On the existence of positive solutions of singular nonlinear elliptic equations with Dirichlet boundary conditions

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Abstract

This paper is concerned with the existence of positive solutions of the singular nonlinear elliptic equation with a Dirichlet boundary condition

$$\begin{cases} \Delta u = F(x,u) & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases}$$

where F is a Borel measurable function in $\Omega \times (0, +\infty)$ such that $|F(x,u)| \leq V(x)u^{-\alpha}$ for some $\alpha > 0$ and V satisfying some appropriate conditions. In particular, we show that the above problem has positive solutions whenever $\inf_{\partial\Omega} \phi$ is greater than a positive quantity given by α and V.

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1 Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\partial^\infty\Omega$ denote the boundary of Ω in the one-point compactification $\mathbb{R}^n \cup \{\infty\}$. In this paper, we study the existence of positive continuous solutions of the following nonlinear elliptic equation with a Dirichlet boundary condition:

$$\begin{cases} \Delta u = F(x,u) & \text{in } \Omega, \\ u = \phi & \text{on } \partial^\infty\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator on $\mathbb{R}^n$, F is a Borel measurable function in $\Omega \times (0, +\infty)$ and ϕ is a nonnegative continuous function on $\partial^\infty\Omega$. The equation $\Delta u = F(x,u)$ is understood in the sense of distributions. In the case that F is negative, we can expect the existence of positive solutions of (1.1) even if ϕ is identical to zero. This case was investigated by many authors [6, 7, 8, 9, 11] in smooth domains or in $\mathbb{R}^n$. In contrast to this, the case that F is nonnegative and ϕ is identical to zero does not guarantee the existence of positive solutions of (1.1), because every positive solution takes its maximum on $\partial^\infty\Omega$. The question is for what ϕ does (1.1) have positive solutions? In [4], Chen, Williams and Zhao studied it in the case when Ω is a Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, with compact boundary, and $|F(x,u)| \leq V(x)u^p$ with $p \geq 1$ and V being the Green-tight function on Ω. They showed that if ϕ is not identical to zero and its supremum norm is small (i.e. bounded by a constant depending only on p, V and Ω), then (1.1) has at least one positive solution. However, in the singular case $p < 0$, the smallness of the supremum norm of ϕ does not imply the existence of positive solutions (see Proposition 3.1 below). In [1], Athreya studied (1.1) with the singular nonlinearity $F(x,u) = u^{-\alpha}$, $0 < \alpha < 1$, in a bounded C^2-domain in $\mathbb{R}^n$, $n \geq 3$. He showed the existence of solutions, bounded below by a given positive harmonic function h_0, under the boundary condition $\phi \geq (1+A)h_0$ with a constant A depending on h_0, α and Ω.

The purpose of this paper is to give the Chen-Williams-Zhao type theorem for a singular nonlinear term $F(x,u)$. More precisely, we shall show that (1.1) has positive solutions whenever $\inf_{\partial^\infty\Omega}\phi$ is greater than a positive quantity depending on F. We impose no assumptions on a domain Ω other than the existence of the Green function and being regular for the Dirichlet problem. Such a domain will be called a *Dirichlet regular* domain. Note that any domains possess the Green function when $n \geq 3$. Let G_Ω stand for the Green function of Ω, i.e., for each $y \in \Omega$, the function $G_\Omega(\cdot,y)$ is a distributional solution of

$$\begin{cases} -\Delta G_\Omega(\cdot,y) = \delta_y & \text{in } \Omega, \\ G_\Omega(\cdot,y) = 0 & \text{on } \partial^\infty\Omega, \end{cases}$$

where δ_y is the Dirac measure at y. By $B(x,r)$ we denote the open ball of center x and radius r. We say that a Borel measurable function f in Ω belongs to $\mathcal{G}(\Omega)$ if, for each $z \in \Omega \cup \partial\Omega$,

$$\lim_{r\to 0}\left(\sup_{x\in\Omega\cap B(z,r)}\int_{\Omega\cap B(z,r)} G_\Omega(x,y)|f(y)|dy\right) = 0, \tag{1.2}$$

and

$$\lim_{R\to+\infty}\left(\sup_{x\in\Omega}\int_{\Omega\setminus B(0,R)} G_\Omega(x,y)|f(y)|dy\right) = 0 \quad \text{(when } \Omega \text{ is unbounded).} \tag{1.3}$$

We define

$$\|f\|_{\mathcal{G}(\Omega)} = \sup_{x\in\Omega}\int_\Omega G_\Omega(x,y)|f(y)|dy.$$

Considering a finite covering of Ω if it is bounded (or $\Omega \cap B(0,R)$ if unbounded), we see that $\|f\|_{\mathcal{G}(\Omega)} < \infty$ whenever $f \in \mathcal{G}(\Omega)$. Note in the case $n \geq 3$ that f is a Green-tight function if and only if it satisfies (1.2) and (1.3) with the Newtonian kernel $|x-y|^{2-n}$ instead of the Green function. Thus all Green-tight functions belong to $\mathcal{G}(\Omega)$. Let $\omega(x,E,D)$ be the harmonic measure of a set E relative to D evaluated at x, and let us define $L^1_\omega(\Omega)$ as the class of every Borel measurable function f in Ω

satisfying

$$\int_{\Omega\setminus B(z,2r)} |f(y)|\omega\big(y, \partial B(z,r)\cap\Omega, \Omega\setminus \overline{B(z,r)}\big)dy < \infty$$

for each $z\in\overline{\Omega}$ and small $0<r<r_z$. Since $0\le\omega\le 1$, we see that $L^1(\Omega)\subset L^1_\omega(\Omega)$. See also Section 3.2.

We consider the following singular nonlinear term $F(x,t)$ defined in $\Omega\times(0,+\infty)$:

(I) $F(x,t)$ is continuous with respect to t for each $x\in\Omega$,

(II) $0\ne |F(x,t)|\le V(x)t^{-\alpha}$ for a.e. $x\in\Omega$ and $t>0$, where $\alpha>0$ and $V\in\mathcal{G}(\Omega)\cap L^1_\omega(\Omega\cap B(0,R))$ for each $R>0$.

Note that $|F|\ne 0$ implies $\|V\|_{\mathcal{G}(\Omega)}>0$. Our results are as follows.

Theorem 1.1. *Let Ω be an arbitrary Dirichlet regular domain in $\mathbb{R}^n$, $n\ge 2$. Suppose that F is a Borel measurable function in $\Omega\times(0,+\infty)$ satisfying* (I) *and* (II). *Then, for every continuous function ϕ on $\partial^\infty\Omega$ satisfying*

$$\inf_{\partial^\infty\Omega}\phi \ge \frac{1+\alpha}{\alpha^{\alpha/(1+\alpha)}}\|V\|_{\mathcal{G}(\Omega)}^{1/(1+\alpha)}, \tag{1.4}$$

the Dirichlet problem (1.1) *has at least one positive continuous solution u such that*

$$\inf_{\Omega} u \ge (\alpha\|V\|_{\mathcal{G}(\Omega)})^{1/(1+\alpha)}.$$

If $V(x)=\kappa$, then we can estimate $\|\kappa\|_{\mathcal{G}(\Omega)}$.

Corollary 1.2. *Suppose that Ω is a Dirichlet regular domain in $\mathbb{R}^n$, $n\ge 2$, with the same volume as the unit ball. Let $\alpha>0$ and $\kappa>0$ be constants. Then, for every continuous function ϕ on $\partial^\infty\Omega$ satisfying*

$$\inf_{\partial^\infty\Omega}\phi \ge \frac{1+\alpha}{\alpha^{\alpha/(1+\alpha)}}\left(\frac{\kappa}{2n}\right)^{1/(1+\alpha)}, \tag{1.5}$$

the Dirichlet problem

$$\begin{cases} \Delta u = \kappa u^{-\alpha} & \text{in } \Omega,\\ u=\phi & \text{on } \partial^\infty\Omega, \end{cases} \tag{1.6}$$

has at least one positive C^2-solution.

We do not know whether the bound (1.5) is sharp to guarantee the existence of positive solutions of (1.6). However we will see that (1.6) does not have positive solutions for any small boundary data ϕ (see Proposition 3.1). As another special case of Theorem 1.1, we obtain the following.

Corollary 1.3. *Let B be the unit ball of $\mathbb{R}^n$, $n\ge 2$, and $\alpha>0$. Then, for every continuous function ϕ on ∂B satisfying*

$$\inf_{\partial B}\phi \ge \frac{(1+\alpha)}{\alpha^{\alpha/(1+\alpha)}}8^{1/(1+\alpha)},$$

the Dirichlet problem

$$\begin{cases} \Delta u(x) = \dfrac{1}{(1-|x|)u(x)^\alpha} & \text{in } B, \\ u = \phi & \text{on } \partial B, \end{cases}$$

has at least one positive C^2-solution.

2 Proofs of Theorem 1.1 and Corollary 1.2

In the proof, we note that $0 < \|V\|_{\mathcal{G}(\Omega)} < \infty$. Let $C(\overline{\Omega})$ denote the Banach space of all bounded continuous functions in $\overline{\Omega} = \Omega \cup \partial^\infty \Omega$ equipped with the supremum norm $\|\cdot\|_\infty$. We define

$$\mathcal{U} = \left\{ u \in C(\overline{\Omega}) : \big(\alpha \|V\|_{\mathcal{G}(\Omega)}\big)^{1/(1+\alpha)} \le u \le \|\phi\|_\infty + \left(\frac{1}{\alpha}\right)^{\alpha/(1+\alpha)} \|V\|_{\mathcal{G}(\Omega)}^{1/(1+\alpha)} \right\}.$$

Since $\alpha^{1/(1+\alpha)} < \alpha^{-\alpha/(1+\alpha)}$ for $0 < \alpha < 1$, we see that $\mathcal{U}$ is non-empty bounded closed convex subset of $C(\overline{\Omega})$. Let $\mathcal{T}$ be the operator on $\mathcal{U}$ defined by

$$\mathcal{T}u(x) = H_\phi(x) - \int_\Omega G_\Omega(x,y) F(y, u(y)) dy, \tag{2.1}$$

where H_ϕ is the unique (Perron-Wiener-Brelot) solution of

$$\begin{cases} \Delta h = 0 & \text{in } \Omega, \\ h = \phi & \text{on } \partial^\infty \Omega. \end{cases}$$

Write $\mathcal{T}(\mathcal{U}) = \{\mathcal{T}u : u \in \mathcal{U}\}$. Note from (II) that

$$|F(y, u(y))| \le V(y) \big(\alpha \|V\|_{\mathcal{G}(\Omega)}\big)^{-\alpha/(1+\alpha)} \quad \text{for a.e. } y \in \Omega, \tag{2.2}$$

whenever $u \in \mathcal{U}$.

Lemma 2.1. *The family $\mathcal{T}(\mathcal{U})$ is equicontinuous in $\overline{\Omega}$. Moreover, if $u \in \mathcal{U}$, then $\mathcal{T}u = \phi$ on $\partial^\infty \Omega$.*

Proof. Let $z \in \Omega$ and let $x_1, x_2 \in \Omega \cap B(z, r/2)$. By (2.2),

$$\begin{aligned} |\mathcal{T}u(x_1) - \mathcal{T}u(x_2)| \le |H_\phi(x_1) - H_\phi(x_2)| \\ + \big(\alpha \|V\|_{\mathcal{G}(\Omega)}\big)^{-\alpha/(1+\alpha)} \int_\Omega |G_\Omega(x_1, y) - G_\Omega(x_2, y)| V(y) dy. \end{aligned}$$

If $r > 0$ is sufficiently small, then it follows from (1.2) and (1.3) that

$$\begin{aligned} &\int_\Omega |G_\Omega(x_1, y) - G_\Omega(x_2, y)| V(y) dy \\ &\quad \le \varepsilon + \int_{\Omega \cap B(0,1/r) \setminus B(z,2r)} |G_\Omega(x_1, y) - G_\Omega(x_2, y)| V(y) dy. \end{aligned} \tag{2.3}$$

Since G_Ω is continuous on $\left(\overline{\Omega \cap B(z,r/2)}\right) \times \left(\overline{\Omega} \cap \partial B(z,r)\right)$, it takes the maximum M on there. By the maximum principle, $G_\Omega(x_i,y) \le M\omega\big(y, \partial B(z,r) \cap \Omega, \Omega \setminus \overline{B(z,r)}\big)$ for $i = 1,2$ and $y \in \Omega \setminus B(z,r)$. Therefore the integrand of the right hand side in (2.3) is bounded by a constant multiple of $V \in L^1_\omega(\Omega \cap B(0,1/r))$. The Lebesgue convergence theorem implies that $\int_\Omega |G_\Omega(x_1,y) - G_\Omega(x_2,y)|V(y)dy \to 0$ as $|x_1 - x_2| \to 0$. Hence $\mathcal{T}u$ is continuous at z uniformly for $u \in \mathcal{U}$. Since $G_\Omega(\cdot,y) = 0$ and $H_\phi = \phi$ on $\partial^\infty\Omega$, the similar argument shows that $\mathcal{T}u(x) \to \phi(x)$ uniformly for $u \in \mathcal{U}$ as $x \to \partial^\infty\Omega$. $\square$

Lemma 2.2. *The operator $\mathcal{T}$ is a continuous mapping from $\mathcal{U}$ into itself such that $\mathcal{T}(\mathcal{U})$ is relatively compact in $C(\overline{\Omega})$.*

Proof. We first show that $\mathcal{T}(\mathcal{U}) \subset \mathcal{U}$. Let $u \in \mathcal{U}$. Then, by (1.4) and (2.2),

$$\begin{aligned}\mathcal{T}u(x) &\ge \frac{1+\alpha}{\alpha^{\alpha/(1+\alpha)}}\|V\|_{\mathcal{G}(\Omega)}^{1/(1+\alpha)} - \left(\frac{1}{\alpha\|V\|_{\mathcal{G}(\Omega)}}\right)^{\alpha/(1+\alpha)} \int_\Omega G_\Omega(x,y)V(y)dy \\ &\ge \big(\alpha\|V\|_{\mathcal{G}(\Omega)}\big)^{1/(1+\alpha)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}u(x) &\le \|\phi\|_\infty + \left(\frac{1}{\alpha\|V\|_{\mathcal{G}(\Omega)}}\right)^{\alpha/(1+\alpha)} \int_\Omega G_\Omega(x,y)V(y)dy \\ &\le \|\phi\|_\infty + \left(\frac{1}{\alpha}\right)^{\alpha/(1+\alpha)} \|V\|_{\mathcal{G}(\Omega)}^{1/(1+\alpha)}.\end{aligned}$$

Since $\mathcal{T}u \in C(\overline{\Omega})$ by Lemma 2.1, we obtain $\mathcal{T}(\mathcal{U}) \subset \mathcal{U}$. Moreover, the Ascoli-Arzelá theorem yields that $\mathcal{T}(\mathcal{U})$ is relatively compact in $C(\overline{\Omega})$.

We next show that $\mathcal{T}$ is continuous on $\mathcal{U}$. Let $\{u_j\}$ be a sequence in $\mathcal{U}$ converging to $u \in \mathcal{U}$ with respect to $\|\cdot\|_\infty$. Then

$$|\mathcal{T}u_j(x) - \mathcal{T}u(x)| \le \int_\Omega G_\Omega(x,y)|F(y,u_j(y)) - F(y,u(y))|dy.$$

In view of (I) and (2.2), it follows from the Lebesgue convergence theorem that $\mathcal{T}u_j(x)$ converges pointwisely to $\mathcal{T}u(x)$ as $j \to \infty$. Hence the relative compactness of $\mathcal{T}(\mathcal{U})$ concludes that $\|\mathcal{T}u_j - \mathcal{T}u\|_\infty \to 0$ as $j \to \infty$. $\square$

Proof of Theorem 1.1. By the Schauder fixed point theorem, there exists $u \in \mathcal{U}$ such that $\mathcal{T}u = u$. We see from the Fubini theorem that

$$\int_\Omega u(x)\Delta\psi(x)dx = \int_\Omega F(y,u(y))\psi(y)dy \quad \text{for } \psi \in C_0^\infty(\Omega),$$

so that u is a distributional solution of $\Delta u = F(x,u)$ in Ω satisfying $u \ge (\alpha\|V\|_{\mathcal{G}(\Omega)})^{1/(1+\alpha)}$. Also, we have by Lemma 2.1 that $u = \phi$ on $\partial^\infty\Omega$. This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.2. Suppose that the volume of Ω is same as the unit ball B. Then the following isoperimetric inequality for the Green function holds (see [3, p. 61]):

$$\sup_{x\in\Omega}\int_\Omega G_\Omega(x,y)dy \le \int_B G_B(0,y)dy = \frac{1}{2n}.$$

Hence $\|\kappa\|_{\mathcal{G}(\Omega)} \le \kappa/2n$. Therefore, if (1.5) is satisfied, then Theorem 1.1 shows that there exists $u \in C(\overline{\Omega})$ such that $u \ge (\alpha\|\kappa\|_{\mathcal{G}(\Omega)})^{1/(1+\alpha)}$ and

$$u(x) = H_\phi(x) - \kappa\int_\Omega G_\Omega(x,y)u(y)^{-\alpha}dy.$$

Since $u^{-\alpha}$ is bounded in Ω, it follows from [10, Theorem 6.6] that $u \in C^1(\Omega)$, and so $u^{-\alpha} \in C^1(\Omega)$. In particular, $u^{-\alpha}$ is locally Hölder continuous in Ω, which concludes that $u \in C^2(\Omega)$ and $\Delta u = \kappa u^{-\alpha}$ in Ω. Thus Corollary 1.2 is proved. □

3 Remarks and Proof of Corollary 1.3

3.1 Nonexistence of positive solutions

The following proposition shows that any small boundary data do not guarantee positive solutions of (1.6).

Proposition 3.1. *Suppose that Ω is a Dirichlet regular domain in $\mathbb{R}^n$, $n \ge 2$, containing the unit ball B. Let $\alpha > 0$ and $\kappa > 0$ be constants. Then, for every continuous function ϕ on $\partial^\infty\Omega$ satisfying*

$$\sup_{\partial^\infty\Omega}\phi \le \left(\frac{\kappa}{2n}\right)^{1/(1+\alpha)},$$

the Dirichlet problem (1.6) *has no positive solutions.*

Proof. Suppose to the contrary that (1.6) has a positive solution u. Then the Riesz decomposition theorem for a subharmonic function yields that

$$0 < u(0) = H_\phi(0) - \kappa\int_\Omega G_\Omega(0,y)u(y)^{-\alpha}dy,$$

where H_ϕ is the harmonic function in Ω determined by ϕ. Observe from the maximum principle that $u \le H_\phi \le (\kappa/2n)^{1/(1+\alpha)}$ in Ω. Therefore

$$\kappa\left(\frac{2n}{\kappa}\right)^{\alpha/(1+\alpha)}\int_B G_B(0,y)dy \le \kappa\int_\Omega G_\Omega(0,y)u(y)^{-\alpha}dy < \left(\frac{\kappa}{2n}\right)^{1/(1+\alpha)},$$

and so

$$\frac{1}{2n} = \int_B G_B(0,y)dy < \frac{1}{2n}.$$

This is a contradiction. □

3.2 Remarks on $\mathcal{G}(\Omega)$ and $L^1_\omega(\Omega)$

Let $\delta_\Omega(x)$ denote the distance from x to the boundary of Ω. For two positive functions f and g in Ω, we write $f \approx g$ if there exists a constant A depending only on Ω such that $f/A \le g \le Af$. Note that if Ω is a bounded Lipschitz domain, then $\mathcal{G}(\Omega) \subset L^1_\omega(\Omega)$. Indeed, let $z \in \overline{\Omega}$ and $r > 0$ be small. The boundary Harnack principle (cf. [2]) yields that there is a constant $A(r, \Omega)$ such that

$$\omega\big(y, \partial B(z,r) \cap \Omega, \Omega \setminus \overline{B(z,r)}\big) \le A(r,\Omega) G_\Omega(x_0, y) \quad \text{for } y \in \Omega \setminus B(z, 2r),$$

where $x_0 \in \Omega$ is fixed. Therefore, if $f \in \mathcal{G}(\Omega)$, then

$$\int_{\Omega \setminus B(z,2r)} |f(y)| \omega\big(y, \partial B(z,r) \cap \Omega, \Omega \setminus \overline{B(z,r)}\big) dy \le A \|f\|_{\mathcal{G}(\Omega)} < \infty,$$

and so $f \in L^1_\omega(\Omega)$. Also, $\mathcal{G}(\Omega)$ is strictly bigger than the Green-tight class.

Example 3.2. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \ge 2$. Then δ_Ω^{-1}belongs to $\mathcal{G}(\Omega)$, but not in the Green-tight class.*

Proof. It is known from [5, 12] that

$$G_\Omega(x,y) \approx \begin{cases} \min\left\{1, \dfrac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right\} |x-y|^{2-n} & \text{if } n \ge 3, \\ \log\left(1 + \dfrac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right) & \text{if } n = 2. \end{cases} \tag{3.1}$$

Let $z \in \overline{\Omega}$ and $r > 0$. When $n \ge 3$, we have for $x \in \Omega \cap B(z,r)$,

$$\begin{aligned} &\int_{\Omega \cap B(z,r)} G_\Omega(x,y) \delta_\Omega(y)^{-1} dy \\ &\le \int_{B(x,2r) \cap B(x, \delta_\Omega(x)/2)} \frac{A\delta_\Omega(x)^{-1} dy}{|x-y|^{n-2}} + \int_{B(x,2r) \setminus B(x,\delta_\Omega(x)/2)} \frac{A dy}{|x-y|^{n-1}} \\ &\le Ar, \end{aligned}$$

and so (1.2) is satisfied. Similarly, we obtain this for $n = 2$. Hence $\delta_\Omega^{-1} \in \mathcal{G}(\Omega)$. On the other hand, it is not difficult to see that the Green-tight class is a subset of $L^1(\Omega)$, and that $\delta_\Omega^{-1} \notin L^1(\Omega)$. □

Let us give a proof of Corollary 1.3.

Proof of Corollary 1.3. Instead of (3.1), we use

$$G_B(x,y) \le \begin{cases} \dfrac{2}{\sigma_n} \dfrac{\delta_B(x)\delta_B(y)}{|x-y|^n} & \text{if } y \notin B(x, \delta_B(x)/2), \\ \dfrac{1}{\sigma_n(n-2)} |x-y|^{2-n} & \text{if } n \ge 3 \text{ and } y \in B(x, \delta_B(x)/2), \\ \dfrac{1}{\sigma_2} \log\left(\dfrac{5\delta_B(x)}{2|x-y|}\right) & \text{if } n = 2 \text{ and } y \in B(x, \delta_B(x)/2), \end{cases}$$

where σ_n denotes the surface area of the unit sphere of $\mathbb{R}^n$. In the same way as above, we obtain $\|\delta_B^{-1}\|_{\mathcal{G}(\Omega)} \leq 8$. Hence the conclusion follows from Theorem 1.1. $\square$

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