Properties of superharmonic functions satisfying nonlinear inequalities in nonsmooth domains *

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Abstract

In a uniform domain Ω, we present a certain reverse mean value inequality and a Harnack type inequality for positive superharmonic functions satisfying a nonlinear inequality $-\Delta u(x) \leq c\delta_\Omega(x)^{-\alpha}u(x)^p$ for $x \in \Omega$, where $c > 0$, $\alpha \geq 0$ and $p > 1$ and $\delta_\Omega(x)$ is the distance from a point x to the boundary of Ω. These are established by refining a boundary growth estimate obtained in our previous paper (2008). Also, we apply them to show the existence of nontangential limits of quotients of such functions and to give an extension of a certain minimum principle studied by Dahlberg (1976).

Keywords: boundary growth, nontangential limit, reverse mean value inequality, Harnack type inequality, convergence property, superharmonic function, semilinear elliptic equation, uniform domain

Mathematics Subject Classifications (2000): Primary 31B05; Secondary 31B25, 31C45, 35J60

1 Introduction

This paper is a continuation of [10, 12]. Therein we studied, from the point of view of potential theory, positive superharmonic functions u satisfying a certain nonlinear inequality, for example, $-\Delta u \leq u^p$, and presented a boundary growth estimate for them in a bounded smooth domain Ω in $\mathbb{R}^n$ $(n \geq 2)$: if $0 < p \leq (n+1)/(n-1)$, then there is a constant $C > 0$ such that $u(x) \leq C\delta_\Omega(x)^{1-n}$ for all $x \in \Omega$, where $\delta_\Omega(x)$ denotes the distance from a point x to the boundary $\partial\Omega$ of Ω. As an application, we showed that if the greatest harmonic minorant of u is the zero function, then u has nontangential limit 0 almost everywhere on $\partial\Omega$. This last result was improved in the recent paper [13], using arguments from minimal fine topology and some techniques from [10]. It was shown, under no additional assumptions on u, that if $0 < p < n/(n-2)$, then u has finite nontangential limits almost everywhere on $\partial\Omega$. Indeed,

*This work was partially supported by Grant-in-Aid for Young Scientists (B) (No. 19740062), Japan Society for the Promotion of Science.

this is valid for nonsmooth domains and the range of p is not affected by the shape of a domain. Concerning this result and the Fatou-Naïm-Doob theorem, we have the following question: if u and v are positive superharmonic functions, each satisfying a nonlinear inequality as above, then does the quotient u/v have finite nontangential limits almost everywhere on $\partial\Omega$? We will see that the range of p depends on the shape of the domain in this case and that, if Ω is a smooth domain, then this question is answered in the affirmative for $p \le (n+1)/(n-1)$ and that this bound is optimal.

As is well known, positive harmonic functions h have many good properties such as the mean value equality, the Harnack inequality, the convergence property and a minimum principle in the sense of Beurling and Dahlberg. In particular, it is noteworthy that the constant C in the Harnack inequality $h(x) \le Ch(y)$ can be taken near 1 whenever x and y are close to each other.

The main purpose of this paper is to extend, in some sense, the above properties for positive harmonic functions to positive superharmonic functions satisfying a nonlinear inequality. As a consequence, we give an answer to the above question about nontangential limits. Many of our results are obtained on nonsmooth domains, after re-studying the relation between a critical exponent of a nonlinear term and a suitable boundary growth estimate.

2 Preliminaries

2.1 Positive superharmonic functions satisfying nonlinear inequalities

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \ge 2)$ and let $\delta_\Omega(x)$ denote the distance from a point x to the boundary $\partial\Omega$ of Ω. A lower semicontinuous function u on Ω taking values in $(-\infty, \infty]$ is called *superharmonic* on Ω if $u \not\equiv \infty$ and u satisfies the following mean value inequality: for any $x \in \Omega$ and $0 < r < \delta_\Omega(x)$,

$$u(x) \ge \frac{1}{\nu_n r^n} \int_{B(x,r)} u(y)\, dy,$$

where $B(x, r)$ denotes the open ball of center x and radius r, and ν_n is the volume of the unit ball in $\mathbb{R}^n$. Let Δ be the Laplacian on $\mathbb{R}^n$. Then, for each superharmonic function u on Ω, there is a unique nonnegative Radon measure μ_u such that $-\Delta u = a_n \mu_u$ in Ω in the sense of distributions, where $a_n = n\nu_n \max\{1, n-2\}$. We call μ_u the *Riesz measure* associated with u. See [4, Section 4.3].

Let $c > 0$, $\alpha \ge 0$ and $p > 1$. We investigate the class $\mathscr{S}_{c,p,\alpha}(\Omega)$ of positive superharmonic functions u on Ω whose Riesz measure μ_u is absolutely continuous with respect to the Lebesgue measure and whose Radon-Nikodým derivative, written f_u, satisfies the nonlinear inequality

$$f_u(x) \le c\delta_\Omega(x)^{-\alpha} u(x)^p \quad \text{for a.e. } x \in \Omega. \tag{2.1}$$

In our results stated below, we need not pay attention to the constant c, so we write $\mathscr{S}_{p,\alpha}(\Omega) = \mathscr{S}_{c,p,\alpha}(\Omega)$ for simplicity. It is obvious that $\mathscr{S}_{p,\alpha}(\Omega)$ includes all positive continuous solutions u of semilinear elliptic equations of the form $-\Delta u = Vu^p$, where V is any nonnegative measurable function satisfying $V(x) \le c\delta_\Omega(x)^{-\alpha}$ for a.e. $x \in \Omega$ and the equation is understood in the sense of distributions. Also, positive continuous solutions u of $-\Delta u = Uu^q + Vu^p$ satisfying $\inf_\Omega u \ge a > 0$ belong to $\mathscr{S}_{p,\alpha}(\Omega)$ when $1 < q < p$ and U and V are nonnegative measurable functions such that $a^{q-p}U(x) + V(x) \le c\delta_\Omega(x)^{-\alpha}$ for a.e. $x \in \Omega$.

2.2 Uniform domains

Many results in this paper will be established in the setting of uniform domains. We say that a domain Ω is *uniform* if there exists a constant $C_\Omega > 1$ such that any pair of points $x, y \in \overline{\Omega}$ can be connected by a rectifiable curve γ in $\overline{\Omega}$ satisfying

$$\ell(\gamma) \le C_\Omega |x - y|, \tag{2.2}$$

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \le C_\Omega \delta_\Omega(z) \quad \text{for all } z \in \gamma, \tag{2.3}$$

where ℓ denotes the length of a curve, and $\gamma(x,z)$, $\gamma(z,y)$ denote the subarcs of γ from x to z and from z to y, respectively. A nontangentially accessible (abbreviated to NTA) domain, as introduced by Jerison and Kenig, is a uniform domain satisfying the *exterior corkscrew condition*: there exists a constant $r_0 > 0$ such that for each $\xi \in \partial\Omega$ and $0 < r < r_0$, we find a point $x \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $|x - \xi| = r$ and $\delta_\Omega(x) \ge r/C_\Omega$. For $\xi \in \partial\Omega$ and $\theta > 1$, we denote a nontangential set at ξ by

$$\Gamma_\theta(\xi) = \{x \in \Omega : |x - \xi| \le \theta\delta_\Omega(x)\}.$$

If Ω is a uniform domain, then we observe from (2.3) that $\Gamma_\theta(\xi)$ is nonempty and that ξ is accessible from $\Gamma_\theta(\xi)$ whenever $\theta \ge C_\Omega$.

Convention: Throughout this paper (except for special cases), we suppose that Ω is a bounded uniform domain in $\mathbb{R}^n$ $(n \ge 3)$ or a bounded NTA domain in $\mathbb{R}^2$.

2.3 Estimates for the Green function and the Martin kernel

Let us recall estimates for the Green function and the Martin kernel. The Martin boundary of a bounded uniform domain coincides with its Euclidean boundary (see Aikawa [1, Corollary 3]). Let $G_\Omega(x,y)$ denote the Green function for Ω and $K_\Omega(x,\xi)$ the Martin kernel of Ω with pole at $\xi \in \partial\Omega$. In arguments below, a point $x_0 \in \Omega$ is fixed and is the reference point of the Martin kernel, i.e. $K_\Omega(x_0,\xi) = 1$ for all $\xi \in \partial\Omega$. For convenience, we assume that $\delta_\Omega(x_0) \ge \operatorname{diam}(\Omega)/4C_\Omega$. By the symbol C, we denote an absolute positive constant whose value is unimportant and may change from line to line. Also, the notation $C = C(a, b, \cdots)$ means that a constant C depends on $a, b, \cdots$. In particular, $C(\Omega)$ stands for a constant depending on C_Ω in (2.2)–(2.3) and the diameter of Ω. We say that two positive functions f_1 and f_2 are *comparable*, written $f_1 \approx f_2$, if there exists a constant $C > 1$ such that $C^{-1}f_1 \le f_2 \le Cf_1$. Then the constant C is called the constant of comparison. The following estimate is found in [9, Corollary 1.5].

Lemma 2.1. *Let $\theta \geq C_\Omega$ and $\xi \in \partial\Omega$. Then there exists a constant $C = C(\theta, n, \Omega) > 1$ such that for all $x \in \Gamma_\theta(\xi)$,*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \geq \frac{1}{C} |x - \xi|^{2-n}.$$

Moreover, the inequality $G_\Omega(x, x_0) K_\Omega(x, \xi) \leq C|x - \xi|^{2-n}$ holds for all $x \in \Gamma_\theta(\xi) \cap B(\xi, \delta_\Omega(x_0)/2)$.

To state a global estimate of the Green function for a nonsmooth domain, we need an auxiliary set. For each pair of points $x, y \in \overline{\Omega}$, let

$$\mathcal{B}(x, y) = \left\{ b \in \Omega : \frac{1}{C_\Omega} \min\{|x - b|, |b - y|\} \leq |x - y| \leq 2C_\Omega \delta_\Omega(b) \right\}.$$

Observe that this set is nonempty for any pair $x, y \in \overline{\Omega}$. Indeed, the midpoint of the curve γ occurring in (2.2)–(2.3) lies in $\mathcal{B}(x, y)$. Let

$$g_\Omega(x) = \min\{1, G_\Omega(x, x_0)\}.$$

The following estimates are found in [11, Theorem 1.2].

Lemma 2.2. *For each $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$G_\Omega(x, y) \approx \begin{cases} \dfrac{g_\Omega(x) g_\Omega(y)}{g_\Omega(b)^2} \left(1 + \log^+ \dfrac{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}{|x - y|} \right) & \text{if } n = 2, \\ \dfrac{g_\Omega(x) g_\Omega(y)}{g_\Omega(b)^2} |x - y|^{2-n} & \text{if } n \geq 3, \end{cases}$$

where $\log^+ t = \max\{0, \log t\}$ and the constant of comparison depends only on n and Ω.

Since the Martin kernel at $\xi \in \partial\Omega$ is given by

$$K_\Omega(x, \xi) = \lim_{\Omega \ni y \to \xi} \frac{G_\Omega(x, y)}{G_\Omega(x_0, y)},$$

we obtain the following estimate (see [11, Lemma 4.2]).

Lemma 2.3. *For each $x \in \Omega$, $\xi \in \partial\Omega$ and $b \in \mathcal{B}(x, \xi)$,*

$$K_\Omega(x, \xi) \approx \frac{g_\Omega(x)}{g_\Omega(b)^2} |x - \xi|^{2-n},$$

where the constant of comparison depends only on n and Ω. Here, in the case $n = 2$, we interpret as $|x - \xi|^{2-n} = 1$.

Also, we have the following.

Lemma 2.4. *There exists a constant $C = C(n, \Omega)$ such that for each $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$\max\{g_\Omega(x), g_\Omega(y)\} \leq C g_\Omega(b).$$

Proof. It is enough to show that $g_\Omega(x) \le C g_\Omega(b)$. Let $r_0 > 0$ and $C_1 > 1$, which are determined by the shape of Ω in the Carleson estimate for harmonic functions (cf. [1, Theorem 1 and Remark 2]). If $|x - y| \le C_1\delta_\Omega(x)$, then the Harnack inequality shows that $g_\Omega(x) \approx g_\Omega(b)$ since

$$|x - b| \le C|x - y| \le C \min\{\delta_\Omega(x), \delta_\Omega(b)\}.$$

If $|x-y| \ge r_0$, then $\delta_\Omega(b) \ge |x-y|/C \ge r_0/C$, and so $g_\Omega(x) \le 1 \approx g_\Omega(b)$. Suppose that $C_1\delta_\Omega(x) < |x - y| < r_0$. Let $\widetilde{x} \in \partial\Omega$ be a point such that $|\widetilde{x} - x| = \delta_\Omega(x)$. Take $z \in \Omega$ with $|z - \widetilde{x}| = |x - y|$ and $\delta_\Omega(z) \ge |x - y|/C_1$. Then the Carleson estimate implies that

$$g_\Omega(x) \le C g_\Omega(z).$$

Since

$$|b - z| \le |b - x| + |x - z| \le C|x - y| \le C \min\{\delta_\Omega(b), \delta_\Omega(z)\},$$

we have $g_\Omega(b) \approx g_\Omega(z)$ by the Harnack inequality. Hence $g_\Omega(x) \le C g_\Omega(b)$. Thus the lemma is proved. $\square$

3 Decay order of the Green function

The behavior of the Green function for a nonsmooth domain is complicated and its decay rate may vary at every boundary point. Nevertheless, we introduce an important number in our study by

$$\tau = \sup\{t > 0 : i(t) = 0\}, \tag{3.1}$$

where

$$i(t) = \inf\left\{ \frac{G_\Omega(x, x_0)}{\delta_\Omega(x)^t} : x \in \Omega \right\}.$$

We give some elementary remarks on τ.

Lemma 3.1. *The following statements hold:*

(i) *If $t < \tau$, then $i(t) = 0$.*

(ii) *If $\tau < \infty$ and $t > \tau$, then $i(t) > 0$.*

In particular, if $\tau < \infty$, then $\tau = \inf\{t > 0 : i(t) > 0\}$.

Proof. If $s < t$, then

$$\frac{G_\Omega(x, x_0)}{\delta_\Omega(x)^s} \le (\operatorname{diam}\Omega)^{t-s} \frac{G_\Omega(x, x_0)}{\delta_\Omega(x)^t},$$

and so $i(s) \le (\operatorname{diam}\Omega)^{t-s} i(t)$. Since Ω is bounded, we have (i). Also, the definition of τ implies (ii). $\square$

Lemma 3.2. *We have $1 \le \tau < \infty$. Moreover, if Ω is a bounded $C^{1,1}$-domain, then $\tau = 1$ and $i(\tau) > 0$.*

Proof. Let us show that $\tau < \infty$. Using the Harnack inequality, we observe that there are constants $\lambda = \lambda(n, C_\Omega) > 1$ and $C = C(n, \Omega) > 1$ such that for all $x \in \Omega$ and $\xi \in \partial\Omega$,

$$K_\Omega(x, \xi) \leq C\delta_\Omega(x)^{-\lambda}.$$

See [2, (5.2) on P. 260]. Therefore, by Lemma 2.1,

$$G_\Omega(x, x_0) \geq \frac{1}{C}\delta_\Omega(x)^{\lambda+2-n}.$$

Hence $\tau \leq \lambda + 2 - n < \infty$. The assertion $\tau \geq 1$ is well known. In fact, we take a ball B_1 so that $\Omega \subset B_1$ and $\partial\Omega \cap \partial B_1 \neq \emptyset$. Let $\xi \in \partial\Omega \cap \partial B_1$. If $x \in \Gamma_\theta(\xi)$ is sufficiently close to ξ, then

$$G_\Omega(x, x_0) \leq G_{B_1}(x, x_0) \leq C|x - \xi| \leq C\delta_\Omega(x).$$

Hence it must be $\tau \geq 1$. Moreover, if Ω is a $C^{1,1}$-domain, then for each $\eta \in \partial\Omega$, there is a ball B_2 such that $B_2 \subset \Omega$, $\eta \in \partial B_2$ and the radius of B_2 is independent of η (see [3]). This implies that $\delta_\Omega(x) \leq CG_\Omega(x, x_0)$ for all $x \in \Omega$, and so $\tau = 1$ in this case. $\square$

It is unknown whether $i(\tau) > 0$ always holds for bounded uniform domains. This is a reason to divide the statements in Theorem 4.1 below.

4 Harmonic growth and exponent of nonlinearity

In this section, we present a boundary growth estimate for functions in $\mathscr{S}_{p,\alpha}(\Omega)$, which generalizes results in [10, 12]. To derive potential theoretic properties, we should pay attention to a maximal growth of positive harmonic functions near the boundary. In view of Lemma 2.1, it is natural to think of $g_\Omega(x)^{-1}\delta_\Omega(x)^{2-n}$ as a maximal growth. The main result of this section is as follows.

Theorem 4.1. *Let τ be as in* (3.1)*. Suppose that*

$$1 < p \leq \frac{n+\tau}{n+\tau-2} \quad \textit{and} \quad \alpha < n + \tau - p(n + \tau - 2). \tag{4.1}$$

If $u \in \mathscr{S}_{p,\alpha}(\Omega)$, then there exist constants $C = C(c, \alpha, p, n, \Omega)$ and

$$\beta = \begin{cases} \beta(p, n) \geq 1 & \textit{if } u(x_0) > 1, \\ 1 & \textit{if } u(x_0) \leq 1, \end{cases}$$

such that for all $x \in \Omega$,

$$u(x) \leq \frac{C}{g_\Omega(x)\delta_\Omega(x)^{n-2}}u(x_0)^\beta. \tag{4.2}$$

Moreover, if $i(\tau) > 0$, then the conclusion holds in the case $\alpha = n + \tau - p(n + \tau - 2)$ as well.

Remark 4.2. Since $(n+\tau)/(n+\tau-2) \le (n+1)/(n-1)$ and $n+\tau-p(n+\tau-2) \le n+1-p(n-1)$, it follows from [10, Theorem 1.1] that u is locally bounded. In particular, $u(x_0)$ is finite.

By Lemma 2.1, we obtain the following corollary.

Corollary 4.3. *Assumptions are the same as in Theorem 4.1. Let $\xi \in \partial\Omega$ and $\theta \ge C_\Omega$. Then there exists a constant $C = C(\theta, c, \alpha, p, n, \Omega)$ such that for all $x \in \Gamma_\theta(\xi)$,*

$$u(x) \le C u(x_0)^\beta K_\Omega(x, \xi), \tag{4.3}$$

where β is as in Theorem 4.1.

Note that the bound $p \le (n+\tau)/(n+\tau-2)$ is optimal for (4.2) to hold. See Section 9. The proof of Theorem 4.1 is similar to that given in [10, 12], but we need additional arguments. We start with an elementary estimate for harmonic functions.

Lemma 4.4. *If h is a nonnegative harmonic function on Ω, then there exists a constant $C = C(n, \Omega)$ such that for all $x \in \Omega$,*

$$h(x) \le \frac{C}{g_\Omega(x)\delta_\Omega(x)^{n-2}} h(x_0).$$

Proof. Lemmas 2.3 and 2.4 imply that for all $x \in \Omega$ and $\xi \in \partial\Omega$,

$$K_\Omega(x, \xi) \le \frac{C}{g_\Omega(x)\delta_\Omega(x)^{n-2}}.$$

Therefore the conclusion follows from the Martin representation. $\square$

In the rest of this section, we let $u \in \mathscr{S}_{p,\alpha}(\Omega)$. By the Riesz decomposition, every nonnegative superharmonic function is decomposed into the sum of a nonnegative harmonic function and a Green potential of its associated Riesz measure. Thus we have for all $x \in \Omega$,

$$u(x) = h(x) + \int_\Omega G_\Omega(x, y) f_u(y)\, dy, \tag{4.4}$$

where h is the greatest harmonic minorant of u on Ω. This yields the following.

Lemma 4.5. *The following inequality holds:*

$$\int_\Omega g_\Omega(y) f_u(y)\, dy \le u(x_0).$$

Lemma 4.6. *Let $n \ge 3$. For each $j \in \mathbb{N}$, there exists a constant $c_j = c(j, n, \Omega)$ such that for any $z \in \Omega$ and $x \in B(z, \delta_\Omega(z)/2^{j+1})$,*

$$u(x) \le \frac{c_j}{g_\Omega(z)\delta_\Omega(z)^{n-2}} u(x_0) + \int_{B(z,\delta_\Omega(z)/2^j)} \frac{f_u(y)}{|x-y|^{n-2}}\, dy.$$

Proof. Let $z \in \Omega$ and $x \in B(z, \delta_\Omega(z)/2^{j+1})$. By Lemmas 2.2 and 2.4, we have for $y \in \Omega \setminus B(z, \delta_\Omega(z)/2^j)$,

$$G_\Omega(x,y) \le C\frac{g_\Omega(y)}{g_\Omega(x)}|x-y|^{2-n} \le \frac{C}{g_\Omega(x)\delta_\Omega(z)^{n-2}}g_\Omega(y),$$

where C depends on j, n and Ω. Since $g_\Omega(x) \approx g_\Omega(z)$ and $G_\Omega(x,y) \le |x-y|^{2-n}$, it follows from Lemma 4.5 that

$$\int_\Omega G_\Omega(x,y) f_u(y)\, dy \le \frac{C}{g_\Omega(z)\delta_\Omega(z)^{n-2}}u(x_0) + \int_{B(z,\delta_\Omega(z)/2^j)} \frac{f_u(y)}{|x-y|^{n-2}}\, dy.$$

Also, since $\delta_\Omega(x) \approx \delta_\Omega(z)$ and $h \le u$, we have by Lemmas 4.4

$$h(x) \le \frac{C}{g_\Omega(z)\delta_\Omega(z)^{n-2}}u(x_0).$$

Therefore the conclusion follows from (4.4). □

Lemma 4.7. *Let $n = 2$. For each $j \in \mathbb{N}$, there exist constants $c_j = c(j,\Omega)$ and $C_2 = C(\Omega)$ such that for any $z \in \Omega$ and $x \in B(z, \delta_\Omega(z)/2^{j+1})$,*

$$u(x) \le \frac{c_j}{g_\Omega(z)}u(x_0) + \int_{B(z,\delta_\Omega(z)/2^j)} f_u(y) \log\frac{C_2\delta_\Omega(z)}{|x-y|}\, dy.$$

Proof. Since Ω is a bounded NTA domain, we observe from the exterior corkscrew condition that there exists a constant $C = C(\Omega)$ such that for any $z \in \Omega$ and $x, y \in B(z, \delta_\Omega(z)/2)$,

$$G_\Omega(x,y) \le \log\frac{C\delta_\Omega(z)}{|x-y|}.$$

The rest of the proof is similar to that of Lemma 4.6. □

Let $z \in \Omega$ be fixed. For $\eta \in B(0,1)$, we define

$$\psi_z(\eta) = g_\Omega(z)\delta_\Omega(z)^n f_u(z + \delta_\Omega(z)\eta).$$

For simplicity, we write $B(r) = B(0,r)$ when the center is the origin.

Lemma 4.8. *Let p and α be as in Theorem 4.1. Then there exists a constant $C = C(c, \alpha, p, n, \Omega)$ such that for a.e. $\eta \in B(1/2)$,*

$$\psi_z(\eta) \le C\bigl\{g_\Omega(z)\delta_\Omega(z)^{n-2}u(z+\delta_\Omega(z)\eta)\bigr\}^p.$$

Proof. First, we consider the case that p and α satisfy (4.1). Let

$$t = \frac{n-\alpha-p(n-2)}{p-1}.$$

Then $t > \tau$, and we therefore find a constant $C = C(t,\Omega) > 1$ such that

$$g_\Omega(z) \ge \frac{1}{C}\delta_\Omega(z)^t.$$

This and (2.1) imply that for a.e. $\eta \in B(1/2)$,

$$\begin{aligned}\psi_z(\eta) &= g_\Omega(z)\delta_\Omega(z)^n f_u(z+\delta_\Omega(z)\eta)\\ &\le C g_\Omega(z)\delta_\Omega(z)^{n-\alpha} u(z+\delta_\Omega(z)\eta)^p\\ &\le C g_\Omega(z)^{1-p}\delta_\Omega(z)^{n-\alpha-p(n-2)}\big\{g_\Omega(z)\delta_\Omega(z)^{n-2}u(z+\delta_\Omega(z)\eta)\big\}^p\\ &\le C\big\{g_\Omega(z)\delta_\Omega(z)^{n-2}u(z+\delta_\Omega(z)\eta)\big\}^p.\end{aligned}$$

If $i(\tau) > 0$, then this holds for $\alpha = n + \tau - p(n + \tau - 2)$ as well. $\square$

The following lemma will play an essential role in the proof of Theorem 4.1.

Lemma 4.9. *Let p and α be as in Theorem 4.1, and let*

$$\frac{n+\tau}{n+\tau-2} < q < \frac{n}{n-2} \quad \textit{and} \quad l = \left[\frac{\log(q/(q-1))}{\log(q/p)}\right] + 1.$$

Let $\kappa \ge 1$. Then there exists a constant $C = C(\kappa, q, c, \alpha, p, n, \Omega)$ such that for each $1 \le j \le l$,

$$\int_{B(1/2^{j+1})} \psi_z(\eta)^{\kappa q/p}\, d\eta \le C u(x_0)^{\kappa q} + C\left(\int_{B(1/2^j)} \psi_z(\eta)^\kappa\, d\eta\right)^q.$$

Proof. We show this lemma for $n \ge 3$. The case $n = 2$ is also proved in the same way. Let $1 \le j \le l$ and let

$$\Psi_{z,j}(\eta) = \int_{B(1/2^j)} \frac{\psi_z(\zeta)}{|\eta-\zeta|^{n-2}}\, d\zeta.$$

Making the change $x = z + \delta_\Omega(z)\eta$ and $y = z + \delta_\Omega(z)\zeta$ in Lemma 4.6, we have that for any $\eta \in B(1/2^{j+1})$,

$$g_\Omega(z)\delta_\Omega(z)^{n-2}u(z+\delta_\Omega(z)\eta) \le c_0 u(x_0) + \Psi_{z,j}(\eta), \tag{4.5}$$

where $c_0 = \max\{c_j : 1 \le j \le l\}$. Let $\kappa \ge 1$. Then, applying the Jensen inequality to the probability measure

$$|\eta-\zeta|^{2-n}\, d\zeta \Big/ \int_{B(1/2^j)} |\eta-\zeta|^{2-n}\, d\zeta \quad \text{on } B(1/2^j),$$

we have

$$\Psi_{z,j}(\eta)^\kappa \le C \int_{B(1/2^j)} \frac{\psi_z(\zeta)^\kappa}{|\eta-\zeta|^{n-2}}\, d\zeta.$$

By the Minkowski inequality for integrals and $q < n/(n-2)$,

$$\left(\int_{B(1/2^j)} \Psi_{z,j}(\eta)^{\kappa q}\, d\eta\right)^{1/q} \le C \int_{B(1/2^j)} \psi_z(\zeta)^\kappa\, d\zeta. \tag{4.6}$$

Also, it follows from Lemma 4.8 and (4.5) that for a.e. $\eta \in B(1/2^{j+1})$,

$$\psi_z(\eta) \le C\{c_0 u(x_0) + \Psi_{z,j}(\eta)\}^p,$$

and so $\psi_z(\eta)^{\kappa q/p} \le Cu(x_0)^{\kappa q} + C\Psi_{z,j}(\eta)^{\kappa q}$. Therefore, by (4.6),

$$\int_{B(1/2^{j+1})} \psi_z(\eta)^{\kappa q/p}\, d\eta \le Cu(x_0)^{\kappa q} + C\left(\int_{B(1/2^j)} \psi_z(\zeta)^\kappa\, d\zeta\right)^q.$$

Thus the lemma is proved. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. First, we consider the case $n \ge 3$. Let $z \in \Omega$ be fixed and let q and l be as in Lemma 4.9. By Lemma 4.6,

$$g_\Omega(z)\delta_\Omega(z)^{n-2}u(z) \le c_{l+1}u(x_0) + \int_{B(1/2^{l+1})} \frac{\psi_z(\eta)}{|\eta|^{n-2}}\, d\eta.$$

Let $s = q/p > 1$. Since $s^l/(s^l - 1) \le q < n/(n-2)$, we have by the Hölder inequality

$$g_\Omega(z)\delta_\Omega(z)^{n-2}u(z) \le c_{l+1}u(x_0) + C\left(\int_{B(1/2^{l+1})} \psi_z(\eta)^{s^l}\, d\eta\right)^{1/s^l}.$$

Applying Lemma 4.9 l times, we have

$$\begin{aligned}\int_{B(1/2^{l+1})} \psi_z(\eta)^{s^l}\, d\eta &\le Cu(x_0)^{s^{l-1}q} + C\left(\int_{B(1/2^l)} \psi_z(\eta)^{s^{l-1}}\, d\eta\right)^q \\ &\le \cdots \\ &\le CU + C\left(\int_{B(1/2)} \psi_z(\eta)\, d\eta\right)^{q^l},\end{aligned}$$

where

$$U = u(x_0)^{s^{l-1}q} + u(x_0)^{s^{l-2}q^2} + \cdots + u(x_0)^{q^l}.$$

Since Lemma 4.5 implies

$$\int_{B(1/2)} \psi_z(\eta)\, d\eta \le Cu(x_0),$$

we obtain

$$g_\Omega(z)\delta_\Omega(z)^{n-2}u(z) \le Cu(x_0)^\beta.$$

Here $\beta = q^l$ if $u(x_0) > 1$; $\beta = 1$ if $u(x_0) \le 1$. Hence (4.2) is proved for $n \ge 3$. When $n = 2$, we can let $l = 1$ in the above by taking a large q, since $\log(1/|\eta|) \in L^r(B(1))$ for any $r > 0$. See [12]. This completes the proof of Theorem 4.1. □

5 Reverse mean value inequality

In Sections 5 – 8, we suppose that $p > 1$ and $\alpha \geq 0$ are as in Theorem 4.1, that is, $u \in \mathscr{S}_{p,\alpha}(\Omega)$ satisfies (4.2). This section presents a reverse mean value inequality for functions in $\mathscr{S}_{p,\alpha}(\Omega)$. Let σ_n be the area of the unit sphere in $\mathbb{R}^n$, and let ν_n be the volume of the unit ball in $\mathbb{R}^n$. Denote

$$\mathcal{M}(v;x,r) = \frac{1}{\sigma_n r^{n-1}} \int_{\partial B(x,r)} v(y)\, d\sigma(y),$$
$$\mathcal{A}(v;x,r) = \frac{1}{\nu_n r^n} \int_{B(x,r)} v(y)\, dy,$$

where σ is the surface area measure on $\partial B(x,r)$. By definition, every superharmonic function v on Ω satisfies the following mean value inequalities: for each $x \in \Omega$ and $0 < r < \delta_\Omega(x)$,

$$v(x) \geq \mathcal{M}(v;x,r) \quad \text{and} \quad v(x) \geq \mathcal{A}(v;x,r). \tag{5.1}$$

Moreover, $\mathcal{A}(v;x,r) \geq \mathcal{M}(v;x,r)$ (see [4, Corollary 3.2.6]). We are interested in the opposite inequalities of (5.1) in some sense.

Theorem 5.1. *Let $u \in \mathscr{S}_{p,\alpha}(\Omega)$ and let d be any function on Ω such that $d(x) \geq 2$ for all $x \in \Omega$. Then there exists a constant $C = C(c,\alpha,p,n,\Omega)$ such that if we put*

$$\rho_d(x,r) = Cu(x_0)^{\beta(p-1)} \frac{r^{2-\alpha-(p-1)(n-2)}}{g_\Omega(x)^{p-1} d(x)^{\alpha+(p-1)(n-2)}},$$

where β is the constant in Theorem 4.1, then the following inequalities hold for any $x \in \Omega$ and $0 < r \leq \delta_\Omega(x)/d(x)$:

$$\{1 - \rho_d(x,r)\} u(x) \leq \mathcal{M}(u;x,r) \leq \mathcal{A}(u;x,r). \tag{5.2}$$

Remark 5.2. In Lemma 6.2 below, we will show that $\rho_d(x,r)$ can be arbitrary small by taking $d(x)$ large enough. Thus (5.2) is meaningful.

To prove Theorem 5.1, we recall the following lemma (see [4, Corollary 4.4.4]).

Lemma 5.3. *Let v be a superharmonic function on an open set which contains $\overline{B(x,r)}$. Then*

$$v(x) = \mathcal{M}(v;x,r) + a_n \int_0^r t^{1-n} \mu_v(B(x,t))\, dt,$$

where $a_n = \max\{n-2, 1\}$ and μ_v is the Riesz measure associated with v.

Proof of Theorem 5.1. Let $x \in \Omega$, $0 < r \leq \delta_\Omega(x)/d(x)$ and $y \in B(x,r)$. Since $d(x) \geq 2$, we have $\delta_\Omega(y) \geq \delta_\Omega(x)/2 \geq d(x)r/2$ and $g_\Omega(y) \geq g_\Omega(x)/C$. It follows from Theorem 4.1 that

$$\begin{aligned} f_u(y) &\leq c\delta_\Omega(y)^{-\alpha} u(y)^{p-1} u(y) \\ &\leq \frac{Cu(x_0)^{\beta(p-1)}}{g_\Omega(x)^{p-1} (d(x)r)^{\alpha+(p-1)(n-2)}} u(y) =: a(x,r)u(y). \end{aligned}$$

Therefore, by Lemma 5.3 and (5.1),

$$\begin{aligned} u(x) &\le \mathcal{M}(u;x,r) + a_n a(x,r) \int_0^r t^{1-n} \int_{B(x,t)} u(y)\, dydt \\ &\le \mathcal{M}(u;x,r) + \frac{a_n \nu_n}{2} a(x,r) r^2 u(x), \end{aligned}$$

and so

$$u(x) \le \mathcal{M}(u;x,r) + Cu(x_0)^{\beta(p-1)} \frac{r^{2-\alpha-(p-1)(n-2)}}{g_\Omega(x)^{p-1} d(x)^{\alpha+(p-1)(n-2)}} u(x).$$

Thus Theorem 5.1 is proved. □

6 Harnack type inequality

As a consequence of Theorem 5.1, we obtain the following Harnack type inequality.

Theorem 6.1. *Let $u \in \mathscr{S}_{p,\alpha}(\Omega)$, and let d and ρ_d be functions as in Theorem 5.1. Then, for each $x \in \Omega$, $0 < r \le \delta_\Omega(x)/2d(x)$ and $y \in B(x,r)$,*

$$\{1 - \rho_d(x,r)\} u(x) \le \left(1 + \frac{|x-y|}{r}\right)^n u(y).$$

Proof. Let $x \in \Omega$, $0 < r \le \delta_\Omega(x)/2d(x)$ and $y \in B(x,r)$. Then $B(x,r) \subset B(y, r + |x-y|) \subset \Omega$. By Theorem 5.1,

$$\begin{aligned} \{1 - \rho_d(x,r)\} u(x) &\le \mathcal{A}(u;x,r) \\ &\le \frac{(r+|x-y|)^n}{r^n} \mathcal{A}(u;y,r+|x-y|). \end{aligned}$$

Hence this theorem follows from (5.1). □

Lemma 6.2. *Let $u \in \mathscr{S}_{p,\alpha}(\Omega)$ and let ρ_d be a function as in Theorem 5.1. Then there exists a constant $C = C(c,\alpha,p,n,\Omega)$ with the following property: Let $\varepsilon > 0$. If a function d satisfies*

$$d(x) \ge C \sqrt{\frac{u(x_0)^{\beta(p-1)}}{\varepsilon}} \quad \textit{for all } x \in \Omega,$$

where β is the constant in Theorem 4.1, then for any $x \in \Omega$ and $0 < r \le \delta_\Omega(x)/d(x)$,

$$\rho_d(x,r) \le \varepsilon.$$

Proof. Let $x \in \Omega$ and $0 < r \le \delta_\Omega(x)/d(x)$. Consider the case that p and α satisfy (4.1). If we let

$$t = \frac{n - \alpha - p(n-2)}{p-1},$$

then $t > \tau$, and so $g_\Omega(x) \geq \delta_\Omega(x)^t/C \geq (d(x)r)^t/C$ for all $x \in \Omega$. Then

$$\rho_d(x,r) \leq Cu(x_0)^{\beta(p-1)} \frac{r^{2-\alpha-(p-1)(n+t-2)}}{d(x)^{\alpha+(p-1)(n+t-2)}} = \frac{Cu(x_0)^{\beta(p-1)}}{d(x)^2}.$$

Therefore, $\rho_d(x,r) \leq \varepsilon$ whenever $d(x) \geq \sqrt{Cu(x_0)^{\beta(p-1)}/\varepsilon}$. Moreover, if $i(\tau) > 0$, then the conclusion holds for $\alpha = n + \tau - p(n + \tau - 2)$ as well. $\square$

Corollary 6.3. *Let $M > 0$ and $0 < \kappa < 1$. Then for each $0 < \varepsilon < 1$, there exists a constant $d_\varepsilon = d(\varepsilon, M, c, \alpha, p, n, \Omega) \geq 2$ such that for any $x \in \Omega$, $0 < r \leq \delta_\Omega(x)/4d_\varepsilon$ and $y \in B(x, \kappa r)$,*

$$\frac{1-\varepsilon}{(1+\kappa)^n} u(x) \leq u(y) \leq \frac{(1+\kappa)^n}{1-\varepsilon} u(x),$$

whenever $u \in \mathscr{S}_{p,\alpha}(\Omega)$ satisfies $u(x_0) \leq M$.

Proof. Let $d_\varepsilon = \max\{C\sqrt{\max\{1,M\}^{\beta(p-1)}/\varepsilon}, 2\}$. Then Lemma 6.2 implies that $\rho_{d_\varepsilon}(z,t) \leq \varepsilon$ for all $z \in \Omega$ and $0 < t \leq \delta_\Omega(z)/d_\varepsilon$. Let $x \in \Omega$, $0 < r \leq \delta_\Omega(x)/4d_\varepsilon$ and $y \in B(x, \kappa r)$. Then, by Theorem 6.1,

$$u(x) \leq \frac{(1+\kappa)^n}{1-\varepsilon} u(y).$$

Also, since $\delta_\Omega(x) \leq 2\delta_\Omega(y)$, we have $r \leq \delta_\Omega(y)/2d_\varepsilon$, and so

$$u(y) \leq \frac{(1+\kappa)^n}{1-\varepsilon} u(x).$$

Thus the corollary is proved. $\square$

Recall the quasi-hyperbolic metric $k_\Omega(x,y)$ on Ω:

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x and y in Ω and ds stands for the line element on γ. Now, let $d_0 = 4d_{1/2}$ for simplicity. A sequence of balls $\{B(x_j, \delta_\Omega(x_j)/d_0)\}_{j=1}^N$ is said to be a *Harnack chain* connecting x and y if $x_1 = x$, $x_N = y$ and $x_{j-1} \in B(x_j, \delta_\Omega(x_j)/d_0)$ for $j = 2, \cdots, N$. It is well known that the smallest number N among Harnack chains connecting x and y is comparable to $k_\Omega(x,y) + 1$, where the constant of comparison depends only on n. Thus, by using Corollary 6.3 $N - 1$ times, we have the following.

Corollary 6.4. *Let $M > 0$. Then there exists a constant $C = C(M, c, \alpha, p, n, \Omega) > 1$ such that for any $x, y \in \Omega$,*

$$\exp\{-C(k_\Omega(x,y)+1)\} \leq \frac{u(x)}{u(y)} \leq \exp\{C(k_\Omega(x,y)+1)\},$$

whenever $u \in \mathscr{S}_{p,\alpha}(\Omega)$ satisfies $u(x_0) \leq M$.

In Sections 7 and 8, we present three applications of Corollary 6.3: the existence of nontangential limits for quotients of two functions in $\mathscr{S}_{p,\alpha}(\Omega)$, an extension of a minimum principle for positive harmonic functions due to Dahlberg to functions in $\mathscr{S}_{p,\alpha}(\Omega)$, and a Harnack type convergence theorem for a class of solutions of a certain semilinear elliptic equation.

7 The existence of nontangential limits and a minimum principle

The boundary behavior of superharmonic functions in a very general setting was studied by Naïm [15] and Doob [7]. Nowadays, their results are known as the Fatou-Naïm-Doob theorem. In our situation, it asserts that for two positive superharmonic functions u and v on Ω, the quotient u/v has finite minimal fine limits ν-almost everywhere on $\partial\Omega$, where ν is a measure on $\partial\Omega$ appearing in the Martin representation of the greatest harmonic minorant of v. For the definition of minimal fine limits and further details, see [4, Section 9]. Note that the approach regions are not defined geometrically and practically impossible to visualise. Applying their results, we give a nontangential limit theorem for functions in $\mathscr{S}_{p,\alpha}(\Omega)$. A function f on Ω is said to have *nontangential limit* a at $\xi \in \partial\Omega$ if

$$\lim_{\Gamma_\theta(\xi)\ni x\to\xi} f(x) = a$$

for each $\theta \geq C_\Omega$.

Theorem 7.1. *Let $u, v \in \mathscr{S}_{p,\alpha}(\Omega)$ and let ν be a measure on $\partial\Omega$ appearing in the Martin representation of the greatest harmonic minorant of v. Then u/v has finite nontangential limits ν-almost everywhere on $\partial\Omega$.*

Proof. By the Fatou-Naïm-Doob theorem, we find a subset E of $\partial\Omega$ with $\nu(E) = 0$ such that u/v has finite minimal fine limit, a say, at each $\xi \in \partial\Omega \setminus E$. Let $0 < \varepsilon < 1$, $0 < \kappa < 1$, $\theta \geq C_\Omega$, and let d_ε be the constant in Corollary 6.3, where $M = \max\{u(x_0), v(x_0)\}$. Take an arbitrary sequence $\{x_j\}$ in $\Gamma_\theta(\xi)$ converging to ξ. Since the set $\bigcup_j B(x_j, \kappa\delta_\Omega(x_j)/4d_\varepsilon)$ is not minimally thin at ξ (see [1, Lemma 5]), there is $z_j \in B(x_j, \kappa\delta_\Omega(x_j)/4d_\varepsilon)$ such that $u(z_j)/v(z_j) \to a$ as $j \to \infty$. Then Corollary 6.3 yields that

$$\frac{(1-\varepsilon)^2}{(1+\kappa)^{2n}} a \leq \liminf_{j\to\infty} \frac{u(x_j)}{v(x_j)} \leq \limsup_{j\to\infty} \frac{u(x_j)}{v(x_j)} \leq \frac{(1+\kappa)^{2n}}{(1-\varepsilon)^2} a.$$

Letting $\kappa \to 0$ and $\varepsilon \to 0$, we obtain

$$\lim_{j\to\infty} \frac{u(x_j)}{v(x_j)} = a.$$

This completes the proof. $\square$

The following is a special case of Theorem 7.1.

Corollary 7.2. *Let $u \in \mathscr{S}_{p,\alpha}(\Omega)$ and $\xi \in \partial\Omega$. Then $u/K_\Omega(\cdot,\xi)$ has a finite nontangential limit at ξ.*

Remark 7.3. If $p > (n+\tau)/(n+\tau-2)$ or $\alpha > n+\tau-p(n+\tau-2)$, then we can construct a function $u \in \mathscr{S}_{p,\alpha}(\Omega)$ such that the upper limit of $u/K_\Omega(\cdot,\xi)$ along a nontangential set at ξ is infinite. See Section 9. Hence the bounds $p \le (n+\tau)/(n+\tau-2)$ and $\alpha \le n+\tau-p(n+\tau-2)$ are optimal to obtain the results in Sections 6 and 7.

Next, we mention an extension of a certain minimum principle for positive harmonic functions studied by Dahlberg [6]. See also Beurling [5]. Let $\mathcal{F}$ be some class of positive functions on Ω and let $\xi \in \partial\Omega$. We say that a subset E of Ω is *equivalent* at ξ for $\mathcal{F}$ if the equality

$$\inf_{x\in E} \frac{f(x)}{K_\Omega(x,\xi)} = \inf_{x\in\Omega} \frac{f(x)}{K_\Omega(x,\xi)} \tag{7.1}$$

holds for all functions $f \in \mathcal{F}$. Dahlberg gave characterizations for a set E to satisfy (7.1) for the class $\mathcal{F}$ of all positive harmonic functions. Indeed, he proved the equivalence of (ii)–(v) in Theorem 7.4 below. We assert that his result can be extended to the wider class $\mathscr{S}_{p,\alpha}(\Omega)$.

Theorem 7.4. *Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \ge 3)$ and let $E \subset D$ and $\xi \in \partial D$. Suppose that*

$$1 < p \le \frac{n+1}{n-1} \quad \textit{and} \quad \alpha \le n+1-p(n-1).$$

Then the following statements are equivalent:

(i) *E is equivalent at ξ for $\mathscr{S}_{p,\alpha}(D)$;*

(ii) *E is equivalent at ξ for the class of all positive harmonic functions on D;*

(iii) *there exists a number $0 < a < 1$ such that*

$$\int_{E_a} |x-\xi|^{-n}\, dx = \infty, \tag{7.2}$$

where $E_a = \bigcup_{x\in E} B(x, a\delta_D(x))$;

(iv) (7.2) *holds for any $0 < a < 1$;*

(v) *there exist a number $a > 0$ and a sequence $\{x_j\}$ in E converging to ξ such that $|x_j - x_k| \ge a\delta_D(x_j)$ whenever $j \ne k$, and that*

$$\sum_{j=1}^{\infty} \left(\frac{\delta_D(x_j)}{|x_j-\xi|} \right)^n = \infty.$$

Proof. We will show that (iv) implies (i). Indeed, the proof follows the argument in [6, P. 249], because we have Corollary 6.3. Suppose to the contrary that (i) fails to hold. Then we find $u \in \mathscr{S}_{p,\alpha}(D)$ with

$$\inf_{x\in E} \frac{u(x)}{K_D(x,\xi)} =: m > s := \inf_{x\in D} \frac{u(x)}{K_D(x,\xi)}.$$

Let $v(x) = u(x) - sK_D(x, \xi)$. Then

$$\inf_{x \in D} \frac{v(x)}{K_D(x, \xi)} = 0. \tag{7.3}$$

Let $C_3 > 1$ be a constant satisfying $m/C_3^2 > s$. By Corollary 6.3, we find a constant $a > 0$ such that for all $x \in D$ and $y \in B(x, a\delta_D(x))$,

$$u(x) \le C_3 u(y) \quad \text{and} \quad K_D(y, \xi) \le C_3 K_D(x, \xi).$$

Let $y \in E_a$. Then $y \in B(x, a\delta_D(x))$ for some $x \in E$. Since $u(x) \ge mK_D(x, \xi)$, it follows that

$$u(y) \ge \frac{1}{C_3} u(x) \ge \frac{m}{C_3} K_D(x, \xi) \ge \frac{m}{C_3^2} K_D(y, \xi),$$

and so

$$v(y) = u(y) - sK_D(y, \xi) \ge \left(\frac{m}{C_3^2} - s \right) K_D(y, \xi).$$

Then assumption (iv) and [6, Theorem 2] imply that the last inequality holds on the whole of D. This contradicts (7.3). □

Remark 7.5. Of course, the above result holds for a bounded Liapunov-Dini domain as well (see [6] for the definition of a Liapunov-Dini domain).

8 Harnack type convergence theorem

Let $M > 0$ be a constant and let V be a nonnegative measurable function on Ω such that $V(x) \le c\delta_\Omega(x)^{-\alpha}$ for a.e. $x \in \Omega$. In this section, we suppose that p and α satisfy (4.1); if $i(\tau) > 0$, then we permit $\alpha = n + \tau - p(n + \tau - 2)$. Let $\mathscr{S}_{p,V}^M(\Omega)$ be the class of all positive continuous solutions u of

$$-\Delta u = Vu^p \quad \text{in } \Omega \quad \text{(in the sense of distributions)}$$

such that $u(x_0) \le M$. Note that $\mathscr{S}_{p,V}^M(\Omega) \subset \mathscr{S}_{p,\alpha}(\Omega)$. Also, it is not difficult to see that $\mathscr{S}_{p,V}^M(\Omega) \ne \emptyset$.

Lemma 8.1. *$\mathscr{S}_{p,V}^M(\Omega)$ is locally uniformly bounded and locally uniformly equicontinuous on Ω.*

Proof. The local boundedness of $\mathscr{S}_{p,V}^M(\Omega)$ follows from Theorem 4.1. Let us show the local uniform equicontinuity of $\mathscr{S}_{p,V}^M(\Omega)$. Let E be a compact subset of Ω and let $\eta > 0$. Write

$$M_0 = \sup_{x \in E, u \in \mathscr{S}_{p,V}^M(\Omega)} u(x) < \infty,$$

and consider a constant function d satisfying

$$d(x) \equiv d \ge \max\left\{ 2, C\sqrt{\frac{M_0 u(x_0)^{\beta(p-1)}}{\eta}} \right\},$$

where C and β are constants in Lemma 6.2 and Theorem 4.1, respectively. Apply Theorem 6.1 with $\varepsilon = \eta/M_0$ and $r = \operatorname{dist}(E, \partial\Omega)/2d$. Then

$$(1-\varepsilon)u(z) \le \left(1 + \frac{|z-w|}{r}\right)^n u(w), \tag{8.1}$$

whenever $z, w \in E$ satisfy $|z-w| < r$. Take $0 < \delta \le r$ with

$$\left(1 + \frac{\delta}{r}\right)^n - 1 \le \frac{\eta}{M_0}.$$

Then (8.1) implies that for any $x, y \in E$ with $|x-y| < \delta$,

$$u(x) - u(y) \le \varepsilon u(x) + \left\{\left(1 + \frac{\delta}{r}\right)^n - 1\right\} u(y) \le 2\eta.$$

Interchanging the roles of x and y, we have $|u(x) - u(y)| \le 2\eta$. Hence $\mathscr{S}_{p,V}^M(\Omega)$ is locally uniformly equicontinuous on Ω. $\square$

Lemma 8.2. *If $\{u_j\}$ is a sequence in $\mathscr{S}_{p,V}^M(\Omega)$ converging pointwisely to a function u on Ω, then the convergence is locally uniform on Ω and $u \in \mathscr{S}_{p,V}^M(\Omega) \cup \{0\}$.*

Proof. Let $\varepsilon > 0$ and let E be a compact set in Ω. Since $\{u_j\}$ is locally uniformly equicontinuous on Ω, there is $\delta > 0$ such that $|u_j(x) - u_j(y)| < \varepsilon$ for all $j \in \mathbb{N}$ and $x, y \in E$ with $|x-y| < \delta$. Then $|u(x) - u(y)| \le \varepsilon$ and $E \subset \bigcup_{k=1}^m B(x_k, \delta)$ for some $m \in \mathbb{N}$, where $x_1, \cdots, x_m \in E$. By assumption, there is $j_0 \in \mathbb{N}$ such that $|u_j(x_k) - u(x_k)| < \varepsilon$ for all $j \ge j_0$ and $1 \le k \le m$. For any $x \in E$, we find $1 \le k \le m$ with $|x - x_k| < \delta$. Therefore

$$|u_j(x) - u(x)| \le |u_j(x) - u_j(x_k)| + |u_j(x_k) - u(x_k)| + |u(x_k) - u(x)| < 3\varepsilon.$$

Hence the convergence is locally uniform on Ω. Also, for $\phi \in C_0^\infty(\Omega)$,

$$-\int_\Omega u\Delta\phi\, dx = -\lim_{j\to\infty}\int_\Omega u_j\Delta\phi\, dx = \lim_{j\to\infty}\int_\Omega V u_j^p \phi\, dx = \int_\Omega V u^p \phi\, dx.$$

Hence $u \in \mathscr{S}_{p,V}^M(\Omega) \cup \{0\}$ by the minimum principle. $\square$

Theorem 8.3. *Let $\{u_j\}$ be a sequence in $\mathscr{S}_{p,V}^M(\Omega)$. Then there exists a subsequence of $\{u_j\}$ which converges locally uniformly on Ω to a function in $\mathscr{S}_{p,V}^M(\Omega) \cup \{0\}$.*

Proof. This follows from the Ascoli-Arzelá theorem together with Lemmas 8.1 and 8.2. $\square$

9 On the bounds $p \le (n+\tau)/(n+\tau-2)$ and $\alpha \le n+\tau-p(n+\tau-2)$

This section shows that the bounds $p \le (n+\tau)/(n+\tau-2)$ and $\alpha \le n+\tau-p(n+\tau-2)$ are optimal to obtain (4.2) and the results in Sections 5–7.

Theorem 9.1. *Let $n \geq 3$, $c > 0$, and τ be as in* (3.1)*. Suppose that either*

(i) $p > (n+\tau)/(n+\tau-2)$ *and* $\alpha \geq 0$, *or*

(ii) $p > 1$ *and* $\alpha > n+\tau-p(n+\tau-2)$

holds. Let κ be a number such that

$$n+\tau-2 < \kappa < \kappa_p := \begin{cases} \dfrac{2\tau+\alpha(n-2)}{n-(n-2)p} & \text{if } p < \dfrac{n}{n-2}, \\ \infty & \text{if } p \geq \dfrac{n}{n-2}. \end{cases}$$

Then there exist $u \in \mathscr{S}_{c,p,\alpha}(\Omega) \cap C^2(\Omega)$ and a sequence $\{x_j\}$ in Ω with no limit point in Ω such that

$$\lim_{j\to\infty} \delta_\Omega(x_j)^\kappa u(x_j) = \infty.$$

Proof. A proof is similar to that given in [10], but we need additional arguments. For the convenience sake of the reader, we provide a proof. Take κ_0 with $\kappa < \kappa_0 < \kappa_p$, and let

$$\gamma = \frac{\alpha+\kappa_0(p-1)}{2} \quad \text{and} \quad \lambda = \alpha+\kappa_0 p.$$

Then $\gamma > 1$. In fact, if p and α satisfy (i), then

$$\gamma > \frac{n+\tau-2}{2}\left(\frac{n+\tau}{n+\tau-2}-1\right) = 1;$$

if p and α satisfy (ii), then

$$\gamma > \frac{1}{2}\{n+\tau-p(n+\tau-2)+(n+\tau-2)(p-1)\} = 1.$$

Let $t < \tau$ be taken so that

$$\kappa_0 < \frac{2t+\alpha(n-2)}{n-(n-2)p} \quad \text{if } p < \frac{n}{n-2}.$$

Then, in any case,

$$\lambda - n\gamma = \frac{1}{2}\{(2-n)\alpha + (n-(n-2)p)\kappa_0\} < t.$$

Also, $t < \tau$ implies $i(t) = 0$, so that there is a sequence $\{x_j\}$ in Ω with no limit point in Ω such that $\delta_\Omega(x_j) < 1$ and

$$B(x_j, \delta_\Omega(x_j)^\gamma/4) \cap B(x_k, \delta_\Omega(x_k)^\gamma/4) = \emptyset \quad \text{if } j \neq k, \tag{9.1}$$

$$g_\Omega(x_j) \leq \delta_\Omega(x_j)^t \quad \text{for all } j, \tag{9.2}$$

$$\sum_{j=1}^{\infty} \delta_\Omega(x_j)^{t-\lambda+n\gamma} < \infty. \tag{9.3}$$

Note that there exists a constant $C_4 > 1$ such that

$$G_\Omega(x,y) \ge \frac{1}{C_4}|x-y|^{2-n} \quad \text{whenever } |x-y| \le \frac{1}{2}\delta_\Omega(x). \tag{9.4}$$

Let $C_5 > 0$ be a constant such that

$$\frac{c}{2^\alpha}\left(\frac{\nu_n C_5}{2^{n+4}C_4}\right)^p \ge C_5, \tag{9.5}$$

where ν_n is the volume of the unit ball in $\mathbb{R}^n$, and let f_j be a nonnegative smooth function on Ω such that $f_j \le C_5/\delta_\Omega(x_j)^\lambda$ and

$$f_j = \begin{cases} \dfrac{C_5}{\delta_\Omega(x_j)^\lambda} & \text{on } B(x_j, \delta_\Omega(x_j)^\gamma/8), \\ 0 & \text{on } \Omega \setminus B(x_j, \delta_\Omega(x_j)^\gamma/4). \end{cases}$$

Define $f = \sum_{j=1}^\infty f_j$. Since (9.2) and the Harnack inequality imply

$$g_\Omega(y) \le C\delta_\Omega(x_j)^t \quad \text{for all } y \in B(x_j, \delta_\Omega(x_j)^\gamma/4),$$

we have by (9.3)

$$\begin{aligned} \int_\Omega g_\Omega(y)f(y)\,dy &\le C\sum_{j=1}^\infty \int_{B(x_j,\delta_\Omega(x_j)^\gamma/4)} \delta_\Omega(x_j)^t f_j(y)\,dy \\ &\le C\sum_{j=1}^\infty \delta_\Omega(x_j)^{t-\lambda+\gamma n} < \infty. \end{aligned}$$

Therefore $u = \int_\Omega G_\Omega(\cdot,y)f(y)dy$ is positive and superharmonic on Ω. Moreover, the local Hölder continuity of f yields that $u \in C^2(\Omega)$ and $-\Delta u = f$ in Ω (see [16, Theorem 6.6]). By the mean value property and (9.4), we have for $x \in \partial B(x_j, \delta_\Omega(x_j)^\gamma/4)$,

$$\begin{aligned} u(x) &\ge \int_{B(x_j,\delta_\Omega(x_j)^\gamma/8)} G_\Omega(x,y)f_j(y)\,dy = \frac{C_5}{\delta_\Omega(x_j)^\lambda}\frac{\nu_n\delta_\Omega(x_j)^{\gamma n}}{8^n}G_\Omega(x,x_j) \\ &\ge \frac{\nu_n C_5}{2^{n+4}C_4}\delta_\Omega(x_j)^{-\kappa_0}. \end{aligned}$$

Here we used $2\gamma - \lambda = -\kappa_0$. By the minimum principle,

$$u(x) \ge \frac{\nu_n C_5}{2^{n+4}C_4}\delta_\Omega(x_j)^{-\kappa_0} \quad \text{for all } x \in B(x_j, \delta_\Omega(x_j)^\gamma/4). \tag{9.6}$$

Therefore

$$\delta_\Omega(x_j)^\kappa u(x_j) \ge \frac{1}{C}\delta_\Omega(x_j)^{\kappa-\kappa_0} \to \infty \quad \text{as } j \to \infty.$$

To complete the proof, we have to show that $-\Delta u(x) \le c\delta_\Omega(x)^{-\alpha}u(x)^p$ for all $x \in \Omega$. If $x \notin \bigcup_j B(x_j, \delta_\Omega(x_j)^\gamma/4)$, then

$$c\delta_\Omega(x)^{-\alpha}u(x)^p \ge 0 = f(x) = -\Delta u(x).$$

If there is j such that $x \in B(x_j, \delta_\Omega(x_j)^\gamma/4)$, then we have by (9.6), (9.5) and (9.1)

$$\begin{aligned} c\delta_\Omega(x)^{-\alpha}u(x)^p &\geq \frac{c}{2^\alpha}\left(\frac{\nu_n C_5}{2^{n+4}C_4}\right)^p \delta_\Omega(x_j)^{-p\kappa_0-\alpha} \\ &\geq \frac{C_5}{\delta_\Omega(x_j)^\lambda} \geq f_j(x) = f(x) = -\Delta u(x). \end{aligned}$$

Thus Theorem 8.3 is proved. □

Two dimensional case is stated as follows.

Theorem 9.2. *Let* $n = 2$, $c > 0$, *and* τ *be as in* (3.1). *Suppose that either*

(i) $p > (2+\tau)/\tau$ *and* $\alpha \geq 0$, *or*

(ii) $p > 1$ *and* $\alpha > 2 + \tau - p\tau$

holds. Assume that there are a constant $C > 1$ *and a sequence* $\{x_j\}$ *in* Ω *with no limit point in* Ω *such that* $\delta_\Omega(x_j)^\tau/C \leq g_\Omega(x_j) \leq C\delta_\Omega(x_j)^\tau$ *for all* j. *Then there exists* $u \in \mathscr{S}_{c,p,\alpha}(\Omega) \cap C^2(\Omega)$ *such that*

$$\limsup_{j\to\infty} g_\Omega(x_j)u(x_j) = \infty. \tag{9.7}$$

Proof. Let $\lambda = \alpha + \tau p$ and $\gamma = (\lambda - \tau)/2$. Taking a subsequence of $\{x_j\}$ if necessary, we may assume that $\{x_j\}$ satisfies (9.1) and

$$\delta_\Omega(x_j) \leq e^{-j^3} \quad \text{for all } j.$$

Let $C_6 > 0$ be sufficiently large and let f_j be a nonnegative smooth function on Ω such that $f_j \leq C_6/j^2\delta_\Omega(x_j)^\lambda$ and

$$f_j = \begin{cases} \dfrac{C_6}{j^2\delta_\Omega(x_j)^\lambda} & \text{on } B(x_j, \delta_\Omega(x_j)^\gamma/8), \\ 0 & \text{on } \Omega \setminus B(x_j, \delta_\Omega(x_j)^\gamma/4). \end{cases}$$

Define $f = \sum_{j=1}^\infty f_j$ and $u = \int_\Omega G_\Omega(\cdot, y)f(y)dy$. Then the similar arguments to the proof of Theorem 9.1 shows that $u \in \mathscr{S}_{c,p,\alpha}(\Omega) \cap C^2(\Omega)$ and u satisfies (9.7). See also [12, Proof of Theorem 1.2]. □

Remark 9.3. The bounds $p \leq (n+\tau)/(n+\tau-2)$ and $\alpha \leq n + \tau - p(n+\tau-2)$ are optimal to obtain the results in Sections 5–7. In fact, we may consider a uniform domain Ω such that there are $\xi \in \partial\Omega$, $\theta \geq C_\Omega$ and $C > 1$ such that for any $x \in \Gamma_\theta(\xi)$ near ξ,

$$\frac{\delta_\Omega(x)^\tau}{C} \leq g_\Omega(x) \leq C\delta_\Omega(x)^\tau. \tag{9.8}$$

Then we can choose $\{x_j\}$, satisfying (9.1)–(9.3), from $\Gamma_\theta(\xi)$. Hence, if p and α satisfy (i) or (ii), then we can construct $u \in \mathscr{S}_{c,p,\alpha}(\Omega)$ such that the upper limit of $u/K_\Omega(\cdot, \xi)$ along $\Gamma_\theta(\xi)$ is infinite.

Elementary bounded domains satisfying (9.8) are $C^{1,1}$-domains ($\tau = 1$), unions of open balls with fixed size ($\tau = 1$), and polygonal uniform domains. Here we say that a bounded domain Ω is *polygonal* if there are finitely many cones $\Gamma_1, \cdots, \Gamma_m$ with the following property: for each $\xi \in \partial\Omega$, there are $r > 0$ and $1 \leq j \leq m$ such that $\Omega \cap B(\xi, r) = \Gamma_j \cap B(\xi, r)$. In fact, the Martin kernels of uniform cones are homogeneous (see [8, 14]). Hence, in view of the boundary Harnack principle and Lemma 2.1, we see that polygonal uniform domains satisfy (9.8) for some $\tau \geq 1$ and $\xi \in \partial\Omega$.

Acknowledgment

The author is grateful to Professor Noriaki Suzuki for helpful suggestions. Also, he thanks the referee for careful reading and valuable comments.

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