Boundary behavior of solutions of the Helmholtz equation *

Kentaro Hirata

Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan
e-mail: hirata@math.sci.hokudai.ac.jp

Abstract

This paper is concerned with the boundary behavior of solutions of the Helmholtz equation in $\mathbb{R}^n$. In particular, we give a Littlewood-type theorem to show that the approach region introduced by Korányi and Taylor (1983) is best possible.

Keywords: boundary behavior, Helmholtz equation
Mathematics Subject Classifications (2000): 31B25, 35J05

1 Introduction

Let $n \geq 2$ and let us denote a typical point in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$. The usual inner product and norm are written respectively as $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ and $|x| = \sqrt{\langle x, x \rangle}$. The symbol $O(n)$ stands for the set of all orthogonal transformations on $\mathbb{R}^n$. Let $\lambda > 0$. We consider the Helmholtz equation

$$\Delta u = \lambda^2 u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. It is known that the Martin boundary for positive solutions of (1.1) can be identified with the unit sphere S of $\mathbb{R}^n$, and that every positive solution u of (1.1) can be represented as $u = K\mu$ for some Radon measure μ on S, where

$$K\mu(x) = \int_S e^{\lambda \langle x, y \rangle} d\mu(y) \quad \text{for } x \in \mathbb{R}^n. \tag{1.2}$$

See [4, Corollary to Theorem 4] and [9]. Let σ denote the surface measure on S. Since $K\sigma(x) \to +\infty$ as $x \to \infty$ (cf. Lemma 2.1), we investigate the behavior at infinity of the normalization $K\mu/K\sigma$. Let $e = (1, 0, \ldots, 0)$ and let Ω be an unbounded

*This work was partially supported by Grant-in-Aid for Young Scientists (B) (No. 19740062), Japan Society for the Promotion of Science.

Current address: Faculty of Education and Human Studies, Akita University, Akita 010-8502, Japan
e-mail: hirata@math.akita-u.ac.jp

subset of $\mathbb{R}^n$ *converging to* e *at* ∞ in the sense that $|x/|x| - e| \to 0$ as $x \to \infty$ within Ω. We write $\Omega(y)$ for the image of Ω under an element of $O(n)$ mapping e to y. Then $\{\Omega(y) : y \in S\}$ makes a collection of approach regions. By the notation $\Omega(y) \ni x \to \infty$, we mean that $x \to \infty$ within $\Omega(y)$. Korányi and Taylor [9] considered the following approach region. For $\alpha > 0$ and $y \in S$, define

$$\mathcal{A}_\alpha(y) = \left\{ x \in \mathbb{R}^n : \big|x - |x|y\big| \le \alpha\sqrt{|x|} \right\}.$$

Theorem A. *Let $\alpha > 0$ and let μ be a Radon measure on S. Then*

$$\lim_{\mathcal{A}_\alpha(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = \frac{d\mu}{d\sigma}(y) \quad \text{for } \sigma\text{-a.e. } y \in S.$$

This result corresponds to Fatou's theorem [5] for the boundary behavior of harmonic functions in the unit ball or the upper half space of $\mathbb{R}^n$ (see also [8, 12] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$). The result corresponding to Nagel–Stein's theorem [11] was established by Berman and Singman [3] and Gowrisankaran and Singman [6]. These results show that there exists an unbounded subset Ω of $\mathbb{R}^n$ converging to e at ∞ such that

$$\limsup_{\Omega \ni x \to \infty} \frac{\big|x - |x|e\big|}{\sqrt{|x|}} = +\infty$$

and that

$$\lim_{\Omega(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = \frac{d\mu}{d\sigma}(y) \quad \text{for } \sigma\text{-a.e. } y \in S,$$

whenever μ is a Radon measure on S. Berman and Singman also showed its converse (see [3, Theorem B and Remark 1. 13(a)]).

Theorem B. *Let Ω be an unbounded subset of $\mathbb{R}^n$ converging to e at ∞ and satisfying*

$$\limsup_{\Omega \ni x \to \infty} \frac{\big|x - |x|e\big|}{\sqrt{|x|}} = +\infty. \tag{1.3}$$

Suppose in addition that Ω is invariant under all elements of $O(n)$ that preserve the point e. Then there exists a Radon measure μ on S such that

$$\limsup_{\Omega(y) \ni x \to \infty} \frac{K\mu}{K\sigma}(x) = +\infty \quad \text{for every } y \in S.$$

Note that the second assumption on Ω can not be omitted from their construction even if "$\limsup$" in (1.3) is replaced by "$\lim$".

The purpose of this paper is to show the following Littlewood-type theorem. See [10, 1, 2, 7] for harmonic or invariant harmonic functions.

Theorem 1.1. *Let γ be a curve in $\mathbb{R}^n$ converging to e at ∞ and satisfying*

$$\lim_{\gamma \ni x \to \infty} \frac{\big|x - |x|e\big|}{\sqrt{|x|}} = +\infty. \tag{1.4}$$

Then there exists a solution u of (1.1) *such that $u/K\sigma$ is bounded in $\mathbb{R}^n$ and that $u/K\sigma$ admits no limits as $x \to \infty$ along $T\gamma$ for every $T \in O(n)$.*

Remark 1.2. We indeed construct u satisfying $-1 \leq u/K\sigma \leq 1$ and

$$\liminf_{T\gamma \ni x \to \infty} \frac{u}{K\sigma}(x) = -1 \quad \text{and} \quad \limsup_{T\gamma \ni x \to \infty} \frac{u}{K\sigma}(x) = 1$$

for every $T \in O(n)$. Note that "lim" in (1.4) can not be replaced by "$\limsup$" as mentioned above (cf. [3, 6]).

The proof of Theorem 1.1 is based on our previous work [7] for invariant harmonic functions in the unit ball of $\mathbb{C}^n$, which was a refinement of Aikawa's method [1, 2] for harmonic functions in the unit disc or the upper half space of $\mathbb{R}^n$. In Section 4, we remark that our construction and estimates are applicable to show the analogue of Theorem B.

2 Lemmas

The symbol A denotes an absolute positive constant depending only on λ and the dimension n, and may change from line to line. The following estimate is found in [3, Lemma 4.1].

Lemma 2.1. *There exists a constant $A > 1$ such that*

$$\frac{1}{A} e^{\lambda|x|} |x|^{(1-n)/2} \leq K\sigma(x) \leq A e^{\lambda|x|} |x|^{(1-n)/2}$$

whenever $|x| \geq 1$.

The surface ball of center $y \in S$ and radius $r > 0$ is denoted by

$$Q(y, r) = \{x \in S : |x - y| < r\}.$$

Then we observe that

$$\lim_{r \to 0} \frac{\sigma(Q(y, r))}{r^{n-1}} = \nu_{n-1}, \tag{2.1}$$

where ν_{n-1} is the volume of the unit ball of $\mathbb{R}^{n-1}$. Moreover, there exists a constant $A > 1$ such that

$$\frac{1}{A} r^{n-1} \leq \sigma(Q(y, r)) \leq A r^{n-1} \quad \text{for } 0 < r \leq 2. \tag{2.2}$$

Let π be the radial projection onto S, i.e., $\pi(x) = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$. For a Radon measure μ on S, we define the maximal function $M_{(c)}\mu$ with parameter $c \geq 1$ by

$$M_{(c)}\mu(x) = \sup \left\{ \frac{\mu(Q(\pi(x), r))}{r^{n-1}} : r \geq \frac{c}{\sqrt{|x|}} \right\}.$$

Lemma 2.2. *Let $c \geq 1$ and let μ be a Radon measure on S. Then*

$$\frac{K\mu}{K\sigma}(x) \leq A \left(|x|^{(n-1)/2} \mu\big(Q(\pi(x), c/\sqrt{|x|})\big) + \frac{1}{c} M_{(c)}\mu(x) \right)$$

whenever $|x| \geq 1$.

Proof. Let $|x| \geq 1$. Since $|x| - \langle x, y\rangle = |x||\pi(x) - y|^2/2$ for $y \in S$, it follows from Lemma 2.1 that

$$\frac{K\mu}{K\sigma}(x) \leq A|x|^{(n-1)/2} \int_S e^{-(\lambda/2)|x||\pi(x)-y|^2} d\mu(y). \tag{2.3}$$

Let $Q_1 = Q(\pi(x), c/\sqrt{|x|})$ and $Q_j = Q(\pi(x), jc/\sqrt{|x|}) \setminus Q(\pi(x), (j-1)c/\sqrt{|x|})$ for $j = 2, \dots, N$, where N is the smallest integer such that $Nc/\sqrt{|x|} > 2$. Then, for $j = 1, \dots, N$,

$$\int_{Q_j} e^{-(\lambda/2)|x||\pi(x)-y|^2} d\mu(y) \leq e^{-(\lambda/2)((j-1)c)^2} \mu\big(Q(\pi(x), jc/\sqrt{|x|})\big).$$

Therefore the right hand side of (2.3) is bounded by

$$A\left(|x|^{(n-1)/2}\mu\big(Q(\pi(x), c/\sqrt{|x|})\big) + \sum_{j\geq 2} e^{-(\lambda/2)((j-1)c)^2}(jc)^{n-1}M_{(c)}\mu(x)\right).$$

Since $\sum_{j\geq 2} e^{-(\lambda/2)((j-1)c)^2}(jc)^{n-1} \leq A/c$, we obtain the required estimate. $\square$

For an integrable function f on S, we write $Kf = K(f d\sigma)$ and $M_{(c)}f = M_{(c)}(|f|d\sigma)$.

Lemma 2.3. *The following statements hold.*

(i) *Let μ be a Radon measure on S. Then*

$$\frac{K\mu}{K\sigma}(x) \leq AM_{(1)}\mu(x)$$

whenever $|x| \geq 1$.

(ii) *Let $y \in S$, $0 < r < 1$ and $c \geq 1$. Suppose that f is a Borel measurable function on S such that $f = 1$ on $Q(y, cr)$ and $|f| \leq 1$ on S. Then*

$$\frac{Kf}{K\sigma}(ty) \geq 1 - \frac{A}{c}$$

whenever $\sqrt{t} \geq 1/r$.

Proof. Lemma 2.2 with $c = 1$ gives (i). To show (ii), let $g = (1-f)/2$. Then $g = 0$ on $Q(y, cr)$ and $|g| \leq 1$ on S. Observe from Lemma 2.2 and (2.2) that if $\sqrt{t} \geq 1/r$, then

$$\frac{Kg}{K\sigma}(ty) \leq \frac{A}{c}M_{(c)}g(ty) \leq \frac{A}{c}\sup\left\{\frac{\sigma(Q(y,\rho))}{\rho^{n-1}} : \rho \geq \frac{c}{\sqrt{t}}\right\} \leq \frac{A}{c}.$$

Since $Kf = K\sigma - 2Kg$, we obtain (ii). $\square$

For a set E, let $\operatorname{diam} E = \sup\{|x-y| : x, y \in E\}$.

Lemma 2.4. *Let γ be a curve in $\mathbb{R}^n$ converging to e at ∞ and satisfying* (1.4). *Then there exist sequences of numbers $\{a_j\}_{j\geq 1}$, $\{b_j\}_{j\geq 1}$ and subarcs $\{\gamma_j\}_{j\geq 1}$ of γ with the following properties:*

(i) $1 < a_1 < b_1 < \cdots < a_j < b_j < a_{j+1} < b_{j+1} < \cdots \to +\infty$,

(ii) $a_j \le \sqrt{|x|} \le b_j$ *for* $x \in \gamma_j$,

(iii) $b_{j-1} \operatorname{diam} \pi(\gamma_j) \le 1$ *if* $j \ge 2$,

(iv) $\lim_{j\to+\infty} a_j \operatorname{diam} \pi(\gamma_j) = +\infty$.

Proof. Let $\{\alpha_j\}$ be a sequence such that $\alpha_j \to +\infty$ as $j \to +\infty$, and let us choose $\{a_j\}$, $\{b_j\}$ and $\{\gamma_j\}$ inductively. By (1.4), we find $a_1 > \max\{1, \inf_{x\in\gamma} \sqrt{|x|}\}$ with

$$\sqrt{|x|}|\pi(x) - e| \ge \alpha_1 \quad \text{for } x \in \gamma \cap \{\sqrt{|x|} \ge a_1\}.$$

Let γ' be the connected component of $\gamma \cap \{\sqrt{|x|} \ge a_1\}$ which converges to ∞, and let $x_1 \in \gamma' \cap \{\sqrt{|x|} = a_1\}$. Then

$$\operatorname{diam} \pi(\gamma') \ge |\pi(x_1) - e| \ge \frac{\alpha_1}{a_1}.$$

Let γ'' be a subarc of γ' starting from x_1 toward ∞ such that

$$\sup_{x\in\gamma''} \sqrt{|x|} < +\infty \quad \text{and} \quad \operatorname{diam} \pi(\gamma'') \ge \frac{1}{2} \operatorname{diam} \pi(\gamma').$$

We take $b_1 > \sup_{x\in\gamma''} \sqrt{|x|}$. Let γ_1 be the connected component of $\gamma \cap \{a_1 \le \sqrt{|x|} \le b_1\}$ containing γ''. Then

$$\operatorname{diam} \pi(\gamma_1) \ge \frac{\alpha_1}{2a_1}.$$

We next choose a_2, b_2 and γ_2 as follows. By (1.4) and the fact that $|\pi(x) - e| \to 0$ as $x \to \infty$ along γ, we find $a_2 > b_1$ such that

$$\frac{1}{2b_1} \ge |\pi(x) - e| \ge \frac{\alpha_2}{\sqrt{|x|}} \quad \text{for } x \in \gamma \cap \{\sqrt{|x|} \ge a_2\}. \tag{2.4}$$

Repeat the above process to get $b_2 > a_2$ and γ_2 such that $a_2 \le \sqrt{|x|} \le b_2$ for $x \in \gamma_2$ and $\operatorname{diam} \pi(\gamma_2) \ge \alpha_2/2a_2$. Then (2.4) also yields that

$$\operatorname{diam} \pi(\gamma_2) \le 2 \sup_{x\in\gamma_2} |\pi(x) - e| \le \frac{1}{b_1}.$$

Continue this process to obtain the required sequences. $\square$

3 Construction

Throughout this section, we suppose that $\{a_j\}_{j\ge1}$, $\{b_j\}_{j\ge1}$ and $\{\gamma_j\}_{j\ge1}$ are as in Lemma 2.4. Let

$$\ell_j = \frac{\operatorname{diam} \pi(\gamma_j)}{3}, \quad c_j = \sqrt{a_j \operatorname{diam} \pi(\gamma_j)} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j}. \tag{3.1}$$

Then, by Lemma 2.4,

$$\lim_{j\to+\infty}\ell_j=0,\qquad \lim_{j\to+\infty}\frac{\rho_j}{\ell_j}=0\quad\text{and}\quad \lim_{j\to+\infty}c_j=+\infty. \tag{3.2}$$

Therefore, in the construction below, we may assume that $\rho_j<\ell_j$ for every $j\in\mathbb{N}$. For each $j\in\mathbb{N}$, we choose finitely many points $\{y_j^\nu\}_\nu$ in S such that

(I) $S=\bigcup_\nu Q(y_j^\nu,\ell_j)$,

(II) $Q(y_j^\mu,\ell_j/2)\cap Q(y_j^\nu,\ell_j/2)=\emptyset$ if $\mu\neq\nu$.

For example, a maximal family of pairwise disjoint surface balls $\{Q(y_j^\nu,\ell_j/2)\}_\nu$ satisfies (I) and (II). We define

$$M_j=\bigcup_\nu\{y\in S:|y-y_j^\nu|=\ell_j\}, \tag{3.3}$$

$$G_j=\left\{x\in\mathbb{R}^n:a_j\le\sqrt{|x|}\le b_j\text{ and }\pi(x)\in M_j\right\}. \tag{3.4}$$

Then we have the following.

Lemma 3.1. *$T\gamma_j\cap G_j\neq\emptyset$ for any $T\in O(n)$ and $j\in\mathbb{N}$.*

Proof. By (I), we find ν with $\pi(T\gamma_j)\cap Q(y_j^\nu,\ell_j)\neq\emptyset$. Since $\operatorname{diam}\pi(T\gamma_j)=\operatorname{diam}\pi(\gamma_j)=3\ell_j$, we see that $\pi(T\gamma_j)\cap M_j\neq\emptyset$. Therefore it follows from $T\gamma_j\subset\{a_j\le\sqrt{|x|}\le b_j\}$ that $T\gamma_j\cap G_j\neq\emptyset$. $\square$

Let $R_j^\nu=\{y\in S:\ell_j-\rho_j<|y-y_j^\nu|<\ell_j+\rho_j\}$ and define

$$E_j=\bigcup_\nu R_j^\nu. \tag{3.5}$$

Note that $Q(y,\rho_j)\subset E_j$ if $y\in M_j$. By $\mathcal{X}_E$ we denote the characteristic function of E.

Lemma 3.2. *The following properties for the above $\{E_j\}_{j\ge1}$ hold.*

(i) $\displaystyle\lim_{j\to+\infty}\left(\sup\left\{\frac{K\mathcal{X}_{E_j}}{K\sigma}(x):\sqrt{|x|}\le b_{j-1}\right\}\right)=0.$

(ii) $\displaystyle\lim_{j\to+\infty}\sigma(E_j)=0.$

Proof. Since the value $\sigma(R_j^\nu)$ is independent of ν, we write $\sigma_j=\sigma(R_j^\nu)$. For a moment, we fix j and let $\sqrt{|x|}\le b_{j-1}$. By Lemma 2.3(i),

$$\begin{aligned}\frac{K\mathcal{X}_{E_j}}{K\sigma}(x)&\le AM_{(1)}\mathcal{X}_{E_j}(x)\\&\le A\sup\left\{\sum_\nu\frac{\sigma(R_j^\nu\cap Q(\pi(x),r))}{r^{n-1}}:r\ge\frac{1}{\sqrt{|x|}}\right\}\\&\le A\sup\left\{\frac{\sigma_j}{r^{n-1}}N_j:r\ge\frac{1}{\sqrt{|x|}}\right\},\end{aligned}$$

where N_j is the number of ν such that $R_j^\nu \cap Q(\pi(x), r) \neq \emptyset$. If $r \geq 1/\sqrt{|x|}$, then $r \geq 1/b_{j-1} \geq \operatorname{diam} \pi(\gamma_j) = 3\ell_j$ by Lemma 2.4. Therefore $R_j^\nu \cap Q(\pi(x), r) \neq \emptyset$ implies $Q(y_j^\nu, \ell_j/2) \subset Q(\pi(x), 2r)$. It follows from (II) that $N_j \leq A(r/\ell_j)^{n-1}$. Hence we obtain

$$\sup\left\{ \frac{K\mathcal{X}_{E_j}}{K\sigma}(x) : \sqrt{|x|} \leq b_{j-1} \right\} \leq A \frac{\sigma_j}{\ell_j^{n-1}}. \tag{3.6}$$

Observe from (2.1) and (3.2) that

$$\begin{aligned} \frac{\sigma_j}{\ell_j^{n-1}} &= \left(\frac{\ell_j + \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j + \rho_j))}{(\ell_j + \rho_j)^{n-1}} - \left(\frac{\ell_j - \rho_j}{\ell_j} \right)^{n-1} \frac{\sigma(Q(y, \ell_j - \rho_j))}{(\ell_j - \rho_j)^{n-1}} \\ &\to 0 \quad \text{as } j \to +\infty. \end{aligned}$$

This together with (3.6) concludes (i).

Taking $x = 0$ in (i), we obtain

$$\sigma(E_j) = \sigma(S) \frac{K\mathcal{X}_{E_j}}{K\sigma}(0) \to 0 \quad \text{as } j \to +\infty.$$

Thus (ii) follows. $\square$

Proof of Theorem 1.1. In view of Lemma 3.2, taking a subsequence of j if necessary, we may assume that

$$\frac{K\mathcal{X}_{E_j}}{K\sigma}(x) \leq 2^{-j} \quad \text{for } \sqrt{|x|} \leq b_{j-1}, \tag{3.7}$$

and $\sigma(E_j) \leq 2^{-j}$. Then $\sigma(\bigcap_k \bigcup_{i \geq k} E_i) = 0$. For $j \in \mathbb{N}$, let

$$f_j(y) = \begin{cases} (-1)^{I_j(y)} & \text{if } y \in \bigcup_{1 \leq i \leq j} E_i, \\ 0 & \text{if } y \notin \bigcup_{1 \leq i \leq j} E_i, \end{cases}$$

where $I_j(y) = \max\{i : y \in E_i, 1 \leq i \leq j\}$. Then we see that f_j converges σ-a.e. on S to

$$f(y) = \begin{cases} (-1)^{I(y)} & \text{if } y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i, \\ 0 & \text{if } y \notin \bigcup_{i \geq 1} E_i \text{ or } y \in \bigcap_k \bigcup_{i \geq k} E_i, \end{cases}$$

where $I(y) = \max\{i : y \in E_i\}$ for $y \in \bigcup_{i \geq 1} E_i \setminus \bigcap_k \bigcup_{i \geq k} E_i$. Also, we have the following:

$$|f_j| \leq 1, \quad |f_{j+1} - f_j| \leq 2\mathcal{X}_{E_{j+1}} \text{ on } S; \quad f_j = (-1)^j \text{ on } E_j; \quad Kf_j \to Kf \text{ on } \mathbb{R}^n.$$

Let $T \in O(n)$. By Lemma 3.1, we find $x_j \in T\gamma \cap G_j$ for each $j \in \mathbb{N}$. Then $a_j \leq \sqrt{|x_j|} \leq b_j$ and $Q(\pi(x_j), c_j/a_j) \subset E_j$. If j is even, then Lemma 2.3(ii) and

(3.7) give

$$\begin{aligned}\frac{Kf}{K\sigma}(x_j) &= \frac{Kf_j}{K\sigma}(x_j) + \sum_{k\geq j}\frac{K(f_{k+1}-f_k)}{K\sigma}(x_j)\\ &\geq \frac{Kf_j}{K\sigma}(x_j) - 2\sum_{k\geq j}\frac{K\mathcal{X}_{E_{k+1}}}{K\sigma}(x_j)\\ &\geq 1-\frac{A}{c_j}-2^{1-j}.\end{aligned}$$

Similarly, if j is odd, then

$$\frac{Kf}{K\sigma}(x_j) \leq -1+\frac{A}{c_j}+2^{1-j}.$$

Hence we conclude from (3.2) that

$$\liminf_{T\gamma\ni x\to\infty}\frac{Kf}{K\sigma}(x) = -1 < 1 = \limsup_{T\gamma\ni x\to\infty}\frac{Kf}{K\sigma}(x).$$

Obviously, $u = Kf$ is a solution of (1.1) such that $-1 \leq u/K\sigma \leq 1$ on $\mathbb{R}^n$. Thus the proof of Theorem 1.1 is complete. $\square$

4 Remark

Our construction and estimates in Sections 2 and 3 are applicable to show the analogue of Theorem B.

Theorem 4.1. *Let Ω be an unbounded subset of $\mathbb{R}^n$ converging to e at ∞ and satisfying* (1.3)*. Suppose in addition that Ω is invariant under all elements of $O(n)$ that preserve the point e. Then there exists a solution u of* (1.1) *such that $u/K\sigma$ is bounded in $\mathbb{R}^n$ and that $u/K\sigma$ admits no limits as $x\to\infty$ along $\Omega(y)$ for every $y\in S$.*

Proof. We give a sketch of the proof and its detail is left to the reader. By the assumption on Ω, we find a sequence $\{x_j\}$ in Ω converging to e at ∞ such that

$$\lim_{j\to+\infty}\frac{\big|x_j-|x_j|e\big|}{\sqrt{|x_j|}} = +\infty.$$

Taking a subsequence of j if necessary, we may assume that $\sqrt{|x_{j-1}|}|\pi(x_j)-e| \leq 1$. Let $\omega_j = \{T_e(x_j) : T_e \in O(n)$ preserves $e\}$ and let $\omega = \bigcup_j \omega_j$. Note that ω is a subset of Ω converging to e at ∞. Let $a_j = b_j = \sqrt{|x_j|}$ and define

$$\ell_j = \frac{|\pi(x_j)-e|}{3}, \quad c_j = \sqrt{a_j|\pi(x_j)-e|} \quad \text{and} \quad \rho_j = \frac{c_j}{a_j},$$

in place of (3.1). Then these satisfy (3.2) and $3\ell_j \leq 1/b_{j-1}$. Let M_j, G_j and E_j be as in (3.3), (3.4) and (3.5) respectively. Then the conclusions in Lemma 3.2 hold in this setting as well. Note that ω_j and G_j lie on the sphere of center at the origin and radius $|x_j|$. Let $T \in O(n)$. Since $\{y \in S : |y - Te| = 3\ell_j\} \subset \pi(T\omega_j)$, we see that $\pi(T\omega_j) \cap M_j \neq \emptyset$, and so $T\omega_j \cap G_j \neq \emptyset$. Hence we observe the existence of f such that

$$\liminf_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \neq \limsup_{T\omega \ni x \to \infty} \frac{Kf}{K\sigma}(x) \quad \text{for every } T \in O(n).$$

Thus $Kf/K\sigma$ admits no limits as $x \to \infty$ along $\Omega(y)$ for every $y \in S$. $\square$

Acknowledgement

The author would like to thank the referee for helpful suggestions.

References

[1] H. Aikawa, *Harmonic functions having no tangential limits*, Proc. Amer. Math. Soc. **108** (1990), no. 2, 457–464.

[2] ———, *Harmonic functions and Green potentials having no tangential limits*, J. London Math. Soc. (2) **43** (1991), no. 1, 125–136.

[3] R. Berman and D. Singman, *Boundary behavior of positive solutions of the Helmholtz equation and associated potentials*, Michigan Math. J. **38** (1991), no. 3, 381–393.

[4] F. T. Brawn, *The Martin boundary of* $R^n \times]0, 1[$, J. London Math. Soc. (2) **5** (1972), 59–66.

[5] P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), 335–400.

[6] K. Gowrisankaran and D. Singman, *Thin sets and boundary behavior of solutions of the Helmholtz equation*, Potential Anal. **9** (1998), no. 4, 383–398.

[7] K. Hirata, *Sharpness of the Korányi approach region*, Proc. Amer. Math. Soc. **133** (2005), no. 8, 2309–2317.

[8] A. Korányi, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc. **135** (1969), 507–516.

[9] A. Korányi and J. C. Taylor, *Fine convergence and parabolic convergence for the Helmholtz equation and the heat equation*, Illinois J. Math. **27** (1983), no. 1, 77–93.

[10] J. E. Littlewood, *On a theorem of Fatou*, J. London Math. Soc. **2** (1927), 172–176.

[11] A. Nagel and E. M. Stein, *On certain maximal functions and approach regions*, Adv. in Math. **54** (1984), no. 1, 83–106.

[12] J. Sueiro, *On maximal functions and Poisson-Szegö integrals*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 653–669.