Characterizations of $\mathcal{M}$-harmonic α-Bloch and BMO functions on the unit ball of $\mathbb{C}^n$

Kentaro Hirata

Department of Mathematics, Shimane University,
Matsue 690-8504, Japan
e-mail: hirata@math.shimane-u.ac.jp

Dedicated to the Memory of Professor Matts Essén

Abstract

We shall give characterizations of $\mathcal{M}$-harmonic α-Bloch functions in terms of the spherical integral and the mean oscillation with respect to the invariant measure involving a certain weight. We also give characterizations of $\mathcal{M}$-harmonic BMO functions.

Keywords: $\mathcal{M}$-harmonic function, α-Bloch space, bounded mean oscillation
Mathematics Subject Classifications (2000): 32A18, 32A37, 31B05

1 Introduction

The characterizations of several spaces of holomorphic functions on the unit ball B of $\mathbb{C}^n$ were given by many authors. Choa and Choe [1] and Jevtić [6, 7] gave characterizations of BMOA in terms of Carleson measures. In [14], Stoll characterized the p-th Hardy space by

$$\int_B (1-|z|^2)^n |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 d\lambda(z) < \infty, \tag{1.1}$$

where $\widetilde{\nabla}$ and λ are the invariant gradient and invariant measure on B respectively. Ouyang, Yang and Zhao [11] and Nowak [10] also characterized the Bergman space and the Bloch space by several finite integrals similar with (1.1) involving the Green function. The hyperbolic Hardy space was characterized by Kwon [9].

The characterization of the $\mathcal{M}$-harmonic α-Bloch space in this paper was motivated by the following result due to Pavlović [12]: Let $0 < p < \infty$ and $\alpha > 0$. An $\mathcal{M}$-harmonic function f on B satisfies the condition

$$\left(\int_S |\widetilde{\nabla} f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} = O((1-r^2)^{-\alpha}) \quad \text{as } r \to 1, \tag{1.2}$$

if and only if

$$\left(\int_S |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p} = O((1-r^2)^{-\alpha}) \quad \text{as } r \to 1. \tag{1.3}$$

Since the α-Bloch space consists of functions with the strong property rather than (1.2), it does not admit to characterize this space by (1.3). However, we shall give a characterization of this space by using the p-th spherical integral of compositions with Möbius transformations. We shall also characterize the $\mathcal{M}$-harmonic BMO space by using the p-th spherical integral of compositions with Möbius transformations and by the BMO property with respect to "the invariant measure" on B. As corollaries we shall obtain analogous characterizations with (1.1) for the little α-Bloch space and the BMO space.

To state our results we prepare notations. We denote by B the unit ball of $\mathbb{C}^n$ and by S its boundary. The normalized Lebesgue measure on B and the normalized surface measure on S are denoted by ν and σ respectively. Let $d\lambda(z) = (1-|z|^2)^{n+1}d\nu(z)$. Note that this measure is invariant under the group $Aut(B)$ of holomorphic automorphisms of B. For $a \in B$, let φ_a denote the Möbius transformation of B, i.e.,

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2}(z - P_a z)}{1 - \langle z, a\rangle},$$

where $P_a z = \frac{\langle z,a\rangle}{|a|^2}a$ if $a \neq 0$, and $P_0 z = 0$, which satisfies $\varphi_a(0) = a$ and $\varphi_a^{-1} = \varphi_a$. We write $E(a,r) = \{z \in B : |\varphi_a(z)| < r\}$.

The Laplace-Beltrami operator $\widetilde{\Delta}$ on B associated with the Bergman metric is defined by

$$\widetilde{\Delta} = \frac{4}{n+1}(1-|z|^2)\sum_{i,j=1}^{n}(\delta_{i,j} - \overline{z}_i z_j)\frac{\partial^2}{\partial z_j \partial \overline{z}_i},$$

which satisfies that $\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$ for $f \in C^2(B)$ and $\psi \in Aut(B)$. A real valued C^2 function f on B satisfying $\widetilde{\Delta} f = 0$ is said to be $\mathcal{M}$-harmonic on B. Let $\widetilde{\nabla}$ be the invariant gradient on B. Then for a real valued C^1 function f on B, we have

$$|\widetilde{\nabla} f(z)|^2 = \frac{4}{n+1}(1-|z|^2)\sum_{i,j=1}^{n}(\delta_{i,j} - \overline{z}_i z_j)\frac{\partial f}{\partial \overline{z}_i}(z)\frac{\partial f}{\partial z_j}(z),$$

and $|\widetilde{\nabla}(f \circ \psi)| = |(\widetilde{\nabla} f) \circ \psi|$ for $\psi \in Aut(B)$.

Let $\alpha \in \mathbb{R}$. The $\mathcal{M}$-harmonic α-Bloch space, written $\mathcal{HB}_\alpha$, is defined as the class of all $\mathcal{M}$-harmonic functions f on B that satisfy

$$\|f\|_{\mathcal{HB}_\alpha} = \sup_{z \in B}(1-|z|^2)^\alpha |\widetilde{\nabla} f(z)| < \infty.$$

The $\mathcal{M}$-harmonic α-Bloch spaces are characterized as follows.

Theorem 1. *The following statements hold.*

(i) *If $\alpha < -1$, then $\mathcal{HB}_\alpha$ consists only of constant functions.*

(ii) *Let $1 \le p < \infty$ and set*

$$\rho_{\alpha,p}(a,r) = \begin{cases} (1-|a|^2)^\alpha & \text{if } -n < \alpha p < 0, \\ (1-|a|^2)^\alpha (1-r)^{-\alpha - n/p} & \text{if } \alpha p < -n, \\ (1-|a|^2)^\alpha \left(\log \dfrac{1}{1-r}\right)^{-1} & \text{if } \alpha p = -n \text{ or } \alpha = 0, \\ (1-|a|^2)^\alpha (1-r)^\alpha & \text{if } \alpha > 0. \end{cases}$$

Then the following properties for an $\mathcal{M}$-harmonic function f on B are equivalent:

(a) $f \in \mathcal{HB}_\alpha$*;*

(b) $H_{\alpha,p}(f) = \sup\limits_{\substack{0<r<1 \\ a \in B}} \rho_{\alpha,p}(a,r) \left(\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \right)^{1/p} < \infty$*;*

(c) $I_{\alpha,p}(f) = \sup\limits_{\substack{0<r<1 \\ a \in B}} \rho_{\alpha,p}(a,r) \left(\frac{1}{\lambda(E(a,r))} \int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) \right)^{1/p} < \infty$*;*

(d) *There exists $0 < r_0 < 1$ such that*

$$J_{\alpha,p}(f) = \sup_{a \in B} (1-|a|^2)^\alpha \left(\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) \right)^{1/p} < \infty.$$

Moreover, the quantities $\|f\|_{\mathcal{HB}_\alpha}$, $H_{\alpha,p}(f)$, $I_{\alpha,p}(f)$ and $J_{\alpha,p}(f)$ are comparable to each other with a constant depending only on the dimension n, p, α and r_0.

Corollary 1. *Let $1 \le p < \infty$ and f be an $\mathcal{M}$-harmonic function on B. If there exist $0 < r_0 < 1$ and $p < \beta < \infty$ such that*

$$\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) = O((1-|a|^2)^\beta) \quad \textit{as } |a| \to 1,$$

then f is constant.

For $z \in B$ and $0 < r < 1$, let

$$g(r,z) = \frac{n+1}{2n} \int_{|z|}^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} dt,$$

and let $g(z) = g(1,z)$ for simplicity. The Green function for $\widetilde{\Delta}$ is defined by $G(z,w) = g(\varphi_w(z))$ for $z, w \in B$.

As another corollary we obtain the analogous characterization with (1.1) for little α-Bloch spaces.

Corollary 2. *Let $-1 \le \alpha < 0$, $1 < p < -n/\alpha$ and f be an $\mathcal{M}$-harmonic function on B. Then the following are equivalent:*

(i) $f \in \mathcal{HB}_\alpha$;

(ii) $\sup_{a\in B}(1-|a|^2)^{\alpha p}\int_B G(z,a)|\widetilde{\nabla} f(z)|^2|f(z)-f(a)|^{p-2}d\lambda(z) < \infty$.

Let BMOH_p denote the class of all $\mathcal{M}$-harmonic functions f on B that are represented as the Poisson-Szegö integral of functions of bounded mean oscillation on S. That is, each element f in BMOH_p is the form

$$f(z) = \int_S \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}} f^*(\zeta)d\sigma(\zeta)$$

with a corresponding function f^* integrable on S satisfying

$$\|f^*\|_{\mathrm{BMO}_p(\sigma)} = \sup_{\substack{0<r\le 2\\ \xi\in S}}\left(\frac{1}{\sigma(Q(\xi,r))}\int_{Q(\xi,r)}|f^*(\zeta)-f^*_{\xi,r}|^p d\sigma(\zeta)\right)^{1/p} < \infty,$$

where $f^*_{\xi,r} = \frac{1}{\sigma(Q(\xi,r))}\int_{Q(\xi,r)} f^* d\sigma$, the average of f^* over $Q(\xi,r)$. Here $Q(\xi,r) = \{\zeta \in S : |1-\langle \zeta,\xi\rangle| < r\}$, the non-isotropic ball of center ξ and radius r.

The interesting points of the following characterization of BMOH_p are that a solution of Dirichlet problem for $\widetilde{\Delta}$ with boundary data of bounded mean osillation also has bounded mean oscillation with respect to the invariant measure λ on B, and that conversely $\mathcal{M}$-harmonic functions on B of bounded mean oscillation with respect to λ can be represented as the Poisson-Szegö integral of a function of bounded mean oscillation on S.

Theorem 2. *Let $1 < p < \infty$ and f be an $\mathcal{M}$-harmonic function on B. Then the followings are equivalent:*

(i) $f \in \mathrm{BMOH}_p$;

(ii) $\|f\|_{S_p} = \sup_{\substack{0<r<1\\ a\in B}}\left(\int_S |f\circ\varphi_a(r\zeta)-f(a)|^p d\sigma(\zeta)\right)^{1/p} < \infty$;

(iii) $\|f\|_{\mathrm{BMO}_p(\lambda)} = \sup_{\substack{0<r<1\\ a\in B}}\left(\frac{1}{\lambda(E(a,r))}\int_{E(a,r)}|f(z)-f(a)|^p d\lambda(z)\right)^{1/p} < \infty$.

Moreover, the quantities $\|f^*\|_{\mathrm{BMO}_p(\sigma)}$, $\|f\|_{S_p}$ and $\|f\|_{\mathrm{BMO}_p(\lambda)}$ are comparable to each other with a constant depending only on the dimension n and p.

Remark 1. If f is the Poisson-Szegö integral of an integrable function on S, then Theorem 1 holds for $1 \le p < \infty$. Furthermore, if f is the Poisson-Szegö integral and holomorphic on B, then Theorem 1 holds for $0 < p < \infty$. We note, in this case, that the equivalence of (i) and (ii) was proved by Ouyang, Yang and Zhao [11].

Corollary 3. *Let $1 < p < \infty$ and f be an $\mathcal{M}$-harmonic function on B. Then the following are equivalent:*

(i) $f \in \mathrm{BMOH}_p$;

(ii) $\sup_{a \in B} \int_B G(z,a) |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} d\lambda(z) < \infty$.

Throughout the paper we use the symbol C to denote absolute positive constant whose value is unimportant and may change from line to line. If we emphasize the dependencies $a, b, \cdots$, then we write $C(a, b, \cdots)$.

2 Proofs of Theorem 1 and Corollaries 1 and 2

For a real valued C^1 function f on B, let

$$X_j f(z) = \frac{\partial f}{\partial z_j}(z) - \overline{z}_j \sum_{i=1}^{n} \overline{z}_i \frac{\partial f}{\partial \overline{z}_i}(z) \qquad (j = 1, \cdots, n).$$

Then we observe that for $z \in B$,

$$|\widetilde{\nabla} f(z)|^2 \leq \frac{4}{n+1} \sum_{j=1}^{n} |X_j f(z)|^2 \leq \frac{(1+|z|^2)^2}{(1-|z|^2)^2} |\widetilde{\nabla} f(z)|^2, \tag{2.1}$$

and that if f is $\mathcal{M}$-harmonic on B, then $X_j f$ is so. See [15, Proposition 10.4 and Lemma 10.5].

Proof of Theorem 1 (i). Let $\alpha < -1$ and $f \in \mathcal{HB}_\alpha$. Then, for each $j = 1, \cdots, n$, it follows from (2.1) that

$$|X_j f(z)| \leq C \frac{|\widetilde{\nabla} f(z)|}{1 - |z|^2} \leq C \|f\|_{\mathcal{HB}_\alpha} (1 - |z|^2)^{-\alpha - 1} \longrightarrow 0 \quad \text{as } |z| \to 1.$$

Therefore the maximum principle yields that $X_j f \equiv 0$ for every $j = 1, \cdots, n$, and so $|\widetilde{\nabla} f| \equiv 0$ by (2.1). Hence f is constant. □

To prove Theorem 1 (ii), we recall the following lemmas.

Lemma 1. ([15, Lemma 10.8]) *Let f be a real valued C^1 function on B and $a \in B$. Then for each $\zeta \in S$ and $0 < r < 1$,*

$$|f \circ \varphi_a(r\zeta) - f(a)| \leq \sqrt{n+1} \int_0^r \frac{|\widetilde{\nabla} f(\varphi_a(t\zeta))|}{1 - t^2} \, dt.$$

Lemma 2. ([15, Proposition 8.18]) *Let $\beta \in \mathbb{R}$. Then there is a positive constant C depending only on the dimension n such that for $z \in B$,*

$$\int_S \frac{1}{|1 - \langle z, \zeta \rangle|^{n+\beta}} d\sigma(\zeta) \leq \begin{cases} C(1 - |z|^2)^{-\beta} & \text{if } \beta > 0, \\ C \log \dfrac{1}{1 - |z|^2} & \text{if } \beta = 0, \\ C & \text{if } \beta < 0. \end{cases}$$

Lemma 3. *([15, Proposition 10.1 and 10.2]) Let $0 < p < \infty$ and f be an $\mathcal{M}$-harmonic function on B. Then for $a \in B$ and $0 < r < 1$,*

$$|f(a)|^p \leq C(n,p,r) \int_{E(a,r)} |f(z)|^p d\lambda(z),$$

and

$$|\widetilde{\nabla} f(a)|^p \leq C(n,p,r) \int_{E(a,r)} |f(z)|^p d\lambda(z).$$

Proof of Theorem 1 (ii). (a) $\Rightarrow$ (b). Suppose $f \in \mathcal{HB}_\alpha$. Let $a \in B$, $\zeta \in S$ and $0 < r < 1$. Since

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2},$$

we have by Lemma 1

$$\begin{aligned} |f\circ\varphi_a(r\zeta) - f(a)| &\leq C \int_0^r \frac{|\widetilde{\nabla} f(\varphi_a(t\zeta))|}{1-t^2} dt \\ &\leq C\|f\|_{\mathcal{HB}_\alpha} \int_0^r \frac{(1-|\varphi_a(t\zeta)|^2)^{-\alpha}}{1-t^2} dt \\ &= C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha} \int_0^r \frac{|1-\langle ta,\zeta\rangle|^{2\alpha}}{(1-t^2)^{\alpha+1}} dt. \end{aligned}$$

Hence it follows from Minkowski's integral inequality that

$$\begin{aligned} &\left(\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} \\ &\leq C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha} \left(\int_S \left(\int_0^r \frac{|1-\langle ta,\zeta\rangle|^{2\alpha}}{(1-t^2)^{\alpha+1}} dt\right)^p d\sigma(\zeta)\right)^{1/p} \\ &\leq C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha} \int_0^r (1-t^2)^{-\alpha-1} \left(\int_S |1-\langle ta,\zeta\rangle|^{2\alpha p} d\sigma(\zeta)\right)^{1/p} dt. \end{aligned} \tag{2.2}$$

Using Lemma 2, we now calculate the integral

$$F(a,r) := \int_0^r (1-t^2)^{-\alpha-1} \left(\int_S |1-\langle ta,\zeta\rangle|^{2\alpha p} d\sigma(\zeta)\right)^{1/p} dt.$$

If $-n < \alpha p < -n/2$, then

$$F(a,r) \leq C\int_0^1 (1-t^2)^{-\alpha-1}(1-t^2)^{(n+2\alpha p)/p} dt = C\int_0^1 (1-t^2)^{\alpha-1+n/p} dt < \infty.$$

If $\alpha p = -n/2$, then

$$F(a,r) \leq C\int_0^1 (1-t^2)^{-\alpha-1}\left(\log\frac{1}{1-t^2}\right)^{1/p} dt < \infty.$$

If $-n/2 < \alpha p$, then

$$F(a,r) \le C\int_0^r (1-t^2)^{-\alpha-1}dt \le \begin{cases} C & \text{if } -n/2 < \alpha p < 0, \\ C\log\dfrac{1}{1-r} & \text{if } \alpha = 0, \\ C(1-r)^{-\alpha} & \text{if } \alpha > 0. \end{cases}$$

Hence it follows from (2.2) that for $a \in B$ and $0 < r < 1$,

$$\begin{aligned}\left(\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} \\ \le \begin{cases} C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha} & \text{if } -n < \alpha p < 0, \\ C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha}\log\dfrac{1}{1-r} & \text{if } \alpha = 0, \\ C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha}(1-r)^{-\alpha} & \text{if } \alpha > 0. \end{cases}\end{aligned}$$

If $\alpha p = -n$, then

$$F(a,r) \le C\int_0^r (1-t^2)^{-1}dt \le C\log\frac{1}{1-r},$$

so that for $a \in B$ and $0 < r < 1$,

$$\left(\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} \le C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha}\log\frac{1}{1-r}.$$

If $\alpha p < -n$, then

$$F(a,r) \le C\int_0^r (1-t^2)^{\alpha-1+n/p}dt \le C(1-r)^{\alpha+n/p},$$

so that for $a \in B$ and $0 < r < 1$,

$$\left(\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} \le C\|f\|_{\mathcal{HB}_\alpha}(1-|a|^2)^{-\alpha}(1-r)^{\alpha+n/p}.$$

Hence, taking the supremum over $0 < r < 1$ and $a \in B$, we obtain (b).

(b) $\Rightarrow$ (c). Let $a \in B$ and $0 < r < 1$. Since $\rho_{\alpha,p}(a,r)$ is positive and non-increasing for r, we have by integration in polar coordinates

$$\begin{aligned}\int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) &= 2n\int_0^r \frac{t^{2n-1}}{(1-t^2)^{n+1}}\int_S |f\circ\varphi_a(t\zeta) - f(a)|^p d\sigma(\zeta)dt \\ &\le H_{\alpha,p}(f)^p\rho_{\alpha,p}(a,r)^{-p}\lambda(E(a,r)),\end{aligned}$$

and (c) follows.

(c) $\Rightarrow$ (d). Let $a \in B$ and $0 < r_0 < 1$. Then we have

$$(1-|a|^2)^\alpha\left(\int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z)\right)^{1/p} \le C(n,p,\alpha,r_0)I_{\alpha,p}(f),$$

and (d) follows.

(d) $\Rightarrow$ (a). Let $a \in B$. Then it follows from Lemma 3 with $r := r_0$ and $f := f - f(a)$ that

$$|\widetilde{\nabla} f(a)|^p \leq C(n,p,r_0) \int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) \leq C J_{\alpha,p}(f)^p (1-|a|^2)^{-\alpha p}.$$

Therefore we have $f \in \mathcal{HB}_\alpha$. Thus Theorem 1 is established. □

Proof of Corollary 1. It follows form Lemma 3 that

$$|\widetilde{\nabla} f(a)|^p \leq C \int_{E(a,r_0)} |f(z) - f(a)|^p d\lambda(z) \leq C(1-|a|^2)^\beta,$$

where C is a constant depending only on n, p and r_0. Hence we have $f \in \mathcal{HB}_{-\beta/p}$, and so f is constant by Theorem 1 (i). □

Corollary 2 follows from the following.

Lemma 4. *([9, Lemma 3.5]) If $1 < p < \infty$ and f is $\mathcal{M}$-harmonic on B, then for $0 < r < 1$,*

$$\int_S |f(r\zeta)|^p d\sigma(\zeta) - |f(0)|^p = p(p-1) \int_{rB} g(r,z) |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z). \quad (2.3)$$

Letting $r \to 1-$ in (2.3), it follows from the monotone convergence that

$$\lim_{r\to 1-} \int_S |f(r\zeta)|^p d\sigma(\zeta) - |f(0)|^p = p(p-1) \int_B g(z) |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z). \quad (2.4)$$

Proof of Corollary 2. Multiplying the both sides of (2.4) with $f := f \circ \varphi_a - f(a)$ by $(1-|a|^2)^{\alpha p}$ and taking the supremum over $a \in B$, we obtain Corollary 2 from Theorem 1 and the invariance of λ under $Aut(B)$. □

3 Proofs of Theorem 2 and Corollary 3

For simplicity we denote the Poisson-Szegö kernel of B by

$$\mathcal{P}(z,\zeta) = \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}.$$

We recall the change of variables formula [13, Remark in page 44]

$$\int_S \mathcal{P}(z,\zeta) f(\zeta) d\sigma(\zeta) = \int_S f(\varphi_z(\zeta)) d\sigma(\zeta). \quad (3.1)$$

To prove Theorem 2 we need the following.

Lemma 5. *Let $1 \leq p < \infty$ and f be the Poisson-Szegö integral of an integrable function f^* on S. Then the following are equivalent:*

(i) $f \in \mathrm{BMOH}_p$;

(ii) $\|f\|_G = \sup_{a\in B}\left(\int_S \mathcal{P}(a,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta)\right)^{1/p} < \infty.$

Moreover, the quantities $\|f^*\|_{\mathrm{BMO}_p(\sigma)}$ and $\|f\|_G$ are comparable with a constant depending only on the dimension n and p.

Proof. This lemma will be proved in the same way as [2, pp. 224–225].

(i) ⇒ (ii). Splitting S into $Q(a/|a|, 1-|a|)$ and $Q(a/|a|, 2^k(1-|a|)) \setminus Q(a/|a|, 2^{k-1}(1-|a|))$ and using the estimate $\mathcal{P}(a,\cdot) \le C2^{-2kn}(1-|a|)^{-n}$ on each partitions, we obtain

$$\int_S \mathcal{P}(a,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta) \le C\int_S \mathcal{P}(a,\zeta)|f^*(\zeta) - f^*_{\xi,r}|^p d\sigma(\zeta) \le C\|f^*\|^p_{\mathrm{BMO}_p(\sigma)}.$$

(ii) ⇒ (i). Letting $z_{\xi,r} = (1-r/2)\xi$, we have $\mathcal{P}(z_{\xi,r},\cdot) \ge Cr^{-n} \ge C\sigma(Q(\xi,r))^{-1}$ on $Q(\xi,r)$. This yields immediately $\|f^*\|_{\mathrm{BMO}_p(\sigma)} \le C(n,p)\|f\|_G$. □

Proof of Theorem 2. (ii) ⇒ (i). Let us assume that (ii) holds. Taking $a = 0$ we see that $f \in H^p$, the p-th Hardy space, so that f can be represented as the Poisson-Szegö integral of a function $f^* \in L^p(\sigma)$. Thus it suffices to show that $\|f^*\|_{\mathrm{BMO}_p(\sigma)} < \infty$. Let $a \in B$ be fixed. By (3.1) we have

$$\int_S \mathcal{P}(a,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta) = \int_S |f^*\circ\varphi_a(\zeta) - f(a)|^p d\sigma(\zeta).$$

We observe that for a.e. $\zeta \in S$,

$$\lim_{\rho\to 1-} f\circ\varphi_a(\rho\zeta) - f(a) = f^*\circ\varphi_a(\zeta) - f(a).$$

Indeed, this follows from Korányi's result [8] since the inequality

$$|1 - \langle\varphi_a(\rho\zeta), \varphi_a(\zeta)\rangle| = \frac{(1-|a|^2)(1-\rho)}{|1-\langle\rho\zeta, a\rangle||1-\langle a,\zeta\rangle|} \le \frac{1}{1-|a|}(1-|\varphi_a(\rho\zeta)|^2)$$

implies that $\{\varphi_a(\rho\zeta) : 0 < \rho < 1\}$ is contained in the Korányi approach region at $\varphi_a(\zeta)$. Since the function $\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta)$ is non-decreasing for $0 < r < 1$, we obtain $\|f\|_G \le \|f\|_{S_p}$. Hence $f \in \mathrm{BMOH}_p$ by Lemma 5.

(iii) ⇒ (ii). Let $a \in B$ be fixed. By the monotonicity of the spherical integral, it is enough to show that

$$\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \le C\|f\|^p_{\mathrm{BMO}_p(\lambda)} \quad \text{for } \tfrac{1}{2} < r < 1,$$

where C is a constant independent of a and r. Since

$$\lambda(E(a, \tfrac{1+r}{2})) = \frac{(1+r)^{2n}}{(3+r)^n(1-r)^n},$$

it follows from integration in polar coordinates that for $1/2 < r < 1$,

$$
\begin{aligned}
&\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \\
&\qquad \le \frac{2}{1-r}\int_r^{\frac{1+r}{2}}\int_S |f\circ\varphi_a(t\zeta) - f(a)|^p d\sigma(\zeta)dt \\
&\qquad \le \frac{2}{1-r}\frac{(1-r^2)^{n+1}}{r^{2n-1}}\int_r^{\frac{1+r}{2}}\frac{t^{2n-1}}{(1-t^2)^{n+1}}\int_S |f\circ\varphi_a(t\zeta) - f(a)|^p d\sigma(\zeta)dt \\
&\qquad \le \frac{1}{n}\frac{1}{1-r}\frac{(1-r^2)^{n+1}}{r^{2n-1}}\int_{B(0,\frac{1+r}{2})} |f\circ\varphi_a(z) - f(a)|^p d\lambda(z) \\
&\qquad = \frac{1}{n}\frac{(1+r)^{3n+1}}{r^{2n-1}(3+r)^n}\frac{1}{\lambda(E(a,\frac{1+r}{2}))}\int_{E(a,\frac{1+r}{2})} |f(z) - f(a)|^p d\lambda(z) \\
&\qquad \le \frac{2^{4n}}{n}\|f\|^p_{\mathrm{BMO}_p(\lambda)}.
\end{aligned}
$$

Hence we obtain $\|f\|_{S_p} \le (2^{4n}/n)\|f\|_{\mathrm{BMO}_p(\lambda)}$. Thus (ii) follows.

(i) $\Rightarrow$ (iii). We assume that

$$f(z) = \int_S \mathcal{P}(z,\zeta)f^*(\zeta)d\sigma(\zeta)$$

with $\|f^*\|_{\mathrm{BMO}_p(\sigma)} < \infty$. Let $a \in B$ and $0 < r < 1$ be fixed. We put $\xi = a/|a|$ and $\rho = 1 - |a|$. Since $\mathcal{P}(z,\cdot)d\sigma$ is a probability measure on S, we have by Jensen's inequality, Fubini's theorem and the mean value property

$$
\begin{aligned}
\int_{E(a,r)} |f(z) - f(a)|^p d\lambda(z) &= \int_{E(a,r)}\left|\int_S \mathcal{P}(z,\zeta)[f^*(\zeta) - f(a)]d\sigma(\zeta)\right|^p d\lambda(z) \\
&\le \int_{E(a,r)}\int_S \mathcal{P}(z,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta)d\lambda(z) \\
&= \int_S\left(\int_{E(a,r)} \mathcal{P}(z,\zeta)d\lambda(z)\right)|f^*(\zeta) - f(a)|^p d\sigma(\zeta) \\
&= \lambda(E(a,r))\int_S \mathcal{P}(a,\zeta)|f^*(\zeta) - f(a)|^p d\sigma(\zeta).
\end{aligned}
$$

Hence it follows from Lemma 5 that $\|f\|_{\mathrm{BMO}_p(\lambda)} \le C(n,p)\|f^*\|_{\mathrm{BMO}_p(\sigma)}$, and so (i) follows. Thus Theorem 2 is proved. □

Proof of Corollary 3. Let $a \in B$ and apply (2.4) to $f := f\circ\varphi_a - f(a)$. Then, by the change of variable, we have

$$
\begin{aligned}
&\sup_{0<r<1}\int_S |f\circ\varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \\
&\qquad = p(p-1)\int_B G(z,a)|\widetilde{\nabla} f(z)|^2|f(z) - f(a)|^{p-2}d\lambda(z).
\end{aligned}
$$

Hence, taking the supremum over $a \in B$, we obtain Corollary 3. □

4 Further remark

In [3], Hahn and Youssfi considered the $\mathcal{M}$-harmonic Besov space $\mathcal{M}B_p$, that is, the space of all $\mathcal{M}$-harmonic functions on B that satisfy

$$\|f\|_{\mathcal{M}B_p} = \left(\int_B |\widehat{Q}f(z)|^p d\lambda(z) \right)^{1/p} < \infty.$$

Here, letting $\nabla = (\partial/\partial z_1, \cdots, \partial/\partial z_n)$ the complex gradient vector field and $\beta(z,\zeta)$ the Bergman metric on B, i.e,

$$\beta(z,\zeta) = \left(\frac{(1-|z|^2)|\zeta|^2 + |\langle z,\zeta\rangle|^2}{(1-|z|^2)^2} \right)^{1/2},$$

we denote

$$\widehat{Q}f(z) = \sup_{|\zeta|=1} \frac{|\nabla f(z)\zeta + \overline{\nabla \overline{f}(z)\zeta}|}{\beta(z,\zeta)}.$$

They proved the inclusion relationship $\mathcal{M}B_p \subset H^p$, the p-th Hardy space for $2n < p < \infty$. See [3, Theorem 4.5]. As a consequence of Theorem 2 and our previous paper [5], we obtain the following inclusion relationship.

Theorem 3. *Let $2n < p < \infty$. Then*

$$\mathcal{M}B_p \subsetneq \mathrm{BMOH}_p$$

Proof. Let $a \in B$ and $0 < r < 1$. Applying [3, (4.13) in p. 75] to $f := f \circ \varphi_a$, we have

$$\int_S |f \circ \varphi_a(r\zeta) - f(a)|^p d\sigma(\zeta) \le C(n,p) \int_B \widehat{Q}(f \circ \varphi_a)(z) d\lambda(z).$$

Since $\widehat{Q}(f \circ \varphi_a)(z) = (\widehat{Q}f)(\varphi_a(z))$ ([3, Proposition 4.1]), it follows that $\|f\|_{S_p} \le C\|f\|_{\mathcal{M}B_p}$, and so $\mathcal{M}B_p \subset \mathrm{BMOH}_p$. Note from [4] that functions in $\mathcal{M}B_p$ have a tangential limit at almost every point of S. In [5], the author constructed a bounded $\mathcal{M}$-harmonic function on B which fails to have a tangential limit at every boundary point of S. Therefore the inclusion is strict. □

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