The boundary growth of superharmonic functions and positive solutions of nonlinear elliptic equations *

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Abstract

We investigate the boundary growth of positive superharmonic functions u on a bounded domain Ω in $\mathbb{R}^n$, $n \geq 3$, satisfying the nonlinear elliptic inequality

$$0 \leq -\Delta u \leq c\delta_\Omega(x)^{-\alpha} u^p \quad \text{in } \Omega,$$

where $c > 0$, $\alpha \geq 0$ and $p > 0$ are constants, and $\delta_\Omega(x)$ is the distance from x to the boundary of Ω. The result is applied to show a Harnack inequality for such superharmonic functions. Also, we study the existence of positive solutions, with singularity on the boundary, of the nonlinear elliptic equation

$$-\Delta u + Vu = f(x, u) \quad \text{in } \Omega,$$

where V and f are Borel measurable functions conditioned by the generalized Kato class.

1 Introduction

The purpose of this paper is to investigate the boundary growth of positive superharmonic functions satisfying a certain nonlinear elliptic inequality. As applications, we shall obtain a Harnack inequality for positive solutions of nonlinear elliptic equations and an existence theorem for nontangential limits of certain Green potentials.

Let Ω be a domain in $\mathbb{R}^n$ and let $\delta_\Omega(x)$ stand for the distance from x to the boundary $\partial\Omega$ of Ω. A lower semicontinuous function $u : \Omega \to (-\infty, +\infty]$, where $u \not\equiv +\infty$, is called *superharmonic* on Ω if it satisfies the mean value inequality

$$u(x) \geq \frac{1}{\nu_n r^n} \int_{B(x,r)} u(y)dy, \quad \text{whenever } 0 < r < \delta_\Omega(x),$$

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where $B(x,r)$ denotes the open ball of center x and radius r, and ν_n is the volume of the unit ball. Let Δ be the Laplace operator on $\mathbb{R}^n$. It is well known that if u is a superharmonic function on Ω, then there exists a unique (Radon) measure μ_u on Ω such that

$$\int_\Omega \phi(x)d\mu_u(x) = -\int_\Omega u(x)\Delta\phi(x)dx \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ is the collection of all infinitely differentiable functions vanishing outside a compact set in Ω (cf. [2, Section 4.3]). The measure μ_u is called the *Riesz measure associated with* u. If μ_u is absolutely continuous with respect to the Lebesgue measure and $d\mu_u(x) = f_u(x)dx$, where f_u is a nonnegative locally integrable function on Ω, then we call f_u the *Riesz function associated with* u for convenience. It is clear that $f_u = -\Delta u$ when $u \in C^2(\Omega)$.

The classical Littlewood theorem states that every Green potential on the unit ball has radial limit 0 almost everywhere on the boundary. However, the nontangential and tangential limits do not necessarily exist. To avoid this, many authors have imposed weighted integrability conditions on the density functions of Green potentials (cf. [3, 8, 21] and references therein). Such results were concerned with the boundary behavior of solutions of the Poisson equation, but are not applicable to positive solutions of stationary Schrödinger equations or nonlinear elliptic equations. For this reason, we study the boundary behavior of positive superharmonic functions u satisfying the nonlinear inequality

$$0 \le f_u \le c\delta_\Omega(x)^{-\alpha}u^p \quad \text{almost everywhere on } \Omega, \tag{1.1}$$

where f_u is the Riesz function associated with u, and $c > 0$, $\alpha \ge 0$ and $p > 0$ are constants.

First of all, we note from the Poisson integral representation that every positive harmonic function h on the unit ball B of $\mathbb{R}^n$ satisfies

$$\frac{h(0)}{2^n}\delta_B(x) \le h(x) \le 2h(0)\delta_B(x)^{1-n} \quad \text{for } x \in B.$$

As seen in Lemma 3.1 below, the lower estimate is extendable to any positive superharmonic function. However, the upper estimate does not necessarily hold even for positive superharmonic functions satisfying (1.1). Our main purpose is to determine the critical number p^* such that every positive superharmonic function satisfying (1.1) with $p \le p^*$ is bounded by a constant multiple of $\delta_\Omega(x)^{1-n}$. By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line. In what follows, we suppose that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \ge 3$.

Theorem 1.1. *Let $c > 0$. Suppose that $0 < p \le (n+1)/(n-1)$ and $0 \le \alpha \le n+1-p(n-1)$. Let u be a positive superharmonic function on Ω having an associated Riesz function f_u which satisfies* (1.1)*. Then there exists a constant A depending only on u, c, α, p and Ω such that*

$$u(x) \le A\delta_\Omega(x)^{1-n} \quad \textit{for } x \in \Omega. \tag{1.2}$$

Furthermore, $u \in C^1(\Omega)$.

As applications of Theorem 1.1, we have a Harnack inequality and an existence theorem for nontangential limits of Green potentials satisfying (1.1).

Corollary 1.2. *Let* $c > 0$*. Suppose that* $0 < p \le (n+1)/(n-1)$ *and* $0 \le \alpha \le \min\{n+1-p(n-1), 1+p\}$*. Let* u *be a positive superharmonic function on* Ω *having an associated Riesz function* f_u *which satisfies* (1.1)*. Then there exists a constant* A *depending only on* u*,* c*,* α*,* p *and* Ω *such that*

$$\sup_{B(x,r)} u \le A \inf_{B(x,r)} u, \tag{1.3}$$

whenever $B(x, 8r) \subset \Omega$*.*

For $\xi \in \partial\Omega$ and $\theta > 0$, we define

$$\Gamma_\theta(\xi) = \big\{x \in \Omega : |x - \xi| < (1+\theta)\delta_\Omega(x)\big\}.$$

Corollary 1.3. *Let* $c > 0$*. Suppose that* $0 < p \le (n+1)/(n-1)$ *and* $0 \le \alpha \le \min\{n+1-p(n-1), 1+p\}$*. Let* u *be a positive superharmonic function on* Ω *having an associated Riesz function* f_u *which satisfies* (1.1)*. If the greatest harmonic minorant of* u *is the zero function, then for each* $\theta > 0$*,*

$$\lim_{\Gamma_\theta(\xi)\ni x\to\xi} u(x) = 0 \quad \textit{for a.e. } \xi \in \partial\Omega.$$

Remark 1.4. Actually, Corollary 1.2 is valid for arbitrary domains. Therefore Corollary 1.3 can be extended easily to Lipschitz and NTA domains. See proofs of them.

Note again that these results are applicable to positive solutions $u \in C^2(\Omega)$ of

$$0 \le -\Delta u \le c\delta_\Omega(x)^{-\alpha} u^p \quad \text{in } \Omega. \tag{1.4}$$

The following theorem shows that the bound $p \le (n+1)/(n-1)$ is sharp in Theorem 1.1.

Theorem 1.5. *Let* $\xi \in \partial\Omega$ *and* $c > 0$*. Suppose that* p *and* α *satisfy either*

(i) $p > (n+1)/(n-1)$ *and* $\alpha \ge 0$*, or*

(ii) $0 < p \le (n+1)/(n-1)$ *and* $\alpha > n+1-p(n-1)$*.*

Then, for each β *satisfying*

$$n - 1 < \beta < \begin{cases} \dfrac{2+\alpha(n-2)}{(2-n)p+n} & \textit{if } p < \dfrac{n}{n-2}, \\ \infty & \textit{if } p \ge \dfrac{n}{n-2}, \end{cases} \tag{1.5}$$

there exists a positive solution $u \in C^2(\Omega)$ *of* (1.4) *such that*

$$\limsup_{\Gamma_\theta(\xi)\ni x\to\xi} \delta_\Omega(x)^\beta u(x) > 0 \tag{1.6}$$

for any $\theta > 0$*. In particular,* u *does not satisfy* (1.2)*.*

Remark 1.6. From $p > (n+1)/(n-1)$ or $\alpha > n+1-p(n-1)$, we observe that $n-1 < (2+\alpha(n-2))/((2-n)p+n)$. Thus we can take β satisfying (1.5).

Two positive functions f and g are said to be comparable if there exists a constant A such that $A^{-1}f \le g \le Af$. Then we write $f \approx g$ and call A the constant of comparison. Obviously, the Poisson kernel gives the sharpness of (1.2). The following theorem is interesting itself and shows that the growth rate in (1.2) is sharp for positive solutions of nonlinear elliptic equations as well.

Theorem 1.7. *Let $\xi \in \partial\Omega$ and $c > 0$ (assumed to be small enough when $p = 1$ only). Suppose that $0 < p < (n+1)/(n-1)$ and $0 \le \alpha < \min\{n+1-p(n-1), 1+p\}$. If g is a locally Hölder continuous function on Ω such that $|g(x)| \le c\delta_\Omega(x)^{-\alpha}$ for $x \in \Omega$, then there exist infinitely many positive solutions $u \in C^2(\Omega)$ of*

$$-\Delta u = gu^p \quad \textit{in } \Omega \tag{1.7}$$

such that

$$u(x) \approx \frac{\delta_\Omega(x)}{|x-\xi|^n} \quad \textit{for } x \in \Omega. \tag{1.8}$$

In contrast to Theorem 1.7, there are many results concerning the existence and nonexistence of positive solutions of the Lane-Emden equation $-\Delta u = u^p$:

- the critical number for the homogeneous Dirichlet problem is $(n+2)/(n-2)$ (e.g. [20]),
- the critical number for the existence of positive solutions comparable to $|\cdot|^{2-n}$ near the origin is $n/(n-2)$ (cf. [13, 16, 22] and references therein).

Theorems 1.7 and 6.1 below assert that $(n+1)/(n-1)$ is the critical number for the existence of positive solutions comparable to the Poisson or Martin kernel.

The plan of this paper is as follows. In Section 2, we shall prove Theorem 1.1 after showing some elementary lemmas. Corollaries 1.2 and 1.3 will be shown in Section 3. Section 4 includes the proof of Theorem 1.5. In Section 5, we introduce a generalized Kato class and discuss the existence of positive solutions of the nonlinear elliptic equation $-\Delta u + Vu = f(x,u)$ rather than (1.7). As a special case of this, we shall obtain Theorem 1.7 in Section 6. Also, we shall give a remark concerning the sharpness of $p < (n+1)/(n-1)$ in Theorem 1.7.

2 Proof of Theorem 1.1

Let $G(\cdot, y)$ denote the Green function of Ω with pole at $y \in \Omega$, i.e. the distributional solution of

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_y is the Dirac measure at y. Let $\xi \in \partial\Omega$ and $x_0 \in \Omega$. It is known from [12] that the Martin boundary of a bounded $C^{1,1}$-domain Ω coincides with the Euclidean boundary, and therefore the ratio $G(\cdot, y)/G(x_0, y)$ converges to a positive harmonic function on Ω as $y \to \xi$. The limit function, written $K(\cdot, \xi)$, is called the *Martin kernel* of Ω with pole at ξ. The following estimate for the Green function is well known (cf. [5, 23]), and yields an estimate for the Martin kernel after elementary calculations.

Lemma 2.1. *For $x, y \in \Omega$ and $\xi \in \partial\Omega$,*

$$G(x,y) \approx \min\left\{1, \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right\}|x-y|^{2-n}, \tag{2.1}$$

$$K(x,\xi) \approx \frac{\delta_\Omega(x)}{|x-\xi|^n}, \tag{2.2}$$

where the constants of comparison depend only on Ω.

In what follows, let u be a positive superharmonic function on Ω having an associated Riesz function f_u which satisfies (1.1). Then the Riesz decomposition theorem (cf. [2, Theorem 4.4.1]) yields that

$$u(x) = h(x) + \int_\Omega G(x,y) f_u(y)dy \quad \text{for } x \in \Omega, \tag{2.3}$$

where h is the greatest harmonic minorant of u on Ω. Note that h is nonnegative.

Lemma 2.2. *If h is a nonnegative harmonic function on Ω, then there exists a constant A depending only on h and Ω such that*

$$h(x) \le A\delta_\Omega(x)^{1-n} \quad \textit{for } x \in \Omega.$$

Proof. By the Martin representation theorem and (2.2), we have

$$h(x) = \int_{\partial\Omega} K(x,y)d\nu(y) \le A\delta_\Omega(x)^{1-n}\nu(\partial\Omega),$$

where ν is the measure on $\partial\Omega$ associated with h. □

Lemma 2.3. *There exists a constant A depending only on u and Ω such that*

$$\int_\Omega \delta_\Omega(y) f_u(y)dy \le A.$$

Proof. Let $x_0 \in \Omega$ be fixed, where $u(x_0) < \infty$. Then we observe from (2.1) that $G(x_0, y) \ge A^{-1}\delta_\Omega(y)$ for $y \in \Omega$. Hence (2.3) implies that $\int_\Omega \delta_\Omega(y) f_u(y)dy \le Au(x_0)$. □

Lemma 2.4. *For each $j \in \mathbb{N}$, there exists a constant $c_j > 0$ depending only on j, u and Ω such that for $z \in \Omega$ and $x \in B(z, \delta_\Omega(z)/2^{j+1})$,*

$$u(x) \le c_j\delta_\Omega(z)^{1-n} + \int_{B(z,\delta_\Omega(z)/2^j)} \frac{f_u(y)}{|x-y|^{n-2}}dy.$$

Proof. Let $z \in \Omega$ and $x \in B(z, \delta_\Omega(z)/2^{j+1})$. By (2.1), we have

$$G(x,y) \le A\frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^n} \le A2^{nj}\delta_\Omega(z)^{1-n}\delta_\Omega(y) \quad \text{for } y \in \Omega \setminus B(z, \delta_\Omega(z)/2^j).$$

Since $f_u \ge 0$, it follows from Lemma 2.3 that

$$\int_{\Omega\setminus B(z,\delta_\Omega(z)/2^j)} G(x,y) f_u(y)dy \le A2^{nj}\delta_\Omega(z)^{1-n},$$

and therefore

$$\int_\Omega G(x,y)f_u(y)dy \le A2^{nj}\delta_\Omega(z)^{1-n} + \int_{B(z,\delta_\Omega(z)/2^j)} \frac{f_u(y)}{|x-y|^{n-2}}dy.$$

This, together with (2.3) and Lemma 2.2, implies the required estimate. $\square$

Proof of Theorem 1.1. Let $z \in \Omega$ and $j \in \mathbb{N}$. By Lemma 2.3, we have

$$\delta_\Omega(z)\int_{B(z,\delta_\Omega(z)/2)} f_u(y)dy \le A,$$

where A depends only on u and Ω. Let $r = \delta_\Omega(z)$. Making the change of variables $x = z + r\eta$ and $y = z + r\zeta$ and letting $\psi_z(\zeta) = r^{n+1}f_u(z + r\zeta)$, we have

$$\int_{B(0,1/2)} \psi_z(\zeta)d\zeta \le A, \tag{2.4}$$

and by Lemma 2.4,

$$r^{n-1}u(z + r\eta) \le c_j + \int_{B(0,2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}d\zeta \quad \text{for } \eta \in B(0, 2^{-(j+1)}). \tag{2.5}$$

Suppose that $0 < p \le (n+1)/(n-1)$ and $0 \le \alpha \le n+1-p(n-1)$, and let

$$\frac{n+1}{n-1} < q < \frac{n}{n-2}, \quad \ell = \left[\frac{\log(q/(q-1))}{\log(q/p)}\right] + 1 \quad \text{and} \quad c_0 = \max_{1\le j\le \ell+1}\{c_j\}.$$

Define $\Psi_{z,j} : B(0,1) \to [0,+\infty]$ by

$$\Psi_{z,j}(\eta) = c_0 + \int_{B(0,2^{-j})} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}d\zeta.$$

To show (1.2), it is enough to prove that $\Psi_{z,\ell+1}(0)$ is bounded by a constant independent of z since $r^{n-1}u(z) \le \Psi_{z,\ell+1}(0)$ by (2.5). We claim that for $\kappa \ge 1$ there exists a constant A depending only on c, c_0, p, q, κ and Ω such that for $1 \le j \le \ell$,

$$\|\psi_z^{\kappa/p}\|_{L^q(B(0,2^{-(j+1)}))} \le A + A\|\psi_z^\kappa\|_{L^1(B(0,2^{-j}))}. \tag{2.6}$$

Indeed, by the Jensen inequality for the probability measure $|\eta-\zeta|^{2-n}d\zeta/\int_{B(0,2^{-j})}|\eta-\zeta|^{2-n}d\zeta$ on $B(0,2^{-j})$,

$$\left(\int_{B(0,2^{-j})}\frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}d\zeta\right)^{\kappa}\le A\int_{B(0,2^{-j})}\frac{\psi_z(\zeta)^{\kappa}}{|\eta-\zeta|^{n-2}}d\zeta\quad\text{for }\eta\in B(0,1),$$

where A depends only on κ and n. This gives that

$$\|\Psi_{z,j}^{\kappa}\|_{L^q(B(0,2^{-j}))}\le A+A\left\|\int_{B(0,2^{-j})}\frac{\psi_z(\zeta)^{\kappa}}{|\cdot-\zeta|^{n-2}}d\zeta\right\|_{L^q(B(0,2^{-j}))}.$$

By the Minkowski inequality and $q(n-2)<n$,

$$\begin{aligned}\left\|\int_{B(0,2^{-j})}\frac{\psi_z(\zeta)^{\kappa}}{|\cdot-\zeta|^{n-2}}d\zeta\right\|_{L^q(B(0,2^{-j}))}&\le\int_{B(0,2^{-j})}\left(\int_{B(0,2^{-j})}\frac{d\eta}{|\eta-\zeta|^{q(n-2)}}\right)^{1/q}\psi_z(\zeta)^{\kappa}d\zeta\\&\le A\|\psi_z^{\kappa}\|_{L^1(B(0,2^{-j}))}.\end{aligned}$$

Therefore

$$\|\Psi_{z,j}^{\kappa}\|_{L^q(B(0,2^{-j}))}\le A+A\|\psi_z^{\kappa}\|_{L^1(B(0,2^{-j}))}.$$

Since $\delta_\Omega(z+r\eta)\ge r/2$ for $\eta\in B(0,1/2)$, it follows from (1.1), (2.5), $0<p\le(n+1-\alpha)/(n-1)$ and the boundedness of Ω that

$$\begin{aligned}\psi_z(\eta)=r^{n+1}f_u(z+r\eta)&\le cr^{n+1}\delta_\Omega(z+r\eta)^{-\alpha}u(z+r\eta)^p\\&\le A\Psi_{z,j}(\eta)^p\quad\text{for a.e. }\eta\in B(0,2^{-(j+1)}).\end{aligned}$$

Therefore

$$\|\psi_z^{\kappa/p}\|_{L^q(B(0,2^{-(j+1)}))}\le A+A\|\psi_z^{\kappa}\|_{L^1(B(0,2^{-j}))},$$

and so (2.6) holds.

Let $s=q/p>1$. Then (2.6) implies that

$$\int_{B(0,2^{-(j+1)})}\psi_z(\eta)^{s\kappa}d\eta\le A+A\left(\int_{B(0,2^{-j})}\psi_z(\eta)^{\kappa}d\eta\right)^q\quad\text{for }1\le j\le\ell.$$

We use this ℓ times to obtain

$$\begin{aligned}\left(\int_{B(0,2^{-(\ell+1)})}\psi_z(\eta)^{s^\ell}d\eta\right)^{1/s^\ell}&\le A+A\left(\int_{B(0,2^{-\ell})}\psi_z(\eta)^{s^{\ell-1}}d\eta\right)^{q/s^\ell}\\&\le\cdots\\&\le A+A\left(\int_{B(0,1/2)}\psi_z(\eta)d\eta\right)^{q^\ell/s^\ell}.\end{aligned}\tag{2.7}$$

Our choice of ℓ implies that $s^\ell\ge q/(q-1)$, equivalent to $s^\ell\le(s^\ell-1)q$. Therefore

$$\frac{s^\ell}{s^\ell-1}(n-2)\le q(n-2)<n.$$

Hence the Hölder inequality, (2.7) and (2.4) give that

$$\Psi_{z,\ell+1}(0) \le A + A\left(\int_{B(0,2^{-(\ell+1)})} \psi_z(\eta)^{s^\ell} d\eta\right)^{1/s^\ell} \le A,$$

where A is independent of z. Hence we obtain (1.2).

Moreover, (1.1) and (1.2) imply the local boundedness of f_u, which by [18, Theorem 6.6] implies $u \in C^1(\Omega)$. This completes the proof of Theorem 1.1. $\square$

3 Proofs of Corollaries 1.2 and 1.3

We have the following lower estimate for positive superharmonic functions on Ω.

Lemma 3.1. *Let u be a positive superharmonic function on Ω. Then there exists a constant A depending only on u and Ω such that*

$$u(x) \ge \frac{1}{A}\delta_\Omega(x) \quad \textit{for } x \in \Omega. \tag{3.1}$$

Proof. Let μ_u be the Riesz measure associated with u. By the Riesz decomposition theorem, we have

$$u(x) = h(x) + \int_\Omega G(x,y)d\mu_u(y),$$

where h is a nonnegative harmonic function on Ω. If $\mu_u(\Omega) = 0$, then $u = h$. The Martin representation theorem and (2.2) yields that

$$u(x) = \int_{\partial\Omega} K(x,y)d\nu(y) \ge \frac{\delta_\Omega(x)}{A}\nu(\partial\Omega),$$

and so (3.1) holds in this case. If $\mu_u(\Omega) > 0$, then we find $r_0 > 0$ such that $\mu_u(E) > 0$, where $E = \{x \in \Omega : \delta_\Omega(x) \ge r_0\}$. It follows from (2.1) that

$$u(x) \ge \int_E G(x,y)d\mu_u(y) \ge \frac{\delta_\Omega(x)}{A}\mu_u(E) \quad \text{whenever } \delta_\Omega(x) < \frac{r_0}{2}.$$

Also, the lower semicontinuity of u yields that u has a positive minimum on $\{x \in \Omega : \delta_\Omega(x) \ge r_0/2\}$. Hence (3.1) follows. $\square$

The following Harnack inequality for stationary Schrödinger equations is found in [9, Theorem 8.20].

Lemma 3.2. *Let $\nu > 0$ be a constant and let ρ be a measurable function on a domain D such that $|\rho| \le \nu^2$. If $u \in W^{1,2}(D)$ is a nonnegative weak solution of $\Delta u + \rho u = 0$ in D, then there exists a constant A depending only on the dimension n such that*

$$\sup_{B(x,r)} u \le A^{\sqrt{n}+\nu r} \inf_{B(x,r)} u$$

whenever $B(x,4r) \subset D$.

Proof of Corollary 1.2. Let $B(y, 8r) \subset \Omega$ and let $D = B(y, 4r)$. By Theorem 1.1, we have $u \in C^1(\Omega) \subset W^{1,2}(D)$. Let $\rho(x) = f_u(x)/u(x)$. Then it follows from the definition of f_u that for $\phi \in C_0^\infty(D)$,

$$\int_D \rho u \phi dx = \int_D f_u \phi dx = -\int_D u \Delta \phi dx = \int_D \nabla u \cdot \nabla \phi dx.$$

Therefore u is a weak solution of $\Delta u + \rho u = 0$ in D. Also, we observe from (1.1) and (1.2) that if $1 \le p \le (n + 1 - \alpha)/(n - 1)$, then

$$\begin{aligned} 0 \le \rho(x) &\le c\delta_\Omega(x)^{-\alpha} u(x)^{p-1} \\ &\le A\delta_{B(y,8r)}(x)^{-\alpha+(p-1)(1-n)} \le A\delta_{B(y,8r)}(x)^{-2} \le Ar^{-2} \quad \text{for } x \in D. \end{aligned}$$

If $0 < p < 1$, then $0 \le \alpha \le 1 + p$, so that we have by Lemma 3.1

$$0 \le \rho(x) \le c\delta_\Omega(x)^{-\alpha} u(x)^{p-1} \le A\delta_{B(y,8r)}(x)^{-\alpha+p-1} \le Ar^{-2} \quad \text{for } x \in D.$$

Hence (1.3) follows from Lemma 3.2. □

Proof of Corollary 1.3. Let u satisfy the assumption in Corollary 1.3. Then u is the Green potential of the density f_u. By [15, Théorème 21] (cf. [2, Corollary 9.3.8]), we see that u has minimal fine limit 0 at $\xi \in \partial\Omega \setminus E$, where the surface measure of E is zero. Let $\{x_j\}$ be arbitrary sequence in $\Gamma_\theta(\xi)$ converging to ξ. Since the bubble set $\bigcup_j B(x_j, \delta_\Omega(x_j)/8)$ is not minimally thin at ξ (cf. [12, Lemma 5.3]), we find a sequence $y_j \in B(x_j, \delta_\Omega(x_j)/8)$ converging to ξ such that $u(y_j) \to 0$ as $j \to \infty$. By Corollary 1.2,

$$0 \le u(x_j) \le Au(y_j) \to 0.$$

Thus Corollary 1.3 is proved. □

4 Proof of Theorem 1.5

Proof of Theorem 1.5. Let β be as in (1.5) and let

$$\gamma = \frac{\alpha + \beta(p-1)}{2} \quad \text{and} \quad \lambda = \alpha + \beta p. \tag{4.1}$$

Then we observe that $\gamma > 1$ and

$$\lambda < \gamma n + 1. \tag{4.2}$$

Since Ω is a $C^{1,1}$-domain, there exists a ball $B(z, \rho)$ such that $B(z, \rho) \subset \Omega$ and $\xi \in \partial B(z, \rho)$. Without loss of generality, we may assume that ξ is the origin, $z = (10, 0, \dots, 0)$ and $\rho = 10$. For $j \in \mathbb{N}$, let $x_j = (2^{-j+3}, 0, \dots, 0)$ and $r_j = 2^{-\gamma j}$. Note that $B(x_j, 8r_j) \subset \Omega$ and $B(x_j, 2r_j) \cap B(x_k, 2r_k) = \emptyset$ if $j \ne k$. Let A_1 be a constant determined in the sequel and let f_j be a nonnegative smooth function on Ω such that $f_j \le A_1 2^{\lambda j}$ and

$$f_j = \begin{cases} A_1 2^{\lambda j} & \text{on } B(x_j, r_j), \\ 0 & \text{on } \Omega \setminus B(x_j, 2r_j). \end{cases}$$

Define $f = \sum_{j=1}^{\infty} f_j$. Then, by (4.2),

$$\int_{\Omega} \delta_{\Omega}(y) f(y) dy = \sum_{j=1}^{\infty} \int_{B(x_j, 2r_j)} \delta_{\Omega}(y) f_j(y) dy$$
$$\leq A_1 \nu_n 2^{n+4} \sum_{j=1}^{\infty} 2^{j(-1+\lambda-\gamma n)} < \infty.$$

Thus $u := \int_{\Omega} G(\cdot, y) f(y) dy$ is well defined on Ω. Since f is locally Hölder continuous on Ω, it follows from [18, Theorem 6.6] that $u \in C^2(\Omega)$ is a positive solution of $-\Delta u = f$ in Ω. Also, we observe from the mean value property and (2.1) that for $x \in \partial B(x_j, 2r_j)$,

$$u(x) \geq \int_{B(x_j, r_j)} G(x, y) f_j(y) dy = A_1 2^{\lambda j} \nu_n r_j^n G(x, x_j) \geq \frac{A_1 \nu_n}{2^{n-2} A_2} 2^{j(\lambda - 2\gamma)},$$

where A_2 is a constant depending only on Ω such that $G(x, x_j) \geq A_2^{-1} |x - x_j|^{2-n}$. Let $A_3 = (A_1 \nu_n)/(2^{n-2} A_2)$. By the minimum principle,

$$u(x) \geq A_3 2^{j(\lambda - 2\gamma)} \quad \text{for } x \in B(x_j, 2r_j). \tag{4.3}$$

Hence it follows from (4.1) and $\delta_{\Omega}(x_j) = 2^{-j+3}$ that

$$u(x_j) \geq A_3 2^{j(\lambda - 2\gamma)} = A_3 2^{j\beta} = A_3 2^{3\beta} \delta_{\Omega}(x_j)^{-\beta},$$

and so u satisfies (1.6).

We finally show that $-\Delta u \leq c \delta_{\Omega}(x)^{-\alpha} u^p$ on Ω. If $x \notin \bigcup_j B(x_j, 2r_j)$, then

$$c \delta_{\Omega}(x)^{-\alpha} u(x)^p \geq 0 = f(x) = -\Delta u(x).$$

Let $x \in B(x_j, 2r_j)$. Then, by (4.3) and (4.1),

$$c \delta_{\Omega}(x)^{-\alpha} u(x)^p \geq c 2^{-4\alpha} A_3^p 2^{j(\alpha + p(\lambda - 2\gamma))} = c 2^{-4\alpha} A_3^p 2^{j\lambda}. \tag{4.4}$$

Note that if $p \neq 1$, then we can take A_1 (large enough if $p > 1$; small enough if $p < 1$) such that

$$c 2^{-4\alpha} A_3^p \geq A_1. \tag{4.5}$$

Hence we obtain

$$c \delta_{\Omega}(x)^{-\alpha} u(x)^p \geq A_1 2^{\lambda j} \geq f(x) = -\Delta u(x).$$

If $p = 1$, then the above inequality holds for $c \geq 2^{4\alpha + n - 2} A_2 / \nu_n$. For the case $c < 2^{4\alpha + n - 2} A_2 / \nu_n$, see Remark 4.1 below. The proof of Theorem 1.5 is complete. □

Remark 4.1. When $p = 1$, we assumed that c is sufficiently large: $c \geq 2^{4\alpha + n - 2} A_2 / \nu_n$. This assumption can be removed by modifying the above proof. Indeed, we need to replace (4.1) by

$$\gamma = \frac{\alpha + \beta(p-1)}{2} - \frac{\varepsilon}{2} \quad \text{and} \quad \lambda = \alpha + \beta p - \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small so that $\gamma > 1$ and (4.2) hold, and redefine f by the partial sum $\sum_{j=j_0}^{\infty} f_j$, where j_0 is sufficiently large so that

$$\frac{c2^{-4\alpha}\nu_n}{2^{n-2}A_2} 2^{j_0\varepsilon} \geq 1.$$

Then, since $\alpha + p(\lambda - 2\gamma) = \alpha + p\beta = \lambda + \varepsilon$, we can replace (4.4) by

$$c\delta_\Omega(x)^{-\alpha}u(x)^p \geq c2^{-4\alpha}A_3 2^{j(\lambda+\varepsilon)} = c2^{-4\alpha}\frac{A_1\nu_n}{2^{n-2}A_2}2^{j(\lambda+\varepsilon)} \geq A_1 2^{j\lambda},$$

whenever $j \geq j_0$.

5 The existence of positive solutions with singularity on $\partial\Omega$

In this section, we consider the existence of positive solutions, with singularity at $\xi \in \partial\Omega$, of the nonlinear elliptic equation

$$\begin{cases} -\Delta u + Vu = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{\xi\}, \end{cases} \tag{5.1}$$

where V and f are Borel measurable functions satisfying some appropriate conditions, and the equation $-\Delta u + Vu = f(x,u)$ is understood in the sense of distributions. We introduce a new class of Borel measurable functions. Let

$$H_\xi(x,y) = \frac{G(x,y)K(y,\xi)}{K(x,\xi)} \quad \text{for } x, y \in \Omega.$$

We say that a Borel measurable function φ on Ω belongs to *the generalized Kato class* $\mathcal{K}_\xi(\Omega)$ *associated with* ξ if

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(x,r)} H_\xi(x,y)|\varphi(y)|dy\right) = 0, \tag{5.2}$$

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(\xi,r)} H_\xi(x,y)|\varphi(y)|dy\right) = 0. \tag{5.3}$$

Note that the classical Kato class $K(\Omega)$ is the set of all Borel measurable functions φ on Ω satisfying

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(z,r)} \frac{|\varphi(y)|}{|x-y|^{n-2}}dy\right) = 0$$

for each $z \in \mathbb{R}^n$. In view of [7, Theorem 3.1], we see that $K(\Omega) \subset \mathcal{K}_\xi(\Omega)$. Define

$$\|\varphi\|_{\mathcal{K}_\xi(\Omega)} = \sup_{x\in\Omega}\int_\Omega H_\xi(x,y)|\varphi(y)|dy.$$

We impose the following conditions on V and f:

(A1) $V \in \mathcal{K}_\xi(\Omega)$ and $\|V\|_{\mathcal{K}_\xi(\Omega)} < 1/2$,

(A2) f is a Borel measurable function on $\Omega \times (0, \infty)$ such that $f(x, t)$ is continuous with respect to t for each $x \in \Omega$,

(A3) $|f(x, t)| \le t\psi(x, t)$, where ψ is a nonnegative Borel measurable function on $\Omega \times (0, \infty)$ such that for each $x \in \Omega$, $\psi(x, t)$ is nondecreasing with respect to t and $\psi(x, t) \to 0$ as $t \to 0$,

(A4) $\psi(x, \delta_\Omega(x)/|x - \xi|^n) \in \mathcal{K}_\xi(\Omega)$.

Theorem 5.1. *Let $\xi \in \partial\Omega$. Suppose that V and f are Borel measurable functions satisfying* (A1)—(A4). *Then* (5.1) *has infinitely many positive solutions $u \in C(\Omega)$ such that*

$$u(x) \approx \frac{\delta_\Omega(x)}{|x - \xi|^n} \quad \textit{for } x \in \Omega. \tag{5.4}$$

We shall see that Theorem 1.7 is a special case of Theorem 5.1, if $1 \le p < (n + 1)/(n - 1)$. For the case $0 < p < 1$, we need to replace (A3) by

(A3') $|f(x, t)| \le t\psi(x, t)$, where ψ is a nonnegative Borel measurable function on $\Omega \times (0, \infty)$ such that for each $x \in \Omega$, $\psi(x, t)$ is nonincreasing with respect to t and $\psi(x, t) \to 0$ as $t \to \infty$.

Theorem 5.2. *Let $\xi \in \partial\Omega$. Suppose that V and f are Borel measurable functions satisfying* (A1), (A2), (A3') *and* (A4). *Then* (5.1) *has infinitely many positive solutions $u \in C(\Omega)$ satisfying* (5.4).

Remark 5.3. If $\mathcal{K}_\xi(\Omega)$ is replaced by the classical Kato class $K(\Omega)$, then Theorems 5.1 and 5.2 do not cover Theorem 1.7. So we need to consider the generalized Kato class.

Theorems 5.1 and 5.2 will be proved by using some properties of functions in the generalized Kato class and the Schauder fixed point theorem. Note that we do not use 3G inequalities (cf. [1, 7, 10, 19]), which were applied widely to the studies of stationary Schrödinger equations and nonlinear elliptic equations (cf. [4, 6, 11, 14, 17, 22] and references therein). We start with lower and upper estimates for H_ξ.

Lemma 5.4. *Let $r > 0$ and $\xi \in \partial\Omega$. Then, for $|x - y| < r < |x - \xi|/2$,*

$$K(y, \xi)^2 \le \frac{A}{r^n} H_\xi(x, y),$$

where A depends only on Ω.

Proof. It is enough to show that for $|x - y| < r < |x - \xi|/2$,

$$G(x, y) \ge \frac{r^n}{A} K(x, \xi) K(y, \xi). \tag{5.5}$$

Let us assume first that $|x - y| \le \delta_\Omega(x)/2$. Then, by (2.1),

$$G(x, y) \ge \frac{1}{A}|x - y|^{2-n} \ge \frac{1}{A} r^{2-n}.$$

Also, since $\delta_\Omega(y) \le 2\delta_\Omega(x) \le 2|x-\xi|$ and $|x-\xi| \le 2|y-\xi|$, it follows from (2.2) that

$$K(x,\xi)K(y,\xi) \le A\frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-\xi|^n|y-\xi|^n} \le A|x-\xi|^{2(1-n)} \le Ar^{2(1-n)}.$$

Hence (5.5) holds in this case. If $|x-y| \ge \delta_\Omega(x)/2$, then we have by (2.1) and (2.2)

$$G(x,y) \ge \frac{1}{A}\frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^n} \ge \frac{r^n}{A}K(x,\xi)K(y,\xi),$$

since $|y-\xi| \ge r$. Thus the lemma is proved. $\square$

Lemma 5.5. *Let $r > 0$ and $\xi \in \partial\Omega$. Then, for $|x-y| \ge r$,*

$$H_\xi(x,y) \le \frac{A}{r^n}K(y,\xi)^2,$$

where A depends only on Ω.

Proof. By (2.1) and (2.2), we have

$$G(x,y) \le A\frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^n} \le \frac{A}{r^n}K(x,\xi)K(y,\xi),$$

since Ω is bounded. Thus the lemma follows. $\square$

Obviously, if $\varphi \in \mathcal{K}_\xi(\Omega)$, then (5.2) and (5.3) imply that for sufficiently small $\delta > 0$,

$$\begin{aligned} &\sup_{x\in\Omega}\int_{\Omega\cap B(x,\delta)} H_\xi(x,y)|\varphi(y)|dy \le \varepsilon, \\ &\sup_{x\in\Omega}\int_{\Omega\cap B(\xi,\delta)} H_\xi(x,y)|\varphi(y)|dy \le \varepsilon. \end{aligned} \tag{5.6}$$

Lemma 5.6. *If $\varphi \in \mathcal{K}_\xi(\Omega)$, then for each $r > 0$,*

$$\int_{\Omega\setminus B(\xi,r)} K(y,\xi)^2|\varphi(y)|dy < \infty.$$

Moreover, $\|\varphi\|_{\mathcal{K}_\xi(\Omega)} < \infty$.

Proof. Let $0 < \delta < r/2$ be small and let us cover $\Omega \setminus B(\xi,r)$ by finitely many balls $B(x_j,\delta)$, where $x_j \in \Omega \setminus B(\xi,r)$. By Lemma 5.4 and (5.6), we obtain

$$\int_{\Omega\setminus B(\xi,r)} K(y,\xi)^2|\varphi(y)|dy \le \frac{A}{\delta^n}\sum\int_{\Omega\cap B(x_j,\delta)} H_\xi(x_j,y)|\varphi(y)|dy < \infty.$$

Also, this and Lemma 5.5 give

$$\sup_{x\in\Omega}\int_{\Omega\setminus(B(x,\delta)\cup B(\xi,\delta))} H_\xi(x,y)|\varphi(y)|dy < \infty.$$

Combining this and (5.6), we obtain $\|\varphi\|_{\mathcal{K}_\xi(\Omega)} < \infty$. $\square$

Lemma 5.7. *If $\varphi \in \mathcal{K}_\xi(\Omega)$, then for each $z \in \overline{\Omega}$,*

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(z,r)} H_\xi(x,y)|\varphi(y)|dy\right) = 0.$$

Proof. Let $x \in \Omega$ and $r > 0$. Then, by (5.6) and Lemma 5.5,

$$\begin{aligned}\int_{\Omega\cap B(z,r)} H_\xi(x,y)|\varphi(y)|dy &\le 2\varepsilon + \int_{\Omega\cap B(z,r)\setminus(B(x,\delta)\cup B(\xi,\delta))} H_\xi(x,y)|\varphi(y)|dy\\ &\le 2\varepsilon + \frac{A}{\delta^n}\int_{\Omega\cap B(z,r)\setminus B(\xi,\delta)} K(y,\xi)^2|\varphi(y)|dy.\end{aligned}$$

In view of Lemma 5.6, we obtain the required property. □

The proofs of Theorems 5.1 and 5.2 are similar to each other. We give the proof only for Theorem 5.1. For $\lambda > 0$, we let

$$W_\lambda = \left\{ w \in C(\overline{\Omega}) : \frac{2(1-2\|V\|_{\mathcal{K}_\xi(\Omega)})}{3-2\|V\|_{\mathcal{K}_\xi(\Omega)}}\lambda \le w \le \frac{4}{3-2\|V\|_{\mathcal{K}_\xi(\Omega)}}\lambda \right\},$$

and define the operator $\mathcal{T}_\lambda$ on W_λ by

$$\mathcal{T}_\lambda w(x) = \lambda - \int_\Omega \mathcal{H}(x,y,w)dy,$$

where

$$\begin{aligned}\mathcal{H}(x,y,w) &= \frac{G(x,y)}{K(x,\xi)}\big(V(y)w(y)K(y,\xi) - f(y,w(y)K(y,\xi))\big)\\ &= H_\xi(x,y)w(y)\left(V(y) - \frac{f(y,w(y)K(y,\xi))}{w(y)K(y,\xi)}\right).\end{aligned}$$

For simplicity, we write $\varphi(y) = |V(y)| + \psi(y, \delta_\Omega(y)/|y-\xi|^n)$. Let A_4 be the constant of comparison appearing in (2.2). Then it follows from (A1), (A3), (A4) and (2.2) that $\varphi \in \mathcal{K}_\xi(\Omega)$ and that for $w \in W_\lambda$,

$$\begin{aligned}|\mathcal{H}(x,y,w)| &\le H_\xi(x,y)w(y)\left(|V(y)| + \psi(y, \frac{4\lambda}{3-2\|V\|_{\mathcal{K}_\xi(\Omega)}}\frac{A_4\delta_\Omega(y)}{|y-\xi|^n})\right)\\ &\le \frac{4\lambda}{3-2\|V\|_{\mathcal{K}_\xi(\Omega)}} H_\xi(x,y)\varphi(y),\end{aligned} \tag{5.7}$$

whenever $0 < \lambda \le (3 - 2\|V\|_{\mathcal{K}_\xi(\Omega)})/(4A_4)$.

Remark 5.8. If f satisfies (A3') instead of (A3), then

$$|\mathcal{H}(x,y,w)| \le \frac{4\lambda}{3-2\|V\|_{\mathcal{K}_\xi(\Omega)}} H_\xi(x,y)\varphi(y),$$

whenever $\lambda \ge A_4(3 - 2\|V\|_{\mathcal{K}_\xi(\Omega)})/(2 - 4\|V\|_{\mathcal{K}_\xi(\Omega)})$.

Let $\mathcal{T}_\lambda(W_\lambda) = \{\mathcal{T}_\lambda w : w \in W_\lambda\}$.

Lemma 5.9. *$\mathcal{T}_\lambda(W_\lambda)$ is equicontinuous on $\overline{\Omega}$. Moreover, $\mathcal{T}_\lambda w(x) \to \lambda$ as $x \to \xi$.*

Proof. Let $z \in \overline{\Omega} \setminus \{\xi\}$ and let $x_1, x_2 \in \Omega \cap B(z, \delta/2)$, where $0 < \delta < |z - \xi|/2$. If $\delta > 0$ is sufficiently small, then we have by (5.7) and Lemma 5.7

$$
\begin{aligned}
&|\mathcal{T}_\lambda w(x_1) - \mathcal{T}_\lambda w(x_2)| \\
&\quad \le \varepsilon + A \int_{\Omega \setminus (B(z,\delta) \cup B(\xi,\delta))} \left| \frac{G(x_1, y)}{K(x_1, \xi)} - \frac{G(x_2, y)}{K(x_2, \xi)} \right| K(y, \xi)\varphi(y) dy. \qquad (5.8)
\end{aligned}
$$

Note that if $y \in \Omega \setminus B(z, \delta)$, then $G(x, y)/K(x, \xi)$ has a finite limit as $x \to z$ (cf. [2, Theorem 8.8.6]). Since the integrand in (5.8) is bounded by a constant multiple of $K(y, \xi)^2\varphi(y)$ in view of Lemma 5.5, it follows from Lemma 5.6 and the Lebesgue convergence theorem that the second term of the right hand side in (5.8) tends to zero as $|x_1 - x_2| \to 0$. Thus $\mathcal{T}_\lambda w$ is continuous at z uniformly for $w \in W_\lambda$.

Next, let $z = \xi$. Then, by (5.7) and Lemma 5.7,

$$
|\mathcal{T}_\lambda w(x) - \lambda| \le \varepsilon + A \int_{\Omega \setminus B(\xi,\delta)} H_\xi(x, y)\varphi(y) dy.
$$

By the same reasoning as above, the second term of the right hand side tends to zero as $x \to \xi$. Thus $\mathcal{T}_\lambda w(x) \to \lambda$ uniformly for $w \in W_\lambda$ as $x \to \xi$. □

Lemma 5.10. *There exists a constant $\lambda_0 > 0$ such that if $0 < \lambda \le \lambda_0$, then $\mathcal{T}_\lambda(W_\lambda) \subset W_\lambda$. Moreover, $\mathcal{T}_\lambda(W_\lambda)$ is relatively compact in $C(\overline{\Omega})$.*

Proof. Let $w \in W_\lambda$. For $\eta > 0$, we define

$$
\Psi_\eta(x) = \int_\Omega H_\xi(x, y)\psi(y, \eta K(y, \xi)) dy.
$$

As in the proof of Lemma 5.9, we see that $\Psi_\eta \in C(\overline{\Omega})$ for sufficiently small η. Moreover, (A3) implies that for each $x \in \overline{\Omega}$, $\Psi_\eta(x) \to 0$ decreasingly as $\eta \to 0$. By the Dini theorem,

$$
\lim_{\eta \to 0} \left(\sup_{x \in \overline{\Omega}} \Psi_\eta(x) \right) = 0.
$$

Therefore there exists a constant $\lambda_0 > 0$ such that for $0 < \lambda \le \lambda_0$,

$$
\sup_{x \in \overline{\Omega}} \Psi_{(4\lambda)/(3 - 2\|V\|_{\mathcal{K}_\xi(\Omega)})}(x) \le \frac{1 - 2\|V\|_{\mathcal{K}_\xi(\Omega)}}{4}.
$$

Here we note from (A1) that the right hand side is positive. Hence

$$
\begin{aligned}
|\mathcal{T}_\lambda w(x) - \lambda| &\le \frac{4\lambda}{3 - 2\|V\|_{\mathcal{K}_\xi(\Omega)}} \left(\|V\|_{\mathcal{K}_\xi(\Omega)} + \Psi_{(4\lambda)/(3 - 2\|V\|_{\mathcal{K}_\xi(\Omega)})}(x) \right) \\
&\le \frac{1 + 2\|V\|_{\mathcal{K}_\xi(\Omega)}}{3 - 2\|V\|_{\mathcal{K}_\xi(\Omega)}} \lambda.
\end{aligned}
$$

This and $\mathcal{T}_\lambda w \in C(\overline{\Omega})$ imply that $\mathcal{T}_\lambda(W_\lambda) \subset W_\lambda$. The relative compactness follows from Lemma 5.9 and the Ascoli-Arzelá theorem. □

Remark 5.11. If f satisfies (A3') instead of (A3), then the first statement of Lemma 5.10 is replaced by that there exists a constant $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$, then $\mathcal{T}_\lambda(W_\lambda) \subset W_\lambda$.

Lemma 5.12. *If $0 < \lambda \leq \lambda_0$, then $\mathcal{T}_\lambda$ is continuous on W_λ.*

Proof. If $w_j \in W_\lambda$ converges to $w \in W_\lambda$ uniformly on Ω, then we observe from (A2) that $\mathcal{T}_\lambda w_j$ converges pointwisely to $\mathcal{T}_\lambda w$. The relative compactness of $\mathcal{T}_\lambda(W_\lambda)$ implies the uniform convergence. □

Proof of Theorem 5.1. Note that W_λ is a nonempty bounded closed convex subset of $C(\overline{\Omega})$. Since $\mathcal{T}_\lambda$ is a continuous mapping from W_λ into itself such that $\mathcal{T}_\lambda(W_\lambda)$ is relatively compact in $C(\overline{\Omega})$, it follows from the Schauder fixed point theorem (cf. [9]) that there is $w \in W_\lambda$ such that $\mathcal{T}_\lambda w = w$. Let $u(x) = w(x)K(x,\xi)$. Then $u \in C(\Omega)$ satisfies (5.4) in view of (2.2) and

$$u(x) = \lambda K(x,\xi) - \int_\Omega G(x,y)V(y)u(y)dy + \int_\Omega G(x,y)f(y,u(y))dy.$$

Therefore, using the Fubini theorem, we see that

$$\int_\Omega u(x)\Delta\phi(x)dx = \int_\Omega \big(V(y)u(y) - f(y,u(y))\big)\phi(y)dy \quad \text{for } \phi \in C_0^\infty(\Omega),$$

and so u is a distributional solution of (5.1). Moreover, we see from Lemma 5.9 that

$$\lim_{x\to\xi} \frac{u(x)}{K(x,\xi)} = \lim_{x\to\xi} \mathcal{T}_\lambda w(x) = \lambda.$$

Thus the proof of Theorem 5.1 is complete. □

6 Proof of Theorem 1.7

In this section, we prove Theorem 1.7 by applying Theorem 5.1 or 5.2.

Proof of Theorem 1.7. We first show that

$$\delta_\Omega(y)^{-\alpha}\left(\frac{\delta_\Omega(y)}{|y-\xi|^n}\right)^{p-1} \in \mathcal{K}_\xi(\Omega). \tag{6.1}$$

Suppose first that $p \geq 1$. Let $x \in \Omega$ and $r > 0$. Put

$$\begin{aligned}
E_1 &= \Omega \cap B(x,r) \cap B(x,\delta_\Omega(x)/2),\\
E_2 &= \big(\Omega \cap B(x,r) \setminus B(x,\delta_\Omega(x)/2)\big) \setminus B(\xi,|x-\xi|/2),\\
E_3 &= \big(\Omega \cap B(x,r) \setminus B(x,\delta_\Omega(x)/2)\big) \cap B(\xi,|x-\xi|/2).
\end{aligned}$$

Let

$$\varphi(y) = \delta_\Omega(y)^{-\alpha}\left(\frac{\delta_\Omega(y)}{|y-\xi|^n}\right)^{p-1}.$$

By (2.1) and (2.2),

$$H_\xi(x,y)\varphi(y) \le A\frac{1}{|x-y|^{n-2}}\frac{\delta_\Omega(y)^{p-\alpha}}{\delta_\Omega(x)}\frac{|x-\xi|^n}{|y-\xi|^{np}}$$
$$\le \frac{A}{|x-y|^{np+\alpha-1-p}} \quad \text{for } y\in E_1,$$

and

$$H_\xi(x,y)\varphi(y) \le A\frac{1}{|x-y|^{n}}\delta_\Omega(y)^{1+p-\alpha}\frac{|x-\xi|^n}{|y-\xi|^{np}}$$
$$\le \begin{cases} \dfrac{A}{|x-y|^{np+\alpha-1-p}} & \text{for } y\in E_2,\\ \dfrac{A}{|y-\xi|^{np+\alpha-1-p}} & \text{for } y\in E_3. \end{cases}$$

Note that $E_3 \neq \emptyset$ implies that $E_3 \subset B(\xi, r)$. Hence we see that φ satisfies (5.2). Also, (5.3) is shown by using (5.2). Indeed, for sufficiently small $\delta > 0$,

$$\begin{aligned}\int_{\Omega\cap B(\xi,r)} H_\xi(x,y)\varphi(y)dy &\le \varepsilon + \int_{\Omega\cap B(\xi,r)\setminus B(x,\delta)} H_\xi(x,y)\varphi(y)dy\\ &\le \varepsilon + A\frac{|x-\xi|^n}{\delta^n}\int_{B(\xi,r)}|y-\xi|^{1+p-\alpha-np}dy\\ &\le \varepsilon + \frac{A}{\delta^n}r^{n+1+p-\alpha-np}.\end{aligned}$$

Hence (6.1) holds in this case.

Suppose next that $p < 1$. Observe from (2.1) and (2.2) that

$$H_\xi(x,y)\varphi(y) \le A\frac{|x-\xi|^{n(1-p)}}{|x-y|^{n+\alpha-1-p}} \quad \text{for } y\in E_1,$$

and

$$H_\xi(x,y)\varphi(y) \le \begin{cases} A\dfrac{|x-\xi|^{n(1-p)}}{|x-y|^{n+\alpha-1-p}} & \text{for } y\in E_2,\\ A\dfrac{1}{|y-\xi|^{np+\alpha-1-p}} & \text{for } y\in E_3. \end{cases}$$

The same reasoning as above yields (6.1).

Now, let us apply Theorem 5.1 or 5.2.

Case 1: $p \neq 1$. Since $V \equiv 0$ and $f(x,t) = g(x)t^p$ fulfill (A1), (A2), (A4) and either (A3) or (A3'), it follows that (1.7) has infinitely many (distributional) positive solutions $u \in C(\Omega)$ satisfying (1.8). The local boundedness of u and (1.7) yield that $u \in C^1(\Omega)$ (cf. [18, Theorem 6.6]). Since gu^p is locally Hölder continuous on Ω, we conclude that $u \in C^2(\Omega)$ and $-\Delta u = gu^p$ in Ω.

Case 2: $p = 1$. Since $\varphi \in \mathcal{K}_\xi(\Omega)$, it follows from Lemma 5.6 that $\|\varphi\|_{\mathcal{K}_\xi(\Omega)} < \infty$. Therefore if $0 < c < 1/(2\|\varphi\|_{\mathcal{K}_\xi(\Omega)})$, then $\|g\|_{\mathcal{K}_\xi(\Omega)} < 1/2$. Applying Theorem 5.1

with $V = g$ and $f \equiv 0$ and repeating the same argument as above, we conclude that $-\Delta u = gu$ has infinitely many positive solutions $u \in C^2(\Omega)$ satisfying (1.8).

This completes the proof of Theorem 1.7. $\square$

We finally note that the condition $p < (n+1)/(n-1)$ in Theorem 1.7 is sharp.

Theorem 6.1. *Let $\xi \in \partial\Omega$ and $c > 0$. Suppose that $p \geq 1$ and $\alpha \geq n+1-p(n-1)$. Then*

$$-\Delta u = c\delta_\Omega(x)^{-\alpha}u^p \quad \textit{in } \Omega \tag{6.2}$$

has no positive solutions satisfying (1.8).

Proof of Theorem 6.1. Suppose to the contrary that there exists a positive solution u of (6.2) satisfying (1.8). Then it follows from (2.3), (2.1) and (1.8) that for $x \in \Omega$,

$$\begin{aligned} u(x) &\geq \int_{\Omega\setminus B(x,\delta_\Omega(x)/2)} G(x,y)(-\Delta u(y))dy \\ &\geq \frac{1}{A}\int_{\Omega\setminus B(x,\delta_\Omega(x)/2)} \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^n}\delta_\Omega(y)^{-\alpha}\left(\frac{\delta_\Omega(y)}{|y-\xi|^n}\right)^p dy \\ &\geq \frac{1}{A}\frac{\delta_\Omega(x)}{(\operatorname{diam}\Omega)^n}\int_{\Gamma_\theta(\xi)\setminus B(x,\delta_\Omega(x)/2)} \frac{1}{|y-\xi|^{np+\alpha-p-1}}dy. \end{aligned}$$

Since $np+\alpha-p-1 \geq n$, we conclude that $u \equiv \infty$ which is a contradiction. $\square$

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