Removable singularities of semilinear parabolic equations

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Abstract

This paper extends the recent result due to Hsu (2010) about removable singularities of semilinear parabolic equations. Our result is applicable to solutions of equations of the form $-\Delta u + \partial_t u = |u|^{p-1}u$ with $0 \le p < n/(n-2)$. The proof is based on the parabolic potential theory and an iteration argument. Also, we prove that if $0 < p < (n+2)/n$, then integral solutions of semilinear parabolic equations with nonlinearity depending on space and time variables and u^p are locally bounded. This implies that the blow-up for continuous solutions is complete.

Keywords: removable singularities, blow-up, semilinear parabolic equation, heat equation.

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1 Introduction

The classical removability theorem states that a compact polar set is removable for bounded harmonic functions. If the set is singleton, then the boundedness of functions can be weaken. Indeed, it is well known that a harmonic function h has a removable singularity at 0 if and only if

$$|h(x)| = \begin{cases} o(\|x\|^{2-n}) & (n \ge 3), \\ o(\log \|x\|) & (n = 2), \end{cases}$$

as $x \to 0$. Also, there are many investigations about a removable isolated singularity of solutions of semilinear elliptic equations (see [2, 7, 13]).

The parabolic analogue that a compact polar set is removable for bounded solutions of the heat equation was given by Watson [14]. Also, Oswald [9] obtained some results about a removable isolated singularity and the asymptotic behavior near an isolated point of nonnegative solutions of semilinear parabolic equations. See also Taliaferro [11] for semilinear parabolic inequalities. By the way, the fundamental solution of the Laplace equation is also the solution of the heat equation. Thus it is interesting to study removable singularities on $\{0\} \times (0, \infty)$ in the parabolic case. This problem was recently researched by Hsu [5] and Hui [6] for solutions of the heat equation and solutions of semilinear parabolic equations with a bounded nonlinear term. However, it is not known about semilinear parabolic equations of the form $-\Delta u + \partial_t u = |u|^{p-1}u$ for instance. Thus the purpose of this paper is to extend Hsu and Hui's result to such equations.

In this paper, we suppose $n \ge 3$ and denote a typical point in $\mathbb{R}^{n+1}$ by (x, t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Also, let Ω be a domain in $\mathbb{R}^n$ containing the origin 0 and let $T > 0$ be fixed. We study semilinear parabolic equations of the form

$$-\Delta u + \partial_t u = F(x, t, u, \nabla u), \tag{1.1}$$

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where Δ is the Laplacian on $\mathbb{R}^n$, ∇u the gradient of u and $\partial_t = \partial/\partial t$. Assume that F is a measurable function on $\Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^n$ satisfying

$$|F(x,t,u,\nabla u)| \leq C_1(1+|u|^p) \tag{1.2}$$

for some constant $C_1 > 0$ and

$$0 \leq p < \frac{n}{n-2}.$$

By saying a solution of (1.1), we mean a continuous function having continuous first partial derivatives with respect to the spatial variables and satisfying (1.1) in the sense of distributions. A solution u of (1.1) in $(\Omega \setminus \{0\}) \times (0,T)$ is said to have *removable singularities* on $\{0\} \times (0,T)$ if there exists a solution $\overline{u}$ of (1.1) in $\Omega \times (0,T)$ such that $\overline{u} = u$ on $(\Omega \setminus \{0\}) \times (0,T)$. We prove the following theorem.

Theorem 1.1. *Assume that F satisfies* (1.2) *for some $0 \leq p < n/(n-2)$, and suppose that u is a solution of* (1.1) *in $(\Omega \setminus \{0\}) \times (0,T)$. Then u has removable singularities on $\{0\} \times (0,T)$ if and only if for any $0 < t_1 < t_2 < T$ and $0 < \delta \leq 1$ there exists $r > 0$ such that*

$$|u(x,t)| \leq \delta \|x\|^{2-n} \tag{1.3}$$

for any $0 < \|x\| < r$ and $t_1 \leq t \leq t_2$.

Remark 1.2. In Theorem 1.1, the upper bound $p < n/(n-2)$ is optimal. Indeed, if $p > n/(n-2)$, then $u(x) = \|x\|^{-2/(p-1)}$ is a solution of $-\Delta u = V u^p$ in $\mathbb{R}^n \setminus \{0\}$ with

$$V = \frac{2\{(n-2)p - n\}}{(p-1)^2}.$$

Also, if $0 < \gamma \leq (n-2)/2$, then $u(x) = \|x\|^{2-n}(-\log\|x\|)^{-\gamma}$ satisfies $-\Delta u = V u^{n/(n-2)}$ in $B(0,1/10) \setminus \{0\}$, where

$$V(x) = \gamma\{n-2-(\gamma+1)(-\log\|x\|)^{-1}\}(-\log\|x\|)^{-1+2\gamma/(n-2)}$$

is nonnegative and bounded on $B(0,1/10)$. Therefore $u(x,t) = u(x)$ is the stationary solution of the corresponding parabolic equation $-\Delta u + \partial_t u = V u^p$ in $(B(0,1/10) \setminus \{0\}) \times (0,\infty)$, which satisfies (1.3), but can not be extended to $\{0\} \times (0,T)$ as a solution.

We say that u is a *temperature* on $\Omega \times (0,T)$ if $u \in C^{2,1}(\Omega \times (0,T))$ and u satisfies the heat equation $-\Delta u + \partial_t u = 0$ in $\Omega \times (0,T)$. The following corollary is the special case $F \equiv 0$ of Theorem 1.1.

Corollary 1.3. *Suppose that u is a temperature on $(\Omega \setminus \{0\}) \times (0,T)$. Then u has removable singularities on $\{0\} \times (0,T)$ if and only if for any $0 < t_1 < t_2 < T$ and $0 < \delta \leq 1$ there exists $r > 0$ such that* (1.3) *holds for any $0 < \|x\| < r$ and $t_1 \leq t \leq t_2$.*

This corollary and the removability theorem for bounded solutions of (1.1) with F being bounded were recently proved by Hsu [5]. His proofs are based on estimates for the Green functions of a circular cylinder and the exterior, and a careful analysis of the behavior of solutions near singularities using the Duhamel principle. After that, Hui [6] gave another proofs for Corollary 1.3 using the parabolic Schauder estimates and the maximum principle. But the proof of the essential fact $u \in L^\infty_{loc}(\Omega \times (0,T))$ is not easy. Also, we note that the maximum principle argument is not applicable to solutions of (1.1). Thus we give a proof based on the parabolic potential theory and an iteration argument developed in the area of nonlinear analysis. When $F \equiv 0$, it also provides a simple proof for Corollary 1.3.

Theorem 1.1 and its proof have some similarities with Giga and Kohn's result [4] concerning blow-up problems. Let $x_0 \in \Omega$ and $T > 0$. A solution u of (1.1) in $\Omega \times (0,T)$ is said to *blow up* at a point (x_0, T) if u is not locally bounded near (x_0, T). They proved that if u is a $C^{2,1}$-solution of (1.1)–(1.2) in $\Omega \times (0, T)$ for some $p > 1$ satisfying

$$|u(x,t)| \leq \varepsilon (T-t)^{-1/(p-1)} \tag{1.4}$$

for all $(x,t) \in B(x_0, r_0) \times (T - r_0^2, T)$ and for some small $\varepsilon \leq \varepsilon(C_1, p, n)$, then there is $0 < r_1 < r_0$ such that u is bounded on $B(x_0, r_1) \times (T - r_1^2, T)$. In other words, u does not blow up at the point (x_0, T). For the proof, they first used a Duhamel formulation and a Gronwall type inequality to obtain an estimate better than (1.4), and then iterated this argument until getting the boundedness of u. The last step in our proof of Theorem 1.1 is similar to their's. But, before proceeding to an iteration argument, we must first establish a Duhamel formulation on $\Omega \times (0, T)$ because solutions may have singularities on $\{0\} \times (0, T)$. Also, it is difficult to apply a Gronwall inequality in space directions. This will be conquerable by obtaining estimates for potentials of the density $\|\cdot\|^{-\alpha}$ (see Lemma 2.3). This iteration argument also yields the following theorem.

Theorem 1.4. *Assume that F satisfies* (1.2) *for some $p > 1$. Let u be a solution of* (1.1) *in $\Omega \times (0, T)$. If there are constants $q < 2/(p-1)$, $r > 0$ and $\delta > 0$ such that*

$$|u(x,t)| \leq \|x\|^{-q} \tag{1.5}$$

for any $\|x\| < r$ and $T - \delta < t < T$, then u does not blow up at the point $(0, T)$.

Also, concerning blow-up problems, we shall prove in Section 4 that if $p < (n+2)/n$, then integral solutions are locally bounded. This implies that the blow-up is complete.

2 Preliminary

This section collects some known results from the parabolic potential theory (see Doob's book [3] and Watson's paper [14] for details and further informations). We adopt Watson's terminology. Let D be a bounded domain in $\mathbb{R}^{n+1}$. A function $u : D \to (-\infty, +\infty]$ is called a *supertemperature* on D if u is lower semicontinuous on D, u is finite on a dense subset of D, and u satisfies the mean value inequality: for any $(x,t) \in D$ and small $0 < r < r_{(x,t)}$,

$$u(x,t) \geq \frac{1}{2^{n+2}(\pi r)^{n/2}} \int_{B(x,t;r)} \frac{\|x-y\|^2}{(t-s)^2} u(y,s)\, dyds,$$

where

$$B(x,t;r) = \left\{ (y,s) : s < t,\ \frac{1}{(t-s)^{n/2}} \exp\left\{ -\frac{\|x-y\|^2}{4(t-s)} \right\} > \frac{1}{r^{n/2}} \right\}.$$

If $-u$ is a supertemperature on D, then u is said to be a *subtemperature* on D. Also, a set E in $\mathbb{R}^{n+1}$ is called a *polar* set if there exists a supertemperature u defined on a neighborhood of $\overline{E}$ such that $u = +\infty$ on E. Observe that the function $u(x,t) = \|x\|^{2-n}$ is a supertemperature on $\mathbb{R}^{n+1}$, and so $\{0\} \times \mathbb{R}$ is a polar set. The following is the removability theorem for supertemperatures.

Lemma 2.1 ([14, Theorem 29])**.** *Let E be a relatively closed polar set in D. If u is a supertemperature and bounded below on $D \setminus E$, then the function*

$$\overline{u}(x,t) = \begin{cases} u(x,t) & ((x,t) \in D \setminus E), \\ \liminf\limits_{D \setminus E \ni (y,s) \to (x,t)} u(y,s) & ((x,t) \in E), \end{cases}$$

is a supertemperature on D.

We call the function $\overline{u}$ the *lower semicontinuous regularization* of u. Also, a temperature v on D satisfying $v \leq u$ on D is said to be a *thermic minorant* of u on D. The Riesz decomposition theorem for supertemperatures is stated as follows.

Lemma 2.2 ([14, Theorem 22]). *If u is a supertemperature on D, then there exists a unique measure μ on D such that $-\Delta u + \partial_t u = \mu$ in D in the sense of distributions. Moreover, if u is bounded below on D, then u is represented as*

$$u(x,t) = h(x,t) + \int_D G_D(x,t;y,s)\, d\mu(y,s) \quad \textit{for all } (x,t) \in D,$$

where h is the greatest thermic minorant of u on D and G_D is the Green function for D and the heat operator.

Note that for any $(x,t), (y,s) \in D$ with $s < t$,

$$G_D(x,t;y,s) \leq \frac{1}{\{4\pi(t-s)\}^{n/2}} \exp\left\{-\frac{\|x-y\|^2}{4(t-s)}\right\}.$$

Finally, we give an elementary estimate which plays an important role in proving the local boundedness of u in the proof of Theorem 1.1. By the symbol C, we denote an absolute positive constant whose value is unimportant and may change from one occurrence and the next.

Lemma 2.3. *Let $\alpha < n$ and $T > 0$. Then there exists a constant C depending only on α, T and n such that for all $x \in \mathbb{R}^n \setminus \{0\}$ and $0 < t < T$,*

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} \frac{1}{(t-s)^{n/2}} \exp\left\{-\frac{\|x-y\|^2}{4(t-s)}\right\} \|y\|^{-\alpha}\, dyds \\ \leq \begin{cases} C\|x\|^{2-\alpha} & \textit{if } 2 < \alpha < n, \\ C\left(1 + \log^+ \dfrac{1}{\|x\|}\right) & \textit{if } \alpha = 2, \\ C(\|x\|^{2-\alpha} + 1) & \textit{if } \alpha < 2, \end{cases}\end{aligned} \tag{2.1}$$

where $\log^+ a = \max\{\log a, 0\}$.

Proof. Let $x \in \mathbb{R}^n \setminus \{0\}$ be fixed and let $\Omega_1 = \{y : \|y\| \leq \|x\|/2\}$, $\Omega_2 = \{y : \|x\|/2 \leq \|y\| \leq 2\|x\|\}$ and $\Omega_3 = \{y : \|y\| \geq 2\|x\|\}$. Then the integral in (2.1) is not greater than $I_1 + I_2 + I_3$, where

$$I_j = \int_0^t \int_{\Omega_j} \frac{1}{(t-s)^{n/2}} \exp\left\{-\frac{\|x-y\|^2}{4(t-s)}\right\} \|y\|^{-\alpha}\, dyds.$$

Let us estimate I_j. Note that $\int_0^\infty \rho^{(n-4)/2} \exp(-\rho)\, d\rho < \infty$ when $n \geq 3$. Since $\|x - y\| \geq \|x\| - \|y\| \geq \|x\|/2$ for $y \in \Omega_1$, we have

$$\begin{aligned} I_1 &\leq \int_0^t \int_{\Omega_1} \frac{1}{(t-s)^{n/2}} \exp\left\{-\frac{\|x\|^2}{16(t-s)}\right\} \|y\|^{-\alpha}\, dyds \\ &\leq C\|x\|^{n-\alpha} \int_0^t \frac{1}{(t-s)^{n/2}} \exp\left\{-\frac{\|x\|^2}{16(t-s)}\right\} ds \\ &\leq C\|x\|^{2-\alpha} \int_{\|x\|^2/16t}^\infty \rho^{(n-4)/2} \exp(-\rho)\, d\rho \\ &\leq C\|x\|^{2-\alpha}. \end{aligned}$$

Also, $\|y - x\| \leq \|y\| + \|x\| \leq 3\|x\|$ for $y \in \Omega_2$. Therefore

$$\begin{aligned} I_2 &\leq C\|x\|^{-\alpha} \int_0^t \int_{\{y : \|y-x\| \leq 3\|x\|\}} \frac{1}{(t-s)^{n/2}} \exp\left\{-\frac{\|x-y\|^2}{4(t-s)}\right\} dyds \\ &\leq C\|x\|^{-\alpha} \int_0^{3\|x\|} \int_0^t \frac{r^{n-1}}{(t-s)^{n/2}} \exp\left\{-\frac{r^2}{4(t-s)}\right\} dsdr \\ &\leq C\|x\|^{-\alpha} \left(\int_0^{3\|x\|} r\, dr\right)\left(\int_0^\infty \rho^{(n-4)/2} \exp(-\rho)\, d\rho\right) \\ &\leq C\|x\|^{2-\alpha}. \end{aligned}$$

For $y \in \Omega_3$, we have $\|x-y\| \geq \|y\| - \|x\| \geq \|y\|/2$, and so

$$\begin{aligned} I_3 &\leq \int_0^t \int_{\Omega_3} \frac{1}{(t-s)^{n/2}} \exp\left\{-\frac{\|y\|^2}{16(t-s)}\right\} \|y\|^{-\alpha}\, dyds \\ &\leq C \int_{2\|x\|}^{\infty} \int_0^t \frac{r^{n-\alpha-1}}{(t-s)^{n/2}} \exp\left\{-\frac{r^2}{16(t-s)}\right\} dsdr \\ &\leq C \int_{2\|x\|}^{\infty} r^{1-\alpha} \left(\int_{r^2/16t}^{\infty} \rho^{(n-4)/2} \exp(-\rho)\, d\rho \right) dr. \end{aligned} \tag{2.2}$$

If $\alpha > 2$, then $I_3 \leq C\|x\|^{2-\alpha}$. Hence (2.1) follows in this case. Let $\alpha \leq 2$. By (2.2), we have for $0 < t < T$,

$$\begin{aligned} I_3 &\leq C \left(\int_{2\|x\|}^{\infty} r^{1-\alpha} \exp\left(-\frac{r^2}{32T}\right) dr \right) \left(\int_0^{\infty} \rho^{(n-4)/2} \exp\left(-\frac{\rho}{2}\right) d\rho \right) \\ &\leq C \int_{2\|x\|}^{\infty} r^{1-\alpha} \exp\left(-\frac{r^2}{32T}\right) dr. \end{aligned}$$

If $2\|x\| \geq 1$, then $I_3 \leq C$. If $2\|x\| < 1$, then

$$\begin{aligned} I_3 &\leq C \left(\int_{2\|x\|}^{1} r^{1-\alpha}\, dr + \int_1^{\infty} r^{1-\alpha} \exp\left(-\frac{r^2}{32T}\right) dr \right) \\ &\leq \begin{cases} C & (\alpha < 2), \\ C \log \dfrac{1}{\|x\|} + C & (\alpha = 2). \end{cases} \end{aligned}$$

Combining the above estimates yields (2.1) for $\alpha \leq 2$. $\square$

3 Proofs of Theorems 1.1 and 1.4

As given in [5, p. 156], the proof of necessity in Theorem 1.1 is easy because u is bounded near $\{0\} \times (t_1, t_2)$. We provide a proof for sufficiency.

Proof of Theorem 1.1 (sufficiency). Let $0 < \delta \leq 1$ and $0 < t_1 < t_2 < t_3 < t_4 < T$. By assumption, there is $r_0 > 0$ such that (1.3) holds for all $0 < \|x\| < r_0$ and $t_1 \leq t \leq t_4$. Take a bounded open set ω with $\overline{\omega} \subset \Omega$ and write $D = \omega \times (t_2, t_3)$ and $D_0 = (\omega \setminus \{0\}) \times (t_2, t_3)$. Since u is continuous on $(\Omega \setminus \{0\}) \times (0, T)$, it follows from (1.2) and (1.3) with $\delta = 1$ that there is a constant C_2 such that for all $(x,t) \in (\omega \setminus \{0\}) \times (t_1, t_4)$,

$$|F(x,t,u,\nabla u)| \leq C_2 \|x\|^{p(2-n)}. \tag{3.1}$$

Claim 1: We first show that there exists a temperature h on D such that for all $(x,t) \in D_0$,

$$u(x,t) = h(x,t) + \int_{D_0} G_D(x,t;y,s) F(y,s,u,\nabla u)\, dyds. \tag{3.2}$$

To this end, we let

$$v(x,t) = C_2 \int_{D_0} G_D(x,t;y,s) \|y\|^{p(2-n)}\, dyds,$$

and consider

$$u_\delta(x,t) = u(x,t) + v(x,t) + \delta \|x\|^{2-n}. \tag{3.3}$$

Then u_δ is continuous on D_0. Since $\|\cdot\|^{2-n}$ is a temperature on D_0, we observe from (1.1) and (3.1) that $-\Delta u_\delta + \partial_t u_\delta \geq 0$ in D_0 in the sense of distributions. Therefore u_δ is a supertemperature on D_0. Since u_δ is bounded below on D_0, the lower semicontinuous regularization $\overline{u}_\delta$ is a supertemperature on D, and so there exists a unique measure μ_δ such that $-\Delta\overline{u}_\delta + \partial_t\overline{u}_\delta = \mu_\delta$ in D. By the Riesz decomposition theorem, we have for all $(x,t) \in D$,

$$\overline{u}_\delta(x,t) = h_\delta(x,t) + \int_D G_D(x,t;y,s)\, d\mu_\delta(y,s), \tag{3.4}$$

where h_δ is the greatest thermic minorant of $\overline{u}_\delta$ on D. Let $0 < r < \min\{r_0, \sqrt{t_2-t_1}, \sqrt{t_4-t_3}\}$ be small. Write B_r for the open ball of center 0 and radius r in $\mathbb{R}^n$. By (1.3) and Lemma 2.3, we find a constant C independent of r such that for all $(x,t) \in B_r \times (t_1,t_4)$,

$$|\overline{u}_\delta(x,t)| \le \begin{cases} 2\delta\|x\|^{2-n} + C\|x\|^{p(2-n)+2} & \text{if } p(n-2) > 2, \\ 2\delta\|x\|^{2-n} + C\log\dfrac{1}{\|x\|} + C & \text{if } p(n-2) = 2, \\ 2\delta\|x\|^{2-n} + C & \text{if } p(n-2) < 2. \end{cases}$$

Take a nonnegative function $\phi \in C_0^\infty(B_r \times (t_1,t_4))$ such that $\phi = 1$ on $\{0\} \times (t_2,t_3)$ and $|-\Delta\phi - \partial_t\phi| \le Cr^{-2}$. Then

$$\begin{aligned} \mu_\delta(\{0\} \times (t_2,t_3)) &\le \int_{B_r\times(t_1,t_4)} \phi\, d\mu_\delta \\ &= \int_{B_r\times(t_1,t_4)} (-\Delta\phi - \partial_t\phi)\overline{u}_\delta\, dyds \\ &\le \begin{cases} C(\delta + r^{p(2-n)+n}) & \text{if } p(n-2) > 2, \\ C(\delta + r^{n-2}\log\dfrac{1}{r}) & \text{if } p(n-2) = 2, \\ C(\delta + r^{n-2}) & \text{if } p(n-2) < 2. \end{cases} \end{aligned}$$

Since $p(2-n)+n > 0$ and $r > 0$ is arbitrary, we have

$$\mu_\delta(\{0\} \times (t_2,t_3)) \le C\delta.$$

Let $(x,t) \in D_0$. Then $G_D(x,t;\cdot,\cdot)$ is bounded on $\{0\} \times (t_2,t_3)$, and so

$$\lim_{\delta\to 0} \int_{\{0\}\times(t_2,t_3)} G_D(x,t;y,s)\, d\mu_\delta(y,s) = 0.$$

Observe that $\mu_\delta = -\Delta u_\delta + \partial_t u_\delta = F(x,t,u,\nabla u) + C_2\|x\|^{p(2-n)}$ in D_0. The uniqueness of such a measure implies that $d\mu_\delta(y,s) = \{F(y,s,u,\nabla u) + C_2\|y\|^{p(2-n)}\}\, dyds$ on D_0. Also, as proved (3.1), there is a positive constant C independent of δ such that $\overline{u}_\delta(x,t) \ge u(x,t) \ge -C\|x\|^{2-n}$ for all $(x,t) \in D_0$. Since

$$h_\delta(x,t) = \sup\{w(x,t) : w \text{ is a subtemperature on } D \text{ such that } w \le \overline{u}_\delta\},^{1} \tag{3.5}$$

we have $h_\delta(x,t) \ge -C\|x\|^{2-n}$ for all $(x,t) \in D$. Therefore h_δ converges decreasingly to a temperature h on D as $\delta \searrow 0$. Then it follows from (3.4) that for all $(x,t) \in D_0$,

$$u(x,t) + v(x,t) = h(x,t) + \int_{D_0} G_D(x,t;y,s)F(y,s,u,\nabla u)\, dyds + v(x,t).$$

Thus Claim 1 is proved. Note that h is bounded apart from the point $(0,t_2)$.

Claim 2: Next, we show that u is bounded on D_0. We give a proof for the case $1 < p < n/(n-2)$, which actually covers the proof for the case $0 \le p \le 1$ (see Remark 3.1). Then $0 < n-p(n-2) < 2$. Let N be the smallest number satisfying

$$N \ge \frac{\log\dfrac{2}{n-p(n-2)}}{\log p},$$

[1] In (3.5), the set taking the supremum is the saturated family of subtemperatures, so the right hand side is a temperature on D (see [14, Theorem 7]).

which is equivalent to

$$\beta := p^N(2-n) + 2p^{N-1} + \cdots + 2p + 2 \geq 0.$$

To apply Lemma 2.3, we note, in arguments below, that for $j = 2, \cdots, N-1$,

$$-n < p(2-n) < p^j(2-n) + 2p^{j-1} + \cdots + 2p < -2.$$

Take $t_1 < \tau_1 < \tau_2 < \cdots < \tau_{N+2} = t_2$ and let $D^j = \omega \times (\tau_j, t_3)$ and $D_0^j = (\omega \setminus \{0\}) \times (\tau_j, t_3)$. By Claim 1, there is a temperature h_j on D^j such that for all $(x,t) \in D_0^j$,

$$u(x,t) = h_j(x,t) + \int_{D_0^j} G_{D^j}(x,t;y,s)F(y,s,u,\nabla u)\,dyds.$$

Since h_1 is bounded on D^2, it follows from (3.1) and Lemma 2.3 that for all $(x,t) \in D_0^2$,

$$|u(x,t)| \leq C + C\|x\|^{p(2-n)+2} \leq C\|x\|^{p(2-n)+2}. \tag{3.6}$$

Then (1.2) and (3.6) imply that $|F(x,t,u,\nabla u)| \leq C\|x\|^{p^2(2-n)+2p}$ for all $(x,t) \in D_0^2$. Since h_2 is bounded on D^3, it follows from Lemma 2.3 that for all $(x,t) \in D_0^3$,

$$|u(x,t)| \leq C + C\|x\|^{p^2(2-n)+2p+2} \leq C\|x\|^{p^2(2-n)+2p+2}.$$

Repeat this process $N-1$ times. Then, for all $(x,t) \in D_0^N$,

$$|u(x,t)| \leq C\|x\|^{p^{N-1}(2-n)+2p^{N-2}+\cdots+2p+2},$$

and so $|F(x,t,u,\nabla u)| \leq C(1+\|x\|^{\beta-2})$ by (1.2). The boundedness of h_N on D^{N+1} and Lemma 2.3 yield that for all $(x,t) \in D_0^{N+1}$,

$$|u(x,t)| \leq \begin{cases} C & (\beta > 0), \\ C + C\log^+ \dfrac{1}{\|x\|} & (\beta = 0). \end{cases}$$

Therefore, if $\beta > 0$, then Claim 2 follows since $D_0 \subset D_0^{N+1}$. If $\beta = 0$, then the above inequality implies that $|u(x,t)| \leq C\|x\|^{-1/p}$, and so $|F(x,t,u,\nabla u)| \leq C\|x\|^{-1}$ for all $(x,t) \in D_0^{N+1}$. Applying Lemma 2.3 again, we obtain $|u(x,t)| \leq C$ for all $(x,t) \in D_0^{N+2} = D_0$. Thus Claim 2 is proved.

Finally, we observe from (1.2) and Claim 2 that the integral in (3.2) is continuous on D and has continuous first partial derivatives with respect to the spatial variables (see [3, pp. 303–305]). This implies that u has a continuous extension, $\overline{u}$ say, to $\Omega \times (0,T)$ because t_2, t_3 and ω are arbitrary. Also, $\nabla\overline{u}$ exists and, by (3.2), we have for all $(x,t) \in D$,

$$\overline{u}(x,t) = h(x,t) + \int_D G_D(x,t;y,s)F(y,s,\overline{u},\nabla\overline{u})\,dyds.$$

Since t_2, t_3 and ω are arbitrary, this implies that $\overline{u}$ is a solution of (1.1) in $\Omega \times (0,T)$. Hence u has removable singularities on $\{0\} \times (0,T)$. This completes the proof. □

Remark 3.1. When $0 \leq p \leq 1$, we take $1 < q < n/(n-2)$. Since ω is bounded, we have $\|x\|^{-\alpha} \leq C\|x\|^{-\beta}$ for all $x \in \omega$ if $0 \leq \alpha \leq \beta$. This implies that (3.1) and each estimate for $|u|$ or $|F|$ in Claim 2 are valid for q in place of p. Hence Claim 2 is true for $0 \leq p \leq 1$ as well.

Remark 3.2. When $F \equiv 0$, we can remove v from u_δ in (3.3). Also, Claim 1 shows that u can be extended to $\Omega \times (0,T)$ as a temperature. Hence we do not need any arguments in Claim 2. Thus the proof of Corollary 1.3 is simpler.

Proof of Theorem 1.4. The proof is almost the same as Step 2 in the proof of Theorem 1.1. Let $D = B(0,r) \times (T-\delta, T)$. By the Duhamel principle, there exists a temperature h on D such that for all $(x,t) \in D$,

$$u(x,t) = h(x,t) + \int_D G_D(x,t;y,s)F(y,s,u,\nabla u)\,dyds.$$

Since u is bounded on the parabolic boundary of D by the continuity and (1.5), the maximum principle shows that h is bounded on D. For convenience, let $-q = \varepsilon - 2/(p-1)$ for some $\varepsilon > 0$ and let N be the smallest number such that $\varepsilon p^N \geq 2/(p-1)$. Let $(x,t) \in D$. Then, by (1.2) and (1.5), we have $|F(x,t,u,\nabla u)| \leq C\|x\|^{\varepsilon p - 2p/(p-1)}$, and so $|u(x,t)| \leq C\|x\|^{\varepsilon p - 2/(p-1)}$ by Lemma 2.3. Repeat this process $N-1$ times. Then $|u(x,t)| \leq C\|x\|^{\varepsilon p^{N-1} - 2/(p-1)}$, and so

$$|F(x,t,u,\nabla u)| \leq C\{1 + \|x\|^{\varepsilon p^N - 2p/(p-1)}\}.$$

As in the final of Step 2, we can show that u is bounded on D. Hence u does not blow up at the point $(0,T)$. □

Remark 3.3. Now, let u be a nonnegative classical solution of $-\Delta u + \partial_t u = u^p$ in $\mathbb{R}^n \times (0,T)$, which blows up at the point $(0,T)$. As shown by Merle [8], there exists a profile $u(\cdot,T)$ such that $u(\cdot,t)$ converges to $u(\cdot,T)$ uniformly on compact sets of $\mathbb{R}^n \setminus \{0\}$ as $t \to T-0$. Then Velázquez's result [12] implies that there is a unitary vector $a \in \mathbb{R}^n$ such that for small $r > 0$,

$$u(ra,T) \geq \frac{1}{C}\left(\frac{r^2}{|\log r|}\right)^{-1/(p-1)}.$$

Thus it is an interesting question whether one can take $q = 2/(p-1)$ in Theorem 1.4.

4 Integral solutions and complete or incomplete blow-up

This section deals with complete or incomplete blow-up of nonnegative solutions of

$$-\Delta u + \partial_t u = F(x,t,u) \quad \text{in } \Omega \times (0,\infty), \tag{4.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{4.2}$$

$$u(x,0) = u_0(x) \quad \text{for all } x \in \Omega, \tag{4.3}$$

where u_0 is nonnegative and bounded on Ω. Assume that Ω is regular for the Dirichlet problem (to understand (4.2) in a usual sense) and that F is a nonnegative measurable function on $\Omega \times (0,\infty) \times [0,\infty]$ satisfying

$$0 \leq F(x,t,u) \leq C(1 + u^p) \tag{4.4}$$

for some constant $C > 0$. We say that u blows up in a finite time T if u is a continuous function on $\Omega \times (0,T)$ satisfying (4.1) in $\Omega \times (0,T)$ in the sense of distributions and

$$\limsup_{t \to T-0} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Throughout this section, we simply write G_Ω for the Green function $G_{\Omega\times(0,\infty)}$. A nonnegative measurable function u on $\Omega \times (0,\infty)$ is an *integral solution* of (4.1) if there exists a nonnegative temperature h on $\Omega \times (0,\infty)$ such that for a.e. $(x,t) \in \Omega \times (0,\infty)$,

$$u(x,t) = h(x,t) + \int_{\Omega\times(0,\infty)} G_\Omega(x,t;y,s)F(y,s,u)\,dyds. \tag{4.5}$$

If u satisfies (4.2) and (4.3), then $h(x,t) = \int_\Omega G_\Omega(x,t;y,0)u_0(y)\,dy$. Given an integral solution u, we write

$$T^* = T^*(u) = \sup\{t : u \text{ is finite a.e. on } \Omega \times (0,t)\}.$$

Observe that $u = \infty$ on $\Omega \times (T^*, \infty)$. Let us define complete or incomplete blow-up for continuous solutions u of (4.1)–(4.3) in $\Omega \times (0, T)$. We say that u blows up *completely* at a time T if $T = T^*(U)$ for any integral solution U of (4.1)–(4.3) satisfying $U = u$ on $\Omega \times (0, T)$. If $T < T^*(U)$ for some integral solution U of (4.1)–(4.3) satisfying $U = u$ on $\Omega \times (0, T)$, then the blow-up is *incomplete*.

In [1], Baras and Cohen proved that the blow-up is complete when the nonlinear term F is independent of (x, t) and is comparable to u^p with $1 < p < (n+2)/(n-2)$. Quittner and Simondon [10] investigated complete blow-up in the case $F = V(x)u(x)^p$ and gave sufficient conditions for V and p. The next result is applicable to more general nonlinearity.

Theorem 4.1. *Assume that F satisfies* (4.4) *for some $0 < p < (n + 2)/n$. Let u be a nonnegative integral solution of* (4.1)*. Then u is locally bounded and continuous on $\Omega \times (0, T^*)$.*

Proof. If u is locally bounded on $\Omega \times (0, T^*)$, then $F(\cdot, \cdot, u)$ is so by (4.4). Therefore we see that u is continuous there. Let us prove the local boundedness of u. Let $x_0 \in \Omega$ and $0 < t_0 < T^*$. Take $r > 0$ satisfying $t_0 - r^2 > 0$ and $B(x_0, 2r) \subset \Omega$ and write $Q_j = B(x_0, r/2^j) \times (t_0 - r^2/2^{2j}, t_0)$. It suffices to show that u is bounded on Q_m for some m. By the definition of T^*, we find a point $(x_1, t_1) \in B(x_0, r) \times (t_0, T^*)$ such that $u(x_1, t_1)$ is finite. It is known from [15] that there exists a constant $C > 1$ depending only on r, T^* and n such that for all $x, y \in B(x_0, r)$ and $s < t < T^*$,

$$G_\Omega(x, t; y, s) \geq G_{B(x_0,2r)}(x, t; y, s) \geq \frac{1}{C(t - s)^{n/2}} \exp\left\{-\frac{C\|x - y\|^2}{t - s}\right\}.$$

Since h is nonnegative, we have by (4.5)

$$\begin{aligned} \infty > u(x_1, t_1) &\geq \int_{Q_0} G_\Omega(x_1, t_1; y, s)F(y, s, u)\, dyds \\ &\geq \frac{1}{C} \int_{Q_0} F(y, s, u)\, dyds. \end{aligned} \tag{4.6}$$

Let $(x, t) \in Q_{j+1}$ and $(y, s) \in (\Omega \times (0, \infty)) \setminus Q_j$. Since $G_\Omega(\cdot, \cdot; y, s)$ is a nonnegative temperature on $(\Omega \times (0, \infty)) \setminus \{(y, s)\}$, it follows from Harnack's inequality that there exists a constant c_j depending on j such that

$$G_\Omega(x, t; y, s) \leq c_j G_\Omega(x_1, t_1; y, s),$$

and so

$$\int_{(\Omega\times(0,\infty))\setminus Q_j} G_\Omega(x, t; y, s)F(y, s, u)\, dyds \leq c_j u(x_1, t_1).$$

Since h is bounded on Q_0, we have by (4.5) that for a.e. $(x, t) \in Q_{j+1}$,

$$u(x, t) \leq C + c_j u(x_1, t_1) + \int_{Q_j} G_\Omega(x, t; y, s)F(y, s, u)\, dyds. \tag{4.7}$$

Also, in arguments below, we use the elementary fact that $G_\Omega(x, t; \cdot, \cdot)^q$ and $G_\Omega(\cdot, \cdot; y, s)^q$ are locally integrable on $\Omega \times (0, \infty)$ if $q < (n + 2)/n$.

Let

$$\max\{p, 1\} < q < \frac{n + 2}{n} \quad \text{and} \quad \ell = \left[\frac{\log(q/(q - 1))}{\log(q/p)}\right] + 1.$$

For simplicity, we write

$$\Psi_j(x, t) = \int_{Q_j} G_\Omega(x, t; y, s)F(y, s, u)\, dyds.$$

Then (4.7) gives that for $j = 0, 1, \ldots, \ell$ and a.e. $(x, t) \in Q_{j+1}$,

$$u(x, t) \leq C + \Psi_j(x, t). \tag{4.8}$$

Let $\kappa \geq 1$. By Jensen's inequality, we have for a.e. $(x,t) \in Q_0$,

$$\Psi_j(x,t)^\kappa \leq C \int_{Q_j} G_\Omega(x,t;y,s) F(y,s,u)^\kappa \, dyds.$$

This and Minkowski's inequality for integrals give

$$\begin{aligned}\left(\int_{Q_0} \Psi_j(x,t)^{\kappa q} \, dxdt\right)^{1/q} &\leq C \int_{Q_j} \left(\int_{Q_0} G_\Omega(x,t;y,s)^q \, dxdt\right)^{1/q} F(y,s,u)^\kappa \, dyds \\ &\leq C \int_{Q_j} F(y,s,u)^\kappa \, dyds.\end{aligned}$$

By the way, (4.4) and (4.8) imply that for a.e. $(x,t) \in Q_{j+1}$,

$$F(x,t,u) \leq C(1+u^p) \leq C + C\Psi_j(x,t)^p.$$

Let $\alpha = q/p$. Then

$$\int_{Q_{j+1}} F(x,t,u)^{\kappa\alpha} \, dxdt \leq C + C\left(\int_{Q_j} F(y,s,u)^\kappa \, dyds\right)^q.$$

Using this inequality ℓ times, we obtain

$$\int_{Q_\ell} F(y,s,u)^{\alpha^\ell} \, dyds \leq C + C\left(\int_{Q_0} F(y,s,u) \, dyds\right)^{q^\ell} \leq C.$$

Here the last inequality is by (4.6). Since our choice of ℓ implies that

$$\frac{\alpha^\ell}{\alpha^\ell - 1} \leq q < \frac{n+2}{n},$$

it follows from (4.8) and Hölder's inequality that for a.e. $(x,t) \in Q_{\ell+1}$,

$$u(x,t) \leq C + C\left(\int_{Q_\ell} F(y,s,u)^{\alpha^\ell} \, dyds\right)^{1/\alpha^\ell} \leq C.$$

The lower semicontinuity of u concludes that $u \leq C$ on $Q_{\ell+1}$. This completes the proof of Theorem 4.1. □

Corollary 4.2. *Assume that F satisfies* (4.4) *for some $0 < p < (n+2)/n$. Let u be a solution of* (4.1)–(4.3) *which blows up in finite time. Then the blow-up is complete.*

Proof. Let T be a blow-up time and let u blow up at a point (x_0, T). Then $x_0 \in \Omega$ by (4.2). Suppose to the contrary that $T^*(U) > T$ for some integral solution U of (4.1)–(4.3) satisfying $U = u$ on $\Omega \times (0,T)$. Then Theorem 4.1 implies that U is bounded on a neighborhood of the point (x_0, T), and so u is bounded on $B(x_0, r) \times (T - r^2, T)$ for small $r > 0$. This is a contradiction. Hence the blow-up is complete. □

In Corollary 4.2, the upper bound of p is nearly optimal.

Theorem 4.3. *Let $T > 0$. If $p > (n+2)/n$, then there exist $V \in C^\infty((\mathbb{R}^n \times (0,\infty)) \setminus \{(0,T)\})$ with $0 \leq V \leq 1$, a nonnegative bounded continuous function u_0 on $\mathbb{R}^n$ and an integral solution $u \in C^{2,1}((\mathbb{R}^n \times (0,\infty)) \setminus \{(0,T)\})$ of*

$$\begin{aligned} -\Delta u + \partial_t u &= V u^p \quad \textit{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) &= u_0(x) \quad \textit{for all } x \in \mathbb{R}^n, \end{aligned} \tag{4.9}$$

such that u blows up incompletely at the point $(0,T)$. Moreover,

$$\limsup_{\|x\| \to 0} \|x\|^{2/(p-1)} u(x,T) > 0 \quad \textit{and} \quad \limsup_{t \to T-0} (T-t)^{1/(p-1)} u(0,T) > 0.$$

For the proof, we need the following elementary estimate.

Lemma 4.4. *There exists a positive constant C_3 depending only on n such that for each $R > 0$,*

$$\int_{B(0,R)} G_{\mathbb{R}^n}(x,t;y,0)\,dy \geq C_3,$$

whenever $x \in B(0,2R)$ and $R^2/8 \leq t \leq R^2$.

Proof. Let $x \in B(0,2R)$ and $t > 0$. By the change of variables $z = (x-y)/\sqrt{4t}$, we have

$$\begin{aligned}\int_{B(0,R)} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right) dy &= \frac{1}{\pi^{n/2}} \int_{B(x/\sqrt{4t},R/\sqrt{4t})} \exp(-\|z\|^2)\,dz \\ &\geq \frac{1}{C}\left(\frac{R}{\sqrt{4t}}\right)^n \exp\left(-\frac{9R^2}{4t}\right).\end{aligned}$$

The right hand side is bounded below by a positive constant when $R^2/8 \leq t \leq R^2$. $\square$

Proof of Theorem 4.3. By the scaling $u_r(x,t) = r^{2/(p-1)}u(rx, r^2t)$, it suffices to consider the case $T = 1/5$. For $j \in \mathbb{N} \cup \{0\}$, let $R_j = 1/4^j$, $r_j = R_j/4$ and $t_j = (1 - R_j^2)/5$. Then

$$\frac{1}{8}R_j^2 < (t_{j+1} - 3r_{j+1}^2) - t_j < T - (t_j - 2r_j^2) < R_j^2. \tag{4.10}$$

Let $x_j = (4R_j/3)e_1$, where $e_1 = (1, 0, \ldots, 0)$. Then

$$0 \in B(x_{j+1}, 2R_{j+1}) \subset B(x_j, 2R_j). \tag{4.11}$$

Also, the first inequality in (4.10) implies that $\{B(x_j, 2R_j) \times (t_j - 3r_j^2, t_j)\}_{j=1}^{\infty}$ is mutually disjoint. Take a constant $C_4 > 0$ satisfying

$$\left(\frac{C_3C_4}{16}\right)^p \geq 4^{\frac{2p}{p-1}}C_4, \tag{4.12}$$

where C_3 is the constant in Lemma 4.4. Let u_0 be a continuous function on $\mathbb{R}^n$ such that $0 \leq u_0 \leq C_4$ and

$$u_0 = \begin{cases} C_4 & \text{on } B(x_0, R_0), \\ 0 & \text{outside } B(x_0, 2R_0). \end{cases}$$

Also, for $j \in \mathbb{N}$, we take $f_j \in C^\infty(\mathbb{R}^n \times (0,\infty))$ with $0 \leq f_j \leq C_4R_j^{-2p/(p-1)}$ and

$$f_j = \begin{cases} C_4R_j^{-2p/(p-1)} & \text{on } B(x_j, R_j) \times (t_j - 2r_j^2, t_j - r_j^2), \\ 0 & \text{outside } B(x_j, 2R_j) \times (t_j - 3r_j^2, t_j). \end{cases}$$

Define $f = \sum_{j=1}^{\infty} f_j$ and

$$u(x,t) = \int_{\mathbb{R}^n} G_{\mathbb{R}^n}(x,t;y,0)u_0(y)\,dy + \int_{\mathbb{R}^n\times(0,\infty)} G_{\mathbb{R}^n}(x,t;y,s)f(y,s)\,dyds.$$

Since $p > (n+2)/n$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n\times(0,\infty)} f(y,s)\,dyds &= \sum_{j=1}^{\infty}\int_{\mathbb{R}^n\times(0,\infty)} f_j(y,s)\,dyds \\ &\leq C\sum_{j=1}^{\infty} R_j^{n+2-2p/(p-1)} < \infty,\end{aligned}$$

and so u is finite a.e. on $\mathbb{R}^n \times (0,\infty)$. Since f and ∇f are bounded on $\mathbb{R}^n \times (0, T-\varepsilon)$ for $\varepsilon > 0$ and u is a temperature outside the support of f, we see that $u \in C^{2,1}((\mathbb{R}^n \times (0,\infty)) \setminus \{(0,T)\})$.

In order to obtain lower estimates for u, we use Lemma 4.4 after a suitable translation. If $(x,t) \in B(x_1, 2R_1) \times (t_1 - 3r_1^2, T]$, then we have by (4.10), (4.11) and Lemma 4.4

$$u(x,t) \geq \int_{B(x_0,R_0)} G_{\mathbb{R}^n}(x,t;y,0)u_0(y)\,dy \geq C_3C_4.$$

If $(x,t) \in B(x_{j+1}, 2R_{j+1}) \times (t_{j+1} - 3r_{j+1}^2, T]$ for some $j \in \mathbb{N}$, then

$$\begin{aligned} u(x,t) &\geq \int_{t_j-2r_j^2}^{t_j-r_j^2} \int_{B(x_j,R_j)} G_{\mathbb{R}^n}(x,t;y,s)f_j(y,s)\,dyds \\ &\geq C_3C_4R_j^{-2p/(p-1)}r_j^2 = \frac{C_3C_4}{16}R_j^{-2/(p-1)}. \end{aligned} \tag{4.13}$$

These and (4.12) imply that if $(x,t) \in B(x_{j+1}, 2R_{j+1}) \times (t_{j+1} - 3r_{j+1}^2, t_{j+1})$ for some $j \in \mathbb{N} \cup \{0\}$, then

$$f(x,t) = f_{j+1}(x,t) \leq C_4 4^{2p/(p-1)} R_j^{-2p/(p-1)} \leq u(x,t)^p.$$

For $(x,t) \notin \bigcup_{j=0}^{\infty} B(x_{j+1}, 2R_{j+1}) \times (t_{j+1} - 3r_{j+1}^2, t_{j+1})$, we have

$$f(x,t) = 0 \leq u(x,t)^p.$$

Since u is positive, we define $V(x,t) = f(x,t)/u(x,t)^p$. Then $V \in C^\infty((\mathbb{R}^n \times (0,\infty)) \setminus \{(0,T)\})$ and $0 \leq V \leq 1$. By definition, u is an integral solution of (4.9). Moreover, (4.13) gives

$$\liminf_{j\to\infty} \|x_j\|^{\frac{2}{p-1}} u(x_j,T) \geq \left(\frac{1}{3}\right)^{\frac{2}{p-1}} \frac{C_3C_4}{16}$$

and

$$\liminf_{j\to\infty} (T-t_j)^{\frac{1}{p-1}} u(0,t_j) \geq \left(\frac{1}{80}\right)^{\frac{1}{p-1}} \frac{C_3C_4}{16}.$$

Thus Theorem 4.3 is proved. □

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