1$ for the interior maximization of (1) given $s_B^*$ and 0<q_1<1 is the value of $q_1$ for the interior maximization of (2) given $s_A^*$. If any, such $s_B^*$ is derived as in what follows. Thus, a+c<0 must be satisfied when the mixed strategy equilibrium exists. Indeed, if a+c<0 is assumed then 0<p_1=(a+c)/(a-b+c)<1 holds, and $s_A^* = \{p_1^*, 1-p_1^*\}, 0<p_1^*<1, and s_B^* = \{q_1^*, 1-q_1^*\}, 0<q_1^*<1 are the pair of mixed strategy equilibrium.

We show that this mixed strategy is not ESS. In (4), the first element, $s_A^*P_A.s_A^*$ is computed as follows.

In[193]:=
check1 = Simplify[(p_1, 1-p_1).jcp.(q_1, 1-q_1) /. sol1]

Out[193]=
\[
\frac{-b+c}{a-b+c} f + a \frac{-b+f}{a-b+c} \]

In (4), the second element, $s_A^*P_A.s_A^*$, is computed as follows.

In[194]:=
check2 = Simplify[(p_1, 1-p_1).jcp.(p_1, 1-p_1)]

Out[194]=
f - c (1+pl)^2 - (a-b) (-1+pl) pl

In (4), the third element, $s_A^*P_A.s_A^*$, is computed as follows, which is equivalent to the first element.

In[195]:=
check3 = Simplify[(p_1, 1-p_1).jcp.(q_1, 1-q_1) /. sol1 /. sol2]

Out[195]=
\[
\frac{-b+c}{a-b+c} f + a \frac{-b+f}{a-b+c} \]

In (4'), the fourth element, $s_A^*P_A.s_A^*$, is computed as follows.

In[196]:=
check4 = Simplify[(p_1, 1-p_1).jcp.(q_1, 1-q_1) /. sol2] /. q_1 \rightarrow pl

Out[196]=
\[
\frac{cf-b (f+c (-1+pl)) - a^2 (-1+pl) + b^2 pl + a (c+f-cpl)}{a-b+c} \]
The difference between the fourth element and the second element is computed as in what follows. This function is maximized, \textit{not minimized}, at \(0<p_1^*(a+c)/(a-b+c)<1\). The maximum is 0. Thus, the pair of mixed strategies is not ESS.

\begin{verbatim}
In[197]:= check5 = Factor[Simplify[check4 - check2]]
Out[197]= 
\frac{(-a-c+a p_1-b p_1+c p_1)^2}{a-b+c}

In[198]:= Simplify[check5 /. sol2]
Out[198]= 0
\end{verbatim}

\textbf{pure strategy 1: Confess}

When \(a+c<0\), the pure strategy equilibrium, \(\{0,1\},\{0,1\}\), is LESS. In (4), the first element, \(s_A.P_A.s_A^*\) is computed as follows.

\begin{verbatim}
In[199]:= check1 = Simplify[{p1, 1 - p1}.jcp.{q1, 1 - q1} /. {q1 -> 0}]
Out[199]= f + c (-1 + p1) + a p1
\end{verbatim}

In (4), the second element, \(s_A.P_A.s_A\), is computed as follows.

\begin{verbatim}
In[200]:= check2 = Simplify[{p1, 1 - p1}.jcp.{p1, 1 - p1}]
Out[200]= f - c (-1 + p1)^2 - (a - b) (-1 + p1) p1
\end{verbatim}

In (4), the third element, \(s_A^*.P_A.s_A^*\), is computed as follows, which is greater than the first except \(p_1=0\).

\begin{verbatim}
In[201]:= check3 = Simplify[{p1, 1 - p1}.jcp.{q1, 1 - q1} /. {p1 -> 0} /. {q1 -> 0}]
Out[201]= -c + f
\end{verbatim}

In (4), the fourth element, \(s_A^*.P_A.s_A\), is computed as follows.

\begin{verbatim}
In[202]:= check4 = Simplify[{p1, 1 - p1}.jcp.{q1, 1 - q1} /. {p1 -> 0}] /. q1 -> p1
Out[202]= f + c (-1 + p1) - b p1
\end{verbatim}

The difference between the fourth element and the second element is computed as in what follows. This function is not always nonnegative.
The difference between the right hand side and the left hand side is computed as follows.

In[204]:= 
Simplify[e*check4 + (1 - e)*check3 - e*check2 - (1 - e)*check1]

Out[204]= 
p1 (-a - c + a p1 - b p1 + c p1)

When \(e>0\) is small, this function is positive except for \(p_1=0\). Thus, the pure strategy equilibrium, \({\{0,1\},\{0,1\}}\), is LESS.

**pure strategy 2: Hide**

When \(a+c<0\), the pure strategy equilibrium, \({\{1,0\},\{1,0\}}\), is LESS. In (4), the first element, \(s_A \cdot P_A \cdot s_A^*\), is computed as follows.

In[205]:= 
check1 = Simplify[{p1, 1 - p1}].jcp.{q1, 1 - q1} /. {q1 \[Rule] 1}]

Out[205]= 
f + b (-1 + p1)

In (4), the second element, \(s_A \cdot P_A \cdot s_A\), is computed as follows.

In[206]:= 
check2 = Simplify[{p1, 1 - p1}].jcp.{p1, 1 - p1}]

Out[206]= 
f - c (-1 + p1)^2 - (a - b) (-1 + p1) p1

In (4), the third element, \(s_A^* \cdot P_A \cdot s_A^*\), is computed as follows, which is greater than the first except \(p_1=1\).

In[207]:= 
check3 = Simplify[{p1, 1 - p1}].jcp.{q1, 1 - q1} /. {p1 \[Rule] 1} /. {q1 \[Rule] 1}]

Out[207]= 
f

In (4), the fourth element, \(s_A^* \cdot P_A \cdot s_A\), is computed as follows.

In[208]:= 
check4 = Simplify[{p1, 1 - p1}].jcp.{q1, 1 - q1} /. {p1 \[Rule] 1}] /. q1 \[Rule] p1

Out[208]= 
a + f - a p1

The difference between the fourth element and the second element is computed as in what follows. This function is not always nonnegative.
In[209]:= check5 = Factor[Simplify[check4 - check2]]
Out[209]= (-1 + p1) (-a - c + a p1 - b p1 + c p1)

The difference between the right hand side and the left hand side is computed as follows.

In[210]:= Simplify[e * check4 + (1 - e) * check3 - e * check2 - (1 - e) * check1]
Out[210]= -(b (1 + e (-1 + p1)) - (a + c) e (-1 + p1)) (-1 + p1)

When $\epsilon>0$ is small, this function is positive except for $p_1=1$. Thus, the pure strategy equilibrium, $\{\{1,0\},\{1,0\}\}$, is LESS.

3-2B: General case A: $a+c>0$

In this subsection, we consider the general case for the Confess-Hide game when $a+c>0$. When this assumption is satisfied, there exists the unique pure strategy solution, $\{\{1,0\},\{1,0\}\}$. We show that it is GESS.

In[211]:= jcq0 = jcq // MatrixForm;
    {jcpe, jcq0}
Out[212]= 
    \[
    \begin{pmatrix}
    f & a + f \\
    -b + f & -c + f
    \end{pmatrix},
    \begin{pmatrix}
    f & -b + f \\
    a + f & -c + f
    \end{pmatrix}
    \]

In (4'), the first element, $s_A . P_A . s_A^*$, is computed as follows.

In[213]:= check1 = Simplify[(p1, 1 - p1) . jcpe . {q1, 1 - q1} /. {p1 -> 1}]
Out[213]= f + b (-1 + p1)

In (4'), the second element, $s_A . P_A . s_A$, is computed as follows.

In[214]:= check2 = Simplify[(p1, 1 - p1) . jcpe . {p1, 1 - p1}]
Out[214]= f - c (-1 + p1)^2 - (a - b) (-1 + p1) p1

In (4'), the third element, $s_A^* . P_A . s_A^*$, is computed as follows, which is greater than the first except for $p_1=1$.

In[215]:= check3 = Simplify[(p1, 1 - p1) . jcpe . {q1, 1 - q1} /. {p1 -> 1, q1 -> 1}]
Out[215]= f

In (4), the fourth element, $s_A^* . P_A . s_A$, is computed as follows.
In[216]:=
check4 = Simplify[{p1, 1 - p1}.jcp.{ql, 1 - ql} /. {p1 \[Rule] 1, ql \[Rule] p1}]

Out[216]=
a + f - a p1

The difference between the fourth element and the second element is computed as in what follows. It is positive except for $p_1=1$ when $a+c>0$.

In[217]:=
Factor[Simplify[check4 - check2]]

Out[217]=
(-1 + p1) (-a - c + a p1 - b p1 + c p1)

Thus, the unique pure strategy solution, $\{1,0\}, \{1,0\}$ is GESS.

4. Conclusions

The aim of this paper is to examine how the dichotomy of ESS between global stability and local stability enriches the game theory, and examine how ESS is robust to the invasion of mutants.

In section 1 of this paper, Hawk-Dove game in Osborne and Rubinstein [1994] was examined from the viewpoint of global and local stabilities. It was shown that the mixed strategy can be GESS, while when only the pure strategy is equilibrium, it is also GESS. In section 2, the Sir Thomas More's game was constructed from the motion picture, *A man for all seasons* (1966) and examined from the viewpoint of global and local stabilities. The conclusion is the same as in Hawk-Dove game: i.e. the mixed strategy can be GESS, while when only the pure strategy is equilibrium, it is also GESS. In section 3, the Confession-Hide game was constructed from the motion picture, *Gentleman's Agreement* (1947) and examined from the viewpoint of global and local stabilities. The conclusion is different from the above two games: i.e. the mixed strategy cannot be ESS. When the mixed strategy was a solution to the game, two other pure strategy solutions are both LSS, while when only the pure strategy is equilibrium, it is GESS.

References


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