Positive solutions with a time-independent boundary singularity of semilinear heat equations in bounded Lipschitz domains

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Abstract

We study time-global positive solutions of semilinear heat equations of the form $u_t - \Delta u = f(x,u)$ in a bounded Lipschitz domain Ω in $\mathbb{R}^n$. In particular, we show the existence of a positive solution with a time-independent singularity at a boundary point ξ of Ω which converges to a positive solution, with the behavior like the Martin kernel at ξ, of the corresponding elliptic equation at time infinity. A nonlinear term f is conditioned in terms of a certain Lipschitz continuity with respect to the second variable and a generalized Kato class associated with the Martin kernel at ξ, and admits not only usual one $V(x)u^p(\log(1+u))^q$, but also one with variable exponents.

Keywords: singular solution, semilinear heat equation, semilinear elliptic equation, heat kernel, Green function, Martin kernel, Kato class.

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1 Introduction

During the last few decades, the existence and the asymptotic behavior of time-global positive solutions, with a time-independent singularity or a time-dependent singularity, of the heat equation with a nonlinear reaction term in the whole space $\mathbb{R}^n$ or in a bounded domain in $\mathbb{R}^n$ have been studied extensively. Now, let Ω be a domain in $\mathbb{R}^n$ $(n \geq 3)$ containing the origin 0, and consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = V(x)u^p & \text{in } (\Omega \setminus \{0\}) \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for all } x \in \Omega. \end{cases} \tag{1.1}$$

Here, $u = u(x,t)$, Δ is the Laplacian with respect to $x \in \mathbb{R}^n$, $u_t = \partial u/\partial t$, V is a nonnegative Borel measurable function on Ω, $p > 1$, u_0 is a nonnegative continuous function on Ω, and the equation $u_t - \Delta u = V(x)u^p$ is understood in the sense of distributions. In [15], Sato proved that, in the case where $\Omega = \mathbb{R}^n$, $V(x) \equiv 1$ and

$$\frac{n}{n-2} < p < \begin{cases} \dfrac{n+2\sqrt{n-1}}{n-4+2\sqrt{n-1}} & \text{if } n \leq 10, \\ \dfrac{n+2}{n-1} & \text{if } n > 10, \end{cases}$$

the problem (1.1) has a time-global positive solution u with a time-independent singularity at the origin such that for each $t > 0$, $\|x\|^{2/(p-1)}u(x,t)$ converges to some constant L depending only on p and n as $x \to 0$, and for each $x \in \mathbb{R}^n \setminus \{0\}$, $u(x,t)$ converges to the singular steady-state $L\|x\|^{-2/(p-1)}$ as $t \to \infty$, whenever the initial value $u_0(x)$ is not greater than $C\|x\|^{-2/(p-1)}$ for some constant $C > 0$. Also, Sato and Yanagida [16, 17] investigated, for p and V being the same as above, the existence of time-local and time-global positive solutions with a prescribed time-dependent singularity in $\mathbb{R}^n$ and some properties including a comparison principle, when u_0 behaves like the above singular steady-state. In contrast, when $V(x)$ vanishes continuously at the origin or $1 < p < n/(n-2)$, one can get a singular solution with the different behavior from the above singular steady-state. Before their works, Zhang and Zhao [18] proved the existence of a time-global positive solution, with a time-independent singularity at the origin, of the problem (1.1) in a bounded Lipschitz domain Ω which converges to a singular solution, with the behavior like the fundamental solution of Laplace's equation, of the corresponding elliptic problem at time infinity. It is noteworthy that their arguments from the point of view of potential theory enable us to treat a general potential V and a nonsmooth domain Ω. Later, Riahi [14] generalized a potential class and refined their arguments to give a simpler proof. Also, in the recent papers due to Kan and Takahashi [10, 11], the existence and the behavior of positive solutions of $u_t - \Delta u = u^p$ having a prescribed time-dependent singularity in the case $1 < p < n/(n-2)$ are studied. We refer to Karch and Zheng [12] for the Navier-Stokes system.

As far as I know, there is no result concerning the existence of time-global positive solutions with singularities on the boundary. This problem requires more delicate estimates in our analysis, because a singularity is influenced by the shape of a domain, and is more difficult than the problem of an interior singularity. The purpose of this paper is to show the existence of a positive solution $u = u(x,t)$ with a time-independent singularity at $\xi \in \partial\Omega$, the boundary of Ω, of the initial-boundary value problem for the following semilinear heat equation:

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } (\partial\Omega \setminus \{\xi\}) \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for all } x \in \Omega, \end{cases} \tag{1.2}$$

where f is a nonnegative Borel measurable function on $\Omega \times [0,\infty)$ satisfying weak conditions stated below, and the equation $u_t - \Delta u = f(x,u)$ is understood in the sense of distributions. Let $M_\Omega(\cdot,\xi)$ be the Martin (Poisson) kernel on Ω with pole at ξ and let $\mathcal{K}_\xi(\Omega)$ denote the generalized Kato class associated with $M_\Omega(\cdot,\xi)$ (see Section 2.1 and Definition 2.12 below for their definitions). We assume

(A1) f is nonnegative and Borel measurable on $\Omega \times [0,\infty)$,

(A2) there is a nonnegative Borel measurable function ψ on $\Omega \times [0,\infty)$ such that

- for each $x \in \Omega$, $\psi(x,\cdot)$ is nondecreasing on $[0,\infty)$ and $\lim_{u\to 0+} \psi(x,u) = 0$,
- $\psi(\cdot, M_\Omega(\cdot,\xi)) \in \mathcal{K}_\xi(\Omega)$,
- whenever $0 \le u_1 \le u_2$, we have

$$|f(x,u_1) - f(x,u_2)| \le \psi(x,u_2)|u_1 - u_2| \quad \text{for all } x \in \Omega.$$

Our main result is as follows.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \ge 2)$ and let $\xi \in \partial\Omega$. Assume that f satisfies* (A1) *and* (A2). *Then there exists a constant $\lambda_0 > 0$ such that the following statements hold for each $\lambda \in (0,\lambda_0]$:*

(i) *for any nonnegative continuous function* u_0 *satisfying* $u_0(x) \le \lambda M_\Omega(x,\xi)$ *for all* $x \in \Omega$*, the problem* (1.2) *has a positive continuous solution* u *satisfying*

$$u(x,t) \le 3\lambda M_\Omega(x,\xi) \quad \textit{for all } (x,t) \in \Omega \times (0,\infty)$$

and

$$\lim_{x\to\xi} \frac{u(x,t)}{M_\Omega(x,\xi)} = \lambda \quad \textit{for each } t > 0,$$

(ii) *for the solution* u *in* (i)*, there exists a positive continuous solution* u_∞ *of the elliptic problem*

$$\begin{cases} -\Delta u_\infty = f(x,u_\infty) & \textit{in } \Omega, \\ u_\infty = 0 & \textit{on } \partial\Omega \setminus \{\xi\}, \end{cases} \tag{1.3}$$

satisfying

$$\lambda M_\Omega(x,\xi) \le u_\infty(x) \le 3\lambda M_\Omega(x,\xi) \quad \textit{for all } x \in \Omega \tag{1.4}$$

and

$$\lim_{x\to\xi} \frac{u_\infty(x)}{M_\Omega(x,\xi)} = \lambda \tag{1.5}$$

such that

$$\lim_{t\to\infty} \frac{u(x,t)}{M_\Omega(x,\xi)} = \frac{u_\infty(x)}{M_\Omega(x,\xi)} \quad \textit{uniformly for } x \in \Omega. \tag{1.6}$$

Remark 1.2. Since $M_\Omega(\cdot,\xi)$ is bounded on $\Omega \setminus B(\xi,r)$, it follows from (1.6) that for each $r > 0$,

$$\lim_{t\to\infty} u(x,t) = u_\infty(x) \quad \text{uniformly for } x \in \Omega \setminus B(\xi,r).$$

Remark 1.3. When Ω is a bounded smooth domain, we have the following example of f. Let γ be a real number and let $p(x)$ and $V(x)$ be nonnegative Borel measurable functions on Ω such that

$$1 < p(x) \le \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \frac{n+1-\gamma}{n-1} \quad \text{and} \quad V(x) \le C\delta_\Omega(x)^{-\gamma}$$

for a.e. $x \in \Omega$ and some constant $C > 0$, where $\delta_\Omega(x)$ is the distance from x to $\partial\Omega$. Then $f(x,u) = V(x)u^{p(x)}$ satisfies (A1) and (A2). When Ω is a bounded Lipschitz domain, the range of $p(x)$ depends on the boundary decay rate of the Green function for the Laplacian (see Example 4.2 below).

In Section 2, we collect some notation, known results and basic lemmas. A proof of Theorem 1.1 is given in Section 3. Examples of f are presented in Section 4.

2 Preliminary materials

In what follows, we suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \ge 2)$ and we fix $\xi \in \partial\Omega$ and $x_0 \in \Omega$. The notation $\|x\|$ stands for the Euclidean norm of a point x on $\mathbb{R}^n$. The Euclidean distance from x to the boundary $\partial\Omega$ of Ω is denoted by $\delta_\Omega(x)$. By $B(x,r)$ we denote the open ball of center x and radius $r > 0$. Also, the symbol C denotes an absolute positive constant whose value may vary at each occurrence. If necessary, we use $C_1, C_2, \dots$ to specify them. Writing $C = C(a,b,\dots)$ stands for the dependences on $a, b, \dots$ of a constant C. If C depends on the diameter of Ω, the Lipschitz characters of Ω and $\delta_\Omega(x_0)$, then we write $C = C(\Omega)$ simply. Also, we use the notations $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$.

2.1 Green function and Martin kernel for the Laplacian

By G_Ω we denote the (Dirichlet) Green function on Ω for the Laplacian Δ. Let us recall two-sided global estimates for G_Ω from [6]. Since the boundary decay rate of G_Ω varies at each boundary point, we need an auxiliary set to control the boundary behavior of G_Ω. For $x, y \in \overline{\Omega}$, we define

$$\mathcal{B}_e(x,y) = \left\{ b \in \overline{\Omega} : \frac{1}{C_1}(\|b-x\| \vee \|b-y\|) \le \|x-y\| \le C_1 \delta_\Omega(b) \right\}.$$

Here the subscript "e" stands for "elliptic". We see that there is $C_1 = C(n,\Omega) \ge 1$ such that $\mathcal{B}_e(x,y)$ is nonempty for any pair $x, y \in \overline{\Omega}$ and that $\mathcal{B}_e(x,y) \subset \Omega$ except for the case $x = y \in \partial\Omega$. Let

$$g(x) = G_\Omega(x, x_0) \wedge 1.$$

Note that there exist $C = C(n,\Omega) \ge 1$ and $\beta \le 1 \le \alpha$ (both depend only on n and Ω) such that

$$\frac{1}{C}\delta_\Omega(x)^\alpha \le g(x) \le C\delta_\Omega(x)^\beta \quad \text{for all } x \in \Omega. \tag{2.1}$$

When $\partial\Omega$ is smooth, we can take $\alpha = \beta = 1$.

Lemma 2.1. *There exists $C = C(n,\Omega) \ge 1$ such that for all $x, y \in \Omega$ and $b \in \mathcal{B}_e(x,y)$,*

$$\frac{1}{C}\frac{g(x)g(y)}{g(b)^2}N(x,y) \le G_\Omega(x,y) \le C\frac{g(x)g(y)}{g(b)^2}N(x,y),$$

where

$$N(x,y) = \begin{cases} 1 + \log^+ \dfrac{\delta_\Omega(x) \wedge \delta_\Omega(y)}{\|x-y\|} & \text{if } n = 2, \\ \|x-y\|^{2-n} & \text{if } n \ge 3. \end{cases} \tag{2.2}$$

Also, the following estimates are helpful. See [6, Lemma 3.3] and [7, Lemma 2.4] for (i) and (ii). The third one follows immediately from (2.1) and the definition of $\mathcal{B}_e$.

Lemma 2.2. *The following statements hold:*

(i) *if $x, y \in \Omega$ satisfy $\|x-y\| \le k(\delta_\Omega(x) \wedge \delta_\Omega(y))$ for some $k > 0$, then there exists $C = C(k,n,\Omega) > 0$ such that $g(x) \le Cg(y)$,*

(ii) *there exists $C = C(n,\Omega) > 0$ such that $g(x) \vee g(y) \le Cg(b)$ for all $x, y \in \overline{\Omega}$ and $b \in \mathcal{B}_e(x,y)$,*

(iii) *there exists $C = C(n,\Omega) > 0$ such that $g(b) \ge C\delta_\Omega(b)^\alpha \ge C\|x-y\|^\alpha$ for all $x, y \in \overline{\Omega}$ and $b \in \mathcal{B}_e(x,y)$, where $\alpha \ge 1$ is as in* (2.1).

Next, we recall the Martin kernel on Ω with pole at $\xi \in \partial\Omega$. Owing to Hunt–Wheeden's work [9] (see also Aikawa [1]), it is known that there exists a unique positive harmonic function on Ω vanishing continuously on $\partial\Omega \setminus \{\xi\}$ and taking the value 1 at the point x_0. Such a harmonic function is called the *Martin kernel* at ξ and is denoted by $M_\Omega(\cdot,\xi)$. Actually, this can be obtained by

$$M_\Omega(x,\xi) = \lim_{y\to\xi}\frac{G_\Omega(x,y)}{G_\Omega(x_0,y)} \quad \text{for } x \in \Omega. \tag{2.3}$$

The following estimate is found in [6, Lemma 4.2].

Lemma 2.3. *There exists $C = C(n,\Omega) \ge 1$ such that for all $x \in \Omega$ and $b \in \mathcal{B}_e(x,\xi)$,*

$$\frac{1}{C}\frac{g(x)}{g(b)^2}\|x-\xi\|^{2-n} \le M_\Omega(x,\xi) \le C\frac{g(x)}{g(b)^2}\|x-\xi\|^{2-n}.$$

In particular, $g(x) \le CM_\Omega(x,\xi)$ for all $x \in \Omega$.

2.2 Green function and kernel function for the heat operator

Let Γ_Ω denote the (Dirichlet) Green function on $\Omega \times \mathbb{R}$ for the heat operator (in other words, the Dirichlet heat kernel on Ω). This is invariant under translation in time: for all $x, y \in \Omega$ and $s, t, \tau \in \mathbb{R}$,

$$\Gamma_\Omega(x, t + \tau; y, s + \tau) = \Gamma_\Omega(x, t; y, s), \tag{2.4}$$

and has the reproducing property: for all $x, y \in \Omega$ and $s < \tau < t$,

$$\Gamma_\Omega(x, t; y, s) = \int_\Omega \Gamma_\Omega(x, t; z, \tau)\Gamma_\Omega(z, \tau; y, s)\, dz. \tag{2.5}$$

Also, the following connection with the Green function G_Ω for the Laplacian is known: for all $x, y \in \Omega$,

$$G_\Omega(x, y) = \int_0^\infty \Gamma_\Omega(x, \tau; y, 0)\, d\tau. \tag{2.6}$$

Then the change of variables $\tau = t - s$, together with (2.4) and (2.6), gives

$$G_\Omega(x, y) \geq \int_0^t \Gamma_\Omega(x, \tau; y, 0)\, d\tau = \int_0^t \Gamma_\Omega(x, t; y, s)\, ds. \tag{2.7}$$

We need a sharp estimate of Γ_Ω near the boundary. The boundary behavior of Γ_Ω in large time is well known by virtue of Davies' work [4, Theorem 4.2.5]: there exist $T > 0$ and $C = C(n, \Omega) \geq 1$ such that for all $x, y \in \Omega$ and $t \geq T$,

$$\frac{1}{C} g(x)g(y)e^{-C_2 t} \leq \Gamma_\Omega(x, t; y, 0) \leq C g(x)g(y)e^{-C_2 t}, \tag{2.8}$$

where $C_2 > 0$ is the first eigenvalue of the minus Laplacian $-\Delta$. The boundary behavior in small time is more delicate. Let us recall two-sided estimates obtained recently in [8]. Let $T > 0$ be as above. Then there exists $C_3 = C(\Omega, T) > 1$ such that for any $x \in \overline{\Omega}$ and $0 < t < T$, the set

$$\mathcal{B}_p(x, t) = \left\{ b \in \Omega : \frac{1}{C_3}\|b - x\| \leq \sqrt{t} \leq C_3 \delta_\Omega(b) \right\}$$

is nonempty (see [8, Lemma 2.1]). Here the subscript "p" stands for "parabolic". For simplicity, we write

$$\gamma_C(x, t) = \frac{C}{t^{n/2}} \exp\left(-\frac{\|x\|^2}{Ct} \right).$$

Lemma 2.4. *There exists $C = C(n, \Omega, T) > 1$ such that for all $x, y \in \Omega$ and $0 \leq s < t < T$,*

$$\frac{g(x)g(y)}{g(b_x)g(b_y)} \gamma_{\frac{1}{C}}(x - y, t - s) \leq \Gamma_\Omega(x, t; y, s) \leq \frac{g(x)g(y)}{g(b_x)g(b_y)} \gamma_C(x - y, t - s),$$

whenever we choose auxiliary points from $b_x \in \mathcal{B}_p(x, t - s)$ and $b_y \in \mathcal{B}_p(y, t - s)$.

This is just for reference that the part $g(x)g(y)/g(b_x)g(b_y)$ can be estimated as follows:

$$\frac{1}{C}\left(\frac{\delta_\Omega(x)}{\sqrt{t - s}} \wedge 1 \right)^\alpha \left(\frac{\delta_\Omega(y)}{\sqrt{t - s}} \wedge 1 \right)^\alpha \leq \frac{g(x)g(y)}{g(b_x)g(b_y)} \leq C\left(\frac{\delta_\Omega(x)}{\sqrt{t - s}} \wedge 1 \right)^\beta \left(\frac{\delta_\Omega(y)}{\sqrt{t - s}} \wedge 1 \right)^\beta,$$

where α and β are the constants appearing in (2.1) (see [8, Corollary 1.2 and Section 4]). The following estimates of g will be used later (see [8, Lemmas 2.3 and 4.2]).

Lemma 2.5. *There exists $C = C(n, \Omega, T) \geq 1$ with the following properties:*

(i) $g(x) \leq Cg(b)$ *for all* $(x, t) \in \Omega \times (0, T)$ *and* $b \in \mathcal{B}_p(x, t)$,

(ii) $g(b) \geq C^{-1}\delta_\Omega(b)^\alpha \geq C^{-1}t^{\alpha/2}$ *for all* $(x, t) \in \Omega \times (0, T)$ *and* $b \in \mathcal{B}_p(x, t)$, *where* $\alpha \geq 1$ *is as in* (2.1).

Next, let us recall a kernel function with pole at the point $(\xi, 0)$ for the heat equation. Owing to Kemper's work [13], it is known that there exists a unique nonnegative solution of the heat equation in $\Omega \times \mathbb{R}$ vanishing continuously on $(\partial\Omega \times \mathbb{R}) \setminus \{(\xi, 0)\}$ and taking the value 1 at the point (x_0, T). This solution is denoted by $K_\Omega(x, t; \xi, 0)$. As shown in [8, Lemma 5.1], this can be obtained by

$$K_\Omega(x, t; \xi, 0) = \lim_{y \to \xi} \frac{\Gamma_\Omega(x, t; y, 0)}{\Gamma_\Omega(x_0, T; y, 0)} \quad \text{for } (x, t) \in \Omega \times \mathbb{R}. \tag{2.9}$$

From (2.8), we can see the large time behavior of K_Ω: there exists $C = C(n, \Omega) \geq 1$ such that

$$\frac{1}{C} g(x) e^{-C_2(t-T)} \leq K_\Omega(x, t; \xi, 0) \leq Cg(x)e^{-C_2(t-T)} \quad \text{for all } x \in \Omega \text{ and } t \geq T.$$

The following estimate of K_Ω in small time is found in [8, Theorem 5.2].

Lemma 2.6. *There exists $C = C(n, \Omega, T) \geq 1$ such that for all $(x, t) \in \Omega \times (0, T)$,*

$$\frac{g(x)}{g(b_x)g(b_\xi)} \gamma_{\frac{1}{C}}(x - \xi, t) \leq K_\Omega(x, t; \xi, 0) \leq \frac{g(x)}{g(b_x)g(b_\xi)} \gamma_C(x - \xi, t),$$

whenever we choose auxiliary points from $b_x \in \mathcal{B}_p(x, t)$ *and* $b_\xi \in \mathcal{B}_p(\xi, t)$.

Also, K_Ω has the following connection with the Martin kernel.

Lemma 2.7. *For all $x \in \Omega$, we have*

$$M_\Omega(x, \xi) = C_4 \int_0^\infty K_\Omega(x, \tau; \xi, 0)\, d\tau,$$

where $C_4{}^{-1} = \int_0^\infty K_\Omega(x_0, \tau; \xi, 0)\, d\tau$.

Proof. Let $x, y \in \Omega$. By (2.6), we have

$$\frac{G_\Omega(x, y)}{G_\Omega(x_0, y)} = \int_0^\infty \frac{\Gamma_\Omega(x, \tau; y, 0)}{\Gamma_\Omega(x_0, T; y, 0)}\, d\tau \Bigg/ \int_0^\infty \frac{\Gamma_\Omega(x_0, \tau; y, 0)}{\Gamma_\Omega(x_0, T; y, 0)}\, d\tau.$$

As $y \to \xi$, we get the required equality from (2.3), (2.9) and Lebesgue's dominated convergence theorem. This is possible by virtue of (2.8) and Lemma 2.4. In fact, if $x \in \Omega$ is fixed, then using Lemma 2.5 we see that for all $y \in \Omega \setminus B(x, \delta_\Omega(x)/2)$,

$$\frac{\Gamma_\Omega(x, \tau; y, 0)}{\Gamma_\Omega(x_0, T; y, 0)} \leq \begin{cases} \dfrac{C}{\tau^{(n+\alpha)/2}} \exp\left(-\dfrac{\delta_\Omega(x)^2}{C\tau}\right) & \text{if } 0 < \tau < T, \\ Ce^{-C_2\tau} & \text{if } \tau \geq T. \end{cases}$$

The right hand side is integrable for $\tau \in (0, \infty)$. Thus Lebesgue's dominated convergence theorem is applicable. ∎

Lemma 2.8. *For all $x \in \Omega$ and $t > 0$, we have*

$$\int_\Omega \Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)\, dy \leq M_\Omega(x, \xi).$$

Proof. By Lemma 2.7, Fubini–Tonelli's theorem and (2.4), we have

$$\int_\Omega \Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)\, dy = C_4 \int_0^\infty \int_\Omega \Gamma_\Omega(x, t + s; y, s) K_\Omega(y, s; \xi, 0)\, dyds.$$

Also, Fatou's lemma, together with (2.5) and (2.9), yields

$$\int_\Omega \Gamma_\Omega(x, t + s; y, s) K_\Omega(y, s; \xi, 0)\, dy \leq K_\Omega(x, t + s; \xi, 0).$$

Since $C_4 \int_0^\infty K_\Omega(x, t + s; \xi, 0)\, ds \leq C_4 \int_0^\infty K_\Omega(x, s; \xi, 0)\, ds = M_\Omega(x, \xi)$, we obtain the required inequality. □

Lemma 2.9. *Let $t > 0$. Then the following statements hold:*

(i) $\displaystyle\lim_{x \to \eta} \int_\Omega \Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)\, dy = 0$ *for each* $\eta \in \partial\Omega$,

(ii) $\displaystyle\lim_{x \to \xi} \frac{1}{M_\Omega(x, \xi)} \int_\Omega \Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)\, dy = 0.$

Proof. Let $\eta \in \partial\Omega$ and let $t > 0$ be fixed. By Lemmas 2.2–2.5 and $\gamma_C(x - y, t) \leq C t^{-n/2}$, we have

$$\begin{aligned}
\Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi) &\leq C \frac{g(x)}{g(b_{(x,t)}) g(b_{(y,t)})} \left(\frac{g(y)}{g(b_{y\xi})} \right)^2 \gamma_C(x - y, t) \|y - \xi\|^{2-n} \\
&\leq C(t) g(x) \|y - \xi\|^{2-n}
\end{aligned} \tag{2.10}$$

for all $x, y \in \Omega$, where $b_{(x,t)} \in \mathcal{B}_p(x, t)$, $b_{(y,t)} \in \mathcal{B}_p(y, t)$ and $b_{y\xi} \in \mathcal{B}_e(y, \xi)$. Therefore

$$\int_\Omega \Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)\, dy \leq C(t) g(x) \to 0 \quad (x \to \eta),$$

and thus (i) holds.

Next, let $x \in \Omega$, let $t > 0$ and let $b_{x\xi} \in \mathcal{B}_e(x, \xi)$. Then, by Lemma 2.3 and (2.10),

$$\frac{\Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)}{M_\Omega(x, \xi)} \leq C(t) g(b_{x\xi})^2 \|x - \xi\|^{n-2} \|y - \xi\|^{2-n} \quad \text{for all } y \in \Omega,$$

and so

$$\frac{1}{M_\Omega(x, \xi)} \int_\Omega \Gamma_\Omega(x, t; y, 0) M_\Omega(y, \xi)\, dy \leq C(t) g(b_{x\xi})^2 \|x - \xi\|^{n-2}.$$

Since $g(b_{x\xi}) \leq C \delta_\Omega(b_{x\xi})^\beta \leq C \|b_{x\xi} - \xi\|^\beta \leq C \|x - \xi\|^\beta$ by (2.1), we obtain (ii) (even when $n = 2$). □

Lemma 2.10. *For each $t > 0$, the function $x \mapsto \int_0^t K_\Omega(x, \tau; \xi, 0)\, d\tau / M_\Omega(x, \xi)$ is continuous on Ω and has a continuous extension to $\overline{\Omega}$. Moreover, for each $t_0 > 0$,*

$$\lim_{x \to \xi} \frac{\int_0^t K_\Omega(x, \tau; \xi, 0)\, d\tau}{M_\Omega(x, \xi)} = \frac{1}{C_4} \quad \textit{uniformly for } t \geq t_0, \tag{2.11}$$

where C_4 is the constant in Lemma 2.7, and the family of such extended functions with parameter $t \geq t_0$ is equicontinuous on $\overline{\Omega}$.

Proof. Let $t > 0$ be fixed. It is easy to see, using Lebesgue's dominated convergence theorem, that the function $x \mapsto \int_0^t K_\Omega(x,\tau;\xi,0)\,d\tau / M_\Omega(x,\xi)$ is continuous at each point in Ω. We have to prove the continuity at a boundary point. Since

$$\frac{\int_0^t K_\Omega(x,\tau;\xi,0)\,d\tau}{M_\Omega(x,\xi)} = \frac{1}{C_4} - \frac{\int_t^\infty K_\Omega(x,\tau;\xi,0)\,d\tau}{M_\Omega(x,\xi)},$$

it suffices to show the continuity of the right hand side. Let $\eta \in \partial\Omega \setminus \{\xi\}$ and let $x_1, x_2 \in \Omega \cap B(\eta, \|\eta - \xi\|/2)$. Then

$$\left| \frac{\int_t^\infty K_\Omega(x_1,\tau;\xi,0)\,d\tau}{M_\Omega(x_1,\xi)} - \frac{\int_t^\infty K_\Omega(x_2,\tau;\xi,0)\,d\tau}{M_\Omega(x_2,\xi)} \right| \le \int_0^\infty \left| \frac{K_\Omega(x_1,\tau;\xi,0)}{M_\Omega(x_1,\xi)} - \frac{K_\Omega(x_2,\tau;\xi,0)}{M_\Omega(x_2,\xi)} \right| d\tau. \tag{2.12}$$

Note from Athanasopoulos–Caffarelli–Salsa's result [2, Corollary 1] that $K_\Omega(x,\tau;\xi,0)/M_\Omega(x,\xi)$ has a finite limit as $x \to \eta$. Since we have by Lemmas 2.3, 2.5 and 2.6 that for $x \in \Omega$ and $b_{x\xi} \in \mathcal{B}_e(x,\xi)$,

$$\frac{K_\Omega(x,\tau;\xi,0)}{M_\Omega(x,\xi)} \le \begin{cases} Cg(b_{x\xi})^2 \|x-\xi\|^{n-2} \dfrac{1}{\tau^{(n/2)+\alpha}} \exp\left(-\dfrac{\|x-\xi\|^2}{C\tau}\right) & \text{if } 0 < \tau < T, \\ Cg(b_{x\xi})^2 \|x-\xi\|^{n-2} e^{-C_2(\tau - T)} & \text{if } \tau \ge T, \end{cases} \tag{2.13}$$

$$\le \begin{cases} \dfrac{C}{\tau^{(n/2)+\alpha}} \exp\left(-\dfrac{\|x-\xi\|^2}{C\tau}\right) & \text{if } 0 < \tau < T, \\ Ce^{-C_2(\tau-T)} & \text{if } \tau \ge T, \end{cases}$$

it follows from Lebesgue's dominated convergence theorem that the right hand side of (2.12) tends to zero as $\|x_1 - x_2\| \to 0$. Hence the function $x \mapsto \int_t^\infty K_\Omega(x,\tau;\xi,0)\,d\tau/M_\Omega(x,\xi)$ has a continuous extension to $\overline{\Omega} \setminus \{\xi\}$ and the continuity is uniform for $t > 0$.

Next, we show the continuity at ξ. Let $t \ge t_0 > 0$. Write $\rho = \|x - \xi\|$. Then, by (2.13) and $(n/2) + \alpha - 2 \ge 0$,

$$\begin{aligned} \frac{\int_t^\infty K_\Omega(x,\tau;\xi,0)\,d\tau}{M_\Omega(x,\xi)} &\le Cg(b_{x\xi})^2 \rho^{n-2} \left\{ \int_{t_0}^T \frac{1}{\tau^{(n/2)+\alpha}} \exp\left(-\frac{\rho^2}{C\tau}\right) d\tau + \int_T^\infty e^{-C_2(\tau-T)}\,d\tau \right\} \\ &\le Cg(b_{x\xi})^2 \rho^{n-2} \left[\frac{1}{t_0^{(n/2)+\alpha-2}} \frac{C}{\rho^2} \left\{ \exp\left(-\frac{\rho^2}{CT}\right) - \exp\left(-\frac{\rho^2}{Ct_0}\right) \right\} + \frac{1}{C_2} \right]. \end{aligned}$$

The right hand side goes to zero as $\rho = \|x - \xi\| \to 0$. Hence we obtain (2.11). Thus the lemma is proved. □

Remark 2.11. Since $M_\Omega(\cdot,\xi)$ vanishes continuously on $\partial\Omega \setminus \{\xi\}$, we see from Lemma 2.10 that for each $t > 0$, the function $x \mapsto \int_0^t K_\Omega(x,\tau,\xi,0)\,d\tau$ vanishes continuously on $\partial\Omega \setminus \{\xi\}$.

2.3 Generalized Kato class associated with the Martin kernel at ξ

In this subsection, we recall the definition of an admissible function class for the elliptic problem (1.3) introduced in [5] and give some elementary properties used later.

Definition 2.12. A Borel measurable function φ on Ω is said to belong to the *generalized Kato class* $\mathcal{K}_\xi(\Omega)$ associated with $M_\Omega(\cdot,\xi)$ if the following two conditions are satisfied:

$$\lim_{r\to 0} \left(\sup_{x\in\Omega} \int_{\Omega\cap B(x,r)} \frac{M_\Omega(y,\xi) G_\Omega(x,y)}{M_\Omega(x,\xi)} |\varphi(y)|\,dy \right) = 0, \tag{2.14}$$

$$\lim_{r\to 0} \left(\sup_{x\in\Omega} \int_{\Omega\cap B(\xi,r)} \frac{M_\Omega(y,\xi) G_\Omega(x,y)}{M_\Omega(x,\xi)} |\varphi(y)|\,dy \right) = 0. \tag{2.15}$$

For $\varphi \in \mathcal{K}_\xi(\Omega)$, we write

$$\|\varphi\|_{\mathcal{K}_\xi(\Omega)} = \sup_{x\in\Omega} \int_\Omega \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)} |\varphi(y)|\, dy.$$

Here we append a little explanation on $\mathcal{K}_\xi(\Omega)$. The classical Kato class $\mathcal{K}(\Omega)$ in higher dimensions, often adopted in the study of potential theory for the stationary Schrödinger operator $\Delta + \varphi$, is the set of all Borel measurable functions φ on Ω satisfying

$$\lim_{r\to 0}\left(\sup_{x\in\Omega} \int_{\Omega\cap B(x,r)} \frac{|\varphi(y)|}{\|x-y\|^{n-2}}\, dy\right) = 0.$$

The Newton kernel $\|x-y\|^{2-n}$ is independent of the shape of a domain Ω and this fact restricts the growth order of functions in $\mathcal{K}(\Omega)$ near $\partial\Omega$. By Cranston–Fabes–Zhao [3, Theorem 3.1], we know that there is $C = C(n,\Omega) > 0$ such that

$$\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)} \le C\{\|x-y\|^{2-n} + \|y-\xi\|^{2-n}\} \quad \text{for all } x,y\in\Omega.$$

Moreover, if $x \in \Omega$ is fixed, then $M_\Omega(y,\xi)G_\Omega(x,y)/M_\Omega(x,\xi)$ vanishes continuously as $y \to \partial\Omega \setminus \{\xi\}$. Thus we find the inclusion relationship: $L^q_{\mathrm{loc}}(\mathbb{R}^n) \subsetneq \mathcal{K}(\Omega) \subsetneq \mathcal{K}_\xi(\Omega)$ for any $q > n/2$.

We need simple estimates for $M_\Omega(y,\xi)G_\Omega(x,y)/M_\Omega(x,\xi)$ to obtain properties on $\mathcal{K}_\xi(\Omega)$.

Lemma 2.13. *Let $C_5 > 0$. Then there exists $C = C(C_5, n, \Omega) > 0$ such that*

$$g(b_{xy}) \le Cg(b_{x\xi}) \quad \textit{for } b_{xy} \in \mathcal{B}_e(x,y) \textit{ and } b_{x\xi} \in \mathcal{B}_e(x,\xi),$$

whenever $x, y \in \Omega$ satisfy $\|x-y\| \le C_5\|x-\xi\|$.

Proof. Let C_1 be the constant in the definition of $\mathcal{B}_e$. If $\|x-\xi\|/2C_1 \le \|x-y\| \le C_5\|x-\xi\|$, then

$$\begin{aligned}\|b_{xy} - b_{x\xi}\| &\le \|b_{xy} - x\| + \|x - b_{x\xi}\| \le C_1(\|x-y\| + \|x-\xi\|)\\ &\le C(\|x-y\| \wedge \|x-\xi\|) \le C(\delta_\Omega(b_{xy}) \wedge \delta_\Omega(b_{x\xi})).\end{aligned}$$

Therefore, by Lemma 2.2, $g(b_{xy})$ and $g(b_{x\xi})$ are comparable in this case. Consider the case $\|x-y\| \le \|x-\xi\|/2C_1$. Then

$$\|x-\xi\| \le \|x-b_{xy}\| + \|b_{xy}-\xi\| \le C_1\|x-y\| + \|b_{xy}-\xi\| \le \frac{1}{2}\|x-\xi\| + \|b_{xy}-\xi\|,$$

and so $\|x-\xi\| \le 2\|b_{xy}-\xi\|$. Moreover,

$$\|b_{xy}-\xi\| \le \|b_{xy}-x\| + \|x-\xi\| \le C_1\|x-y\| + \|x-\xi\| \le 2\|x-\xi\|.$$

We take $b \in \mathcal{B}_e(b_{xy},\xi)$. Then

$$\begin{aligned}\|b - b_{x\xi}\| &\le \|b-\xi\| + \|\xi - b_{x\xi}\| \le C_1(\|b_{xy}-\xi\| + \|x-\xi\|)\\ &\le C(\|b_{xy}-\xi\| \wedge \|x-\xi\|) \le C(\delta_\Omega(b) \wedge \delta_\Omega(b_{x\xi})).\end{aligned}$$

It then follows from Lemma 2.2 that $g(b_{xy}) \le Cg(b) \le Cg(b_{x\xi})$. Thus the lemma is proved. $\square$

Lemma 2.14. *Let $\alpha \ge 1$ be as in* (2.1). *Then there exists $C = C(n,\Omega) > 0$ such that*

$$M_\Omega(y,\xi)^2 \le \frac{C}{\|x-\xi\|^{n-2+2\alpha}} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)},$$

whenever $x, y \in \Omega$ satisfy $\|x-y\| \le \|x-\xi\|/2$.

Proof. We give a proof in the case $n \geq 3$, because the case $n = 2$ is similar. It suffices to show that if $\|x - y\| \leq \|x - \xi\|/2$, then

$$\frac{M_\Omega(x,\xi)M_\Omega(y,\xi)}{G_\Omega(x,y)} \leq \frac{C}{\|x-\xi\|^{n-2+2\alpha}}. \tag{2.16}$$

By Lemmas 2.1, 2.3 and 2.13, we get

$$\begin{aligned}\frac{M_\Omega(x,\xi)M_\Omega(y,\xi)}{G_\Omega(x,y)} &\leq C\left(\frac{g(b_{xy})}{g(b_{x\xi})g(b_{y\xi})}\right)^2\left(\frac{\|x-y\|}{\|x-\xi\|\|y-\xi\|}\right)^{n-2} \\ &\leq \frac{C}{g(b_{y\xi})^2}\left(\frac{\|x-y\|}{\|x-\xi\|\|y-\xi\|}\right)^{n-2},\end{aligned}$$

where $b_{xy} \in \mathcal{B}_e(x,y)$, $b_{x\xi} \in \mathcal{B}_e(x,\xi)$ and $b_{y\xi} \in \mathcal{B}_e(y,\xi)$. Since $\|x-\xi\|/2 \leq \|y-\xi\| \leq 2\|x-\xi\|$, we have

$$\left(\frac{\|x-y\|}{\|x-\xi\|\|y-\xi\|}\right)^{n-2} \leq \frac{C}{\|x-\xi\|^{n-2}}$$

and $g(b_{y\xi}) \geq C\|y-\xi\|^\alpha \geq C\|x-\xi\|^\alpha$ by Lemma 2.2. Thus (2.16) follows. □

Lemma 2.15. *Let $\alpha \geq 1$ be as in* (2.1)*. Then there exists $C = C(n,\Omega) > 0$ such that*

$$\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)} \leq \frac{CN(x,y)}{\|x-y\|^{2\alpha}}M_\Omega(y,\xi)^2 \quad \textit{for all } x,y \in \Omega,$$

where the function $N(x,y)$ is given by (2.2)*.*

Proof. By Lemmas 2.1 and 2.3, we get

$$\frac{G_\Omega(x,y)}{M_\Omega(x,\xi)M_\Omega(y,\xi)} \leq C\left(\frac{g(b_{x\xi})g(b_{y\xi})}{g(b_{xy})}\right)^2\left(\|x-\xi\|\|y-\xi\|\right)^{n-2}N(x,y) \leq \frac{CN(x,y)}{\|x-y\|^{2\alpha}},$$

since Ω is bounded, $g \leq 1$ and $g(b_{xy}) \geq C\|x-y\|^\alpha$ by Lemma 2.2. Thus the lemma follows. □

Using Lemmas 2.14 and 2.15, we can obtain the following three lemmas by the same way as in the case where Ω is smooth (see [5, Lemmas 5.6, 5.7 and 5.8]). For reader's convenience, we give proofs.

Lemma 2.16. *If $\varphi \in \mathcal{K}_\xi(\Omega)$, then*

$$\int_{\Omega\setminus B(\xi,r)} M_\Omega(x,\xi)^2|\varphi(x)|\,dx < \infty \quad \textit{for each } r > 0. \tag{2.17}$$

Moreover, $\|\varphi\|_{\mathcal{K}_\xi(\Omega)} < \infty$.

Proof. Let $0 < \delta < r/2$ be small enough. Covering $\Omega \setminus B(\xi,r)$ by finitely many balls $B(x_j,\delta)$ with $x_j \in \Omega \setminus B(\xi,r)$, we can obtain from Lemma 2.14 and (2.14) that

$$\int_{\Omega\setminus B(\xi,r)} M_\Omega(y,\xi)^2|\varphi(y)|\,dy \leq \frac{C}{\delta^{n-2+2\alpha}}\sum_j\int_{\Omega\cap B(x_j,\delta)}\frac{M_\Omega(y,\xi)G_\Omega(x_j,y)}{M_\Omega(x_j,\xi)}|\varphi(y)|\,dy < \infty.$$

Thus (2.17) follows. Also, this and Lemma 2.15 give

$$\sup_{x\in\Omega}\int_{\Omega\setminus(B(x,\delta)\cup B(\xi,\delta))}\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}|\varphi(y)|\,dy \leq C(\delta,n,\Omega)\int_{\Omega\setminus B(\xi,\delta)} M_\Omega(y,\xi)^2|\varphi(y)|\,dy < \infty.$$

This, together with (2.14) and (2.15), yields $\|\varphi\|_{\mathcal{K}_\xi(\Omega)} < \infty$. □

Lemma 2.17. *If $\varphi \in \mathcal{K}_\xi(\Omega)$, then*

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(z,r)}\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}|\varphi(y)|\,dy\right)=0 \quad \textit{for each } z\in\overline{\Omega}.$$

Proof. Let $z \in \overline{\Omega}$. From (2.14), (2.15) and Lemma 2.15, we see that for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in \Omega$ and $r > 0$,

$$\begin{aligned}\int_{\Omega\cap B(z,r)}\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}|\varphi(y)|\,dy &\le \varepsilon+\int_{\Omega\cap B(z,r)\setminus(B(x,\delta)\cup B(\xi,\delta))}\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}|\varphi(y)|\,dy\\ &\le \varepsilon + C(\delta,\alpha,n)\int_{\Omega\cap B(z,r)\setminus B(\xi,\delta)}M_\Omega(y,\xi)^2|\varphi(y)|\,dy.\end{aligned}$$

The right hand side is independent of x. In view of Lemma 2.16, we obtain the required property. $\square$

Lemma 2.18. *Let $\varphi \in \mathcal{K}_\xi(\Omega)$. Then the function*

$$\Phi(x)=\int_\Omega\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y)\,dy \quad \textit{for } x\in\Omega$$

is continuous on Ω and has a continuous extension to $\overline{\Omega}$. Moreover, $\lim_{x\to\xi}\Phi(x)=0$.

Proof. Let $\varepsilon > 0$, let $z \in \overline{\Omega}\setminus\{\xi\}$ and let $x_1, x_2 \in \Omega\cap B(z,\delta/2)$. If $\delta > 0$ is small enough, then we have by Lemma 2.17

$$|\Phi(x_1)-\Phi(x_2)|\le\varepsilon+\int_{\Omega\setminus(B(z,\delta)\cup B(\xi,\delta))}\left|\frac{G_\Omega(x_1,y)}{M_\Omega(x_1,\xi)}-\frac{G_\Omega(x_2,y)}{M_\Omega(x_2,\xi)}\right|M_\Omega(y,\xi)|\varphi(y)|\,dy.$$

Note that $G_\Omega(x,y)/M_\Omega(x,\xi)$ has a finite limit as $x \to z$ (see Aikawa [1]). Since the integrand is bounded by $C(\delta,\alpha,n)M_\Omega(y,\xi)^2|\varphi(y)|$ by virtue of Lemma 2.15, it follows from Lemma 2.16 and Lebesgue's dominated convergence theorem that the above integral tends to zero as $\|x_1-x_2\|\to 0$. Thus Φ is continuous at $z \in \overline{\Omega}\setminus\{\xi\}$.

Also, we have

$$|\Phi(x)|\le\varepsilon+\int_{\Omega\setminus B(\xi,\delta)}\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}|\varphi(y)|\,dy,$$

and, using Lemmas 2.1 and 2.3, we observe that $G_\Omega(x,y)/M_\Omega(x,\xi)\to 0$ as $x\to\xi$. By the same reasoning as above, the integral tends to zero as $x \to \xi$. Hence Φ is continuous on $\overline{\Omega}$ and $\lim_{x\to\xi}\Phi(x) = 0$. $\square$

3 Proof of Theorem 1.1

Let $0 < \lambda \le \lambda_0$, where $\lambda_0 > 0$ is chosen later. Assume that $0 \le u_0(x) \le \lambda M_\Omega(x,\xi)$ for all $x \in \Omega$. Let $v(x,t) = \int_\Omega \Gamma_\Omega(x,t;y,0)u_0(y)\,dy$ be a solution of the heat equation with the initial value u_0:

$$\begin{cases} v_t = \Delta v & \text{in } \Omega\times(0,\infty),\\ v = 0 & \text{on } \partial\Omega\times(0,\infty),\\ v(x,0)=u_0(x) & \text{for all } x\in\Omega.\end{cases}$$

Note from Lemmas 2.8 and 2.9 that

$$0\le v(x,t)\le\lambda M_\Omega(x,\xi) \quad \text{for all } (x,t)\in\Omega\times(0,\infty), \tag{3.1}$$

and

$$\lim_{x\to\xi}\frac{v(x,t)}{M_\Omega(x,\xi)}=0 \quad \text{for each } t>0. \tag{3.2}$$

Also, for simplicity, we write

$$h(x,t)=C_4\int_0^t K_\Omega(x,\tau;\xi,0)\,d\tau \quad \text{for } (x,t)\in\Omega\times(0,\infty),$$

where C_4 is the constant in Lemma 2.7. It is easy to see from the definition of K_Ω and Lemma 2.7 that h is a positive solution of the heat equation

$$\begin{cases} h_t=\Delta h & \text{in } \Omega\times(0,\infty),\\ h=0 & \text{on } (\partial\Omega\setminus\{\xi\})\times(0,\infty),\\ h(x,0)=0 & \text{for all } x\in\Omega, \end{cases}$$

satisfying

$$0\le h(x,t)\le M_\Omega(x,\xi) \quad \text{for all } (x,t)\in\Omega\times(0,\infty). \tag{3.3}$$

Therefore, in order to prove the existence of a positive solution of (1.2), it is enough to show that there is a positive continuous solution u in $\Omega\times(0,\infty)$ of the integral equation

$$u(x,t)=\lambda h(x,t)+v(x,t)+\int_0^t\int_\Omega \Gamma_\Omega(x,t;y,s)f(y,u(y,s))\,dyds. \tag{3.4}$$

To this end, for $\lambda\in(0,1/3]$, we let

$$W_\lambda=\left\{w\in C_b(\Omega\times(0,\infty))\ :\ \begin{array}{l} 0\le w(x,t)\le 3\lambda \text{ for all } (x,t)\in\Omega\times(0,\infty) \text{ and}\\ \lim\limits_{t\to\infty} w(x,t) \text{ exists for each } x\in\Omega \end{array}\right\},$$

where $C_b(\Omega\times(0,\infty))$ denotes the space of all bounded continuous functions on $\Omega\times(0,\infty)$ endowed with the uniform norm $\|\cdot\|_\infty$, and consider the operator T_λ on W_λ defined by

$$T_\lambda[w](x,t)=\frac{\lambda h(x,t)+v(x,t)}{M_\Omega(x,\xi)}+\Gamma[w](x,t) \quad \text{for } (x,t)\in\Omega\times(0,\infty),$$

where

$$\Gamma[w](x,t)=\frac{1}{M_\Omega(x,\xi)}\int_0^t\int_\Omega \Gamma_\Omega(x,t;y,s)f(y,w(y,s)M_\Omega(y,\xi))\,dyds.$$

It is easy to see that W_λ is closed in the Banach space $C_b(\Omega\times(0,\infty))$. We will show that T_λ has a fixed point in W_λ by using Banach's fixed point theorem. In the arguments below, we note from (A2) with $u_1=0$ and $u_2=w(y,s)M_\Omega(y,\xi)$ that if $w\in W_\lambda$, then

$$\begin{aligned} f(y,w(y,s)M_\Omega(y,\xi)) &\le 3\lambda M_\Omega(y,\xi)\psi(y,3\lambda M_\Omega(y,\xi)) && (3.5)\\ &\le M_\Omega(y,\xi)\psi(y,M_\Omega(y,\xi)) && (3.6) \end{aligned}$$

for all $(y,s)\in\Omega\times(0,\infty)$.

Claim 1. *There exists $\lambda_1\in(0,1/3]$ such that whenever $\lambda\in(0,\lambda_1]$ and $w\in W_\lambda$, we have*

$$0\le T_\lambda[w](x,t)\le 3\lambda \quad \textit{for all } (x,t)\in\Omega\times(0,\infty).$$

Proof. Consider the family $\{\Psi_\lambda : 0 < \lambda < 1\}$ of functions defined by

$$\Psi_\lambda(x) = \int_\Omega \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\psi(y, \lambda M_\Omega(y,\xi))\,dy \quad \text{for } x \in \Omega.$$

Since $\psi(\cdot, \lambda M_\Omega(\cdot,\xi)) \in \mathcal{K}_\xi(\Omega)$, it follows from Lemma 2.18 that Ψ_λ has a continuous extension, say $\overline{\Psi}_\lambda$, to $\overline{\Omega}$. Also, the function $\lambda \mapsto \overline{\Psi}_\lambda(x)$ is nondecreasing and tends to zero as $\lambda \to 0+$ by (A2). We then see from Dini's theorem that $\overline{\Psi}_\lambda(x) \to 0$ uniformly for $x \in \overline{\Omega}$ as $\lambda \to 0+$; namely,

$$\lim_{\lambda \to 0+} \|\psi(\cdot, \lambda M_\Omega(\cdot,\xi))\|_{\mathcal{K}_\xi(\Omega)} = 0. \tag{3.7}$$

Therefore it follows from (2.7), (3.5) and (3.7) that

$$0 \le \Gamma[w](x,t) \le 3\lambda \|\psi(\cdot, 3\lambda M_\Omega(\cdot,\xi))\|_{\mathcal{K}_\xi(\Omega)} \le \lambda \quad \text{for all } (x,t) \in \Omega \times (0,\infty),$$

whenever $\lambda > 0$ is sufficiently small. This, together with (3.1) and (3.3), concludes the claim. $\square$

Claim 2. *For $w \in W_\lambda$ with $\lambda \in (0, 1/3]$, we have*

(i) $\|\Gamma[w]\|_\infty \le \|\psi(\cdot, M_\Omega(\cdot,\xi))\|_{\mathcal{K}_\xi(\Omega)} < \infty$,

(ii) $\Gamma[w]$ *is continuous on* $\Omega \times (0,\infty)$,

(iii) *for each* $t > 0$, $\Gamma[w](\cdot,t)$ *has a continuous extension, say* $\overline{\Gamma[w]}(\cdot,t)$, *to* $\overline{\Omega}$, *and*

$$\lim_{x \to \xi} \Gamma[w](x,t) = 0, \tag{3.8}$$

(iv) $\{\overline{\Gamma[w]}(\cdot,t) : t > 0\}$ *is equicontinuous on* $\overline{\Omega}$,

(v) $\lim_{t\to\infty} \Gamma[w](x,t)$ *exists for each* $x \in \Omega$.

Proof. For simplicity, we write $\varphi(y) = \psi(y, M_\Omega(y,\xi)) \in \mathcal{K}_\xi(\Omega)$. By (2.7) and (3.6), we have for all $(x,t) \in \Omega \times (0,\infty)$,

$$\Gamma[w](x,t) \le \int_\Omega \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y)\,dy \le \|\varphi\|_{\mathcal{K}_\xi(\Omega)} < \infty.$$

Thus (i) holds.

Next, we show that for each $t > 0$, $\Gamma[w](\cdot,t)$ has a continuous extension to $\overline{\Omega}$ and the continuity is uniform for $t > 0$. Let $z \in \overline{\Omega} \setminus \{\xi\}$ and let $\varepsilon > 0$. By Lemma 2.17, there exists $r > 0$ such that

$$\sup_{x\in\Omega} \int_{\Omega\cap B(z,r)} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y)\,dy < \varepsilon, \tag{3.9}$$

$$\sup_{x\in\Omega} \int_{\Omega\cap B(\xi,r)} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y)\,dy < \varepsilon. \tag{3.10}$$

Let $x_1, x_2 \in \Omega \cap B(z, r/2)$ and $t > 0$. By (3.6), (2.4) and the change of variables $\tau = t - s$, we have

$$\begin{aligned}
|\Gamma[w](x_1,t) - \Gamma[w](x_2,t)| &\le \int_\Omega \left(\int_0^t \left| \frac{\Gamma_\Omega(x_1,t;y,s)}{M_\Omega(x_1,\xi)} - \frac{\Gamma_\Omega(x_2,t;y,s)}{M_\Omega(x_2,\xi)} \right| ds \right) M_\Omega(y,\xi)\varphi(y)\,dy \\
&\le \int_\Omega \left(\int_0^\infty \left| \frac{\Gamma_\Omega(x_1,\tau;y,0)}{M_\Omega(x_1,\xi)} - \frac{\Gamma_\Omega(x_2,\tau;y,0)}{M_\Omega(x_2,\xi)} \right| d\tau \right) M_\Omega(y,\xi)\varphi(y)\,dy \\
&=: I_1 + I_2 + I_3,
\end{aligned}$$

where $I_1 = \int_{\Omega \cap B(z,r)} \dots \, dy$, $I_2 = \int_{\Omega \cap B(\xi,r)} \dots \, dy$ and $I_3 = \int_{\Omega \setminus (B(z,r) \cup B(\xi,r))} \dots \, dy$. If one proves that these integrals are bounded by $C\varepsilon$, then we can conclude that $\Gamma[w]$ is continuous at z uniformly for $t > 0$. From (2.6), (3.9) and (3.10), we get $I_1 \le 2\varepsilon$ and $I_2 \le 2\varepsilon$. Consider I_3. Since we know from [2, Corollary 1] that $\Gamma_\Omega(\cdot, \tau; y, 0)/M_\Omega(\cdot, \xi)$ has a continuous extension to $\overline{\Omega} \setminus \{\xi\}$, we have only to estimate $\Gamma_\Omega(x, \tau; y, 0)/M_\Omega(x, \xi)$ by a function independent of x. Let $x \in \Omega \cap B(z, r/2)$ and $y \in \Omega \setminus (B(z,r) \cup B(\xi, r))$. If $0 < \tau < T$, then by Lemmas 2.3–2.5 we have for $b_{(x,\tau)} \in \mathcal{B}_p(x,\tau)$ and $b_{(y,\tau)} \in \mathcal{B}_p(y,\tau)$,

$$\begin{aligned}\frac{\Gamma_\Omega(x,\tau;y,0)}{M_\Omega(x,\xi)} &\le \frac{g(x)g(y)}{g(b_{(x,\tau)})g(b_{(y,\tau)})} \frac{C}{\tau^{n/2}} \exp\left(-\frac{\|x-y\|^2}{C\tau}\right) \Big/ M_\Omega(x,\xi) \\ &\le \frac{C}{\tau^{\alpha+(n/2)}} \exp\left(-\frac{r^2}{C\tau}\right) M_\Omega(y,\xi). \end{aligned} \tag{3.11}$$

If $\tau \ge T$, then Lemma 2.3 and (2.8) give

$$\frac{\Gamma_\Omega(x,\tau;y,0)}{M_\Omega(x,\xi)} \le C \frac{g(x)g(y)e^{-C_2\tau}}{M_\Omega(x,\xi)} \le C M_\Omega(y,\xi) e^{-C_2\tau}, \tag{3.12}$$

where $C_2 > 0$. Now, let $\rho(y,\tau)$ be defined by the right hand sides of (3.11) and (3.12). Since

$$\int_{\Omega \setminus (B(z,r) \cup B(\xi,r))} \int_0^\infty \rho(y,\tau) M_\Omega(y,\xi) \varphi(y) \, d\tau dy \le C \int_{\Omega \setminus B(\xi,r)} M_\Omega(y,\xi)^2 \varphi(y) \, dy$$

and the right hand side is finite by Lemma 2.16, it follows from Lebesgue's dominated convergence theorem that $I_3 < \varepsilon$ whenever $\|x_1 - x_2\|$ is sufficiently small. Therefore $\Gamma[w](\cdot, t)$ is continuous at $z \in \overline{\Omega} \setminus \{\xi\}$. Moreover, we have by Lemma 2.18

$$\Gamma[w](x,t) \le \int_\Omega \frac{M_\Omega(y,\xi) G_\Omega(x,y)}{M_\Omega(x,\xi)} \varphi(y) \, dy \to 0 \quad (x \to \xi).$$

Hence $\Gamma[w](\cdot, t)$ has a continuous extension to $\overline{\Omega}$ and the continuity is uniform for $t > 0$. Thus (iii) and (iv) hold.

Next, we show that $\Gamma[w]$ is continuous at $(x_1, t_1) \in \Omega \times (0, \infty)$. If $\|x - x_1\|$ is small enough, then by the above we have for all $t > 0$,

$$\begin{aligned}|\Gamma[w](x,t) - \Gamma[w](x_1,t_1)| &\le |\Gamma[w](x,t) - \Gamma[w](x_1,t)| + |\Gamma[w](x_1,t) - \Gamma[w](x_1,t_1)| \\ &\le \varepsilon + |\Gamma[w](x_1,t) - \Gamma[w](x_1,t_1)|.\end{aligned}$$

By (2.4), the change of variables $\tau = t - s$, (A2) and (3.6), we have

$$\begin{aligned}&|\Gamma[w](x_1,t) - \Gamma[w](x_1,t_1)| \\ &\quad\le \frac{1}{M_\Omega(x_1,\xi)} \int_\Omega \left| \int_0^t \Gamma_\Omega(x_1,\tau;y,0) f(y, w(y,t-\tau) M_\Omega(y,\xi)) \, d\tau \right. \\ &\qquad \left. - \int_0^{t_1} \Gamma_\Omega(x_1,\tau;y,0) f(y, w(y,t_1-\tau) M_\Omega(y,\xi)) \, d\tau \right| dy \\ &\quad\le \int_\Omega \left(\int_0^{t_1} \Gamma_\Omega(x_1,\tau;y,0) |w(y,t-\tau) - w(y,t_1-\tau)| \, d\tau \right) \frac{M_\Omega(y,\xi)}{M_\Omega(x_1,\xi)} \varphi(y) \, dy \\ &\qquad + \int_\Omega \left(\int_{t \wedge t_1}^{t \vee t_1} \Gamma_\Omega(x_1,\tau;y,0) \, d\tau \right) \frac{M_\Omega(y,\xi)}{M_\Omega(x_1,\xi)} \varphi(y) \, dy,\end{aligned} \tag{3.13}$$

where we assumed for convenience that $w(y, s) = 0$ for $s \le 0$. By Lebesgue's dominated convergence theorem, both of the integrals of the right hand side converge to zero as $t \to t_1$. Therefore $\Gamma[w]$ is continuous at (x_1, t_1). Thus (ii) holds.

Finally, we show (v). Let $0 < t_1 < t_2$ and $x \in \Omega$. By the same way as (3.13), we have

$$\begin{aligned}
&|\Gamma[w](x,t_1) - \Gamma[w](x,t_2)| \\
&\quad \le \int_\Omega \left(\int_0^\infty \Gamma_\Omega(x,\tau;y,0)|w(y,t_1-\tau) - w(y,t_2-\tau)|\,d\tau \right) \frac{M_\Omega(y,\xi)}{M_\Omega(x,\xi)} \varphi(y)\,dy \\
&\qquad + \int_\Omega \left(\int_{t_1}^{t_2} \Gamma_\Omega(x,\tau;y,0)\,d\tau \right) \frac{M_\Omega(y,\xi)}{M_\Omega(x,\xi)} \varphi(y)\,dy.
\end{aligned}$$

Since $w(y, t)$ has a finite limit as $t \to \infty$, the right hand side becomes smaller if t_1 is bigger. Therefore $\Gamma[w](x, t)$ has a finite limit as $t \to \infty$. □

The above two claims show that T_λ is a mapping from W_λ into W_λ, whenever $0 < \lambda \le \lambda_1$.

Claim 3. *There exists $\lambda_0 > 0$ such that whenever $\lambda \in (0, \lambda_0]$,*

$$\|T_\lambda[w_1] - T_\lambda[w_2]\|_\infty = \|\Gamma[w_1] - \Gamma[w_2]\|_\infty \le \frac{1}{2}\|w_1 - w_2\|_\infty \quad \textit{for all } w_1, w_2 \in W_\lambda.$$

Moreover, there is a unique positive function $w_* \in W_\lambda$ such that $T_\lambda[w_*] = w_*$.

Proof. Note from (3.7) that there is $\lambda_0 \in (0, \lambda_1]$ such that

$$\|\psi(\cdot, 3\lambda M_\Omega(\cdot,\xi))\|_{\mathcal{K}_\xi(\Omega)} \le \frac{1}{2} \quad \text{for all } \lambda \in (0, \lambda_0].$$

Let $0 < \lambda \le \lambda_0$ and let $w_1, w_2 \in W_\lambda$. Then, by (A2),

$$|f(y, w_1(y,s)M_\Omega(y,\xi)) - f(y, w_2(y,s)M_\Omega(y,\xi))| \le \psi(y, 3\lambda M_\Omega(y,\xi))M_\Omega(y,\xi)\|w_1 - w_2\|_\infty,$$

and so

$$\begin{aligned}
\|T_\lambda[w_1] - T_\lambda[w_2]\|_\infty = \|\Gamma[w_1] - \Gamma[w_2]\|_\infty &\le \|\psi(\cdot, 3\lambda M_\Omega(\cdot,\xi))\|_{\mathcal{K}_\xi(\Omega)}\|w_1 - w_2\|_\infty \\
&\le \frac{1}{2}\|w_1 - w_2\|_\infty.
\end{aligned}$$

Since W_λ is closed in the Banach space $C_b(\Omega \times (0,\infty))$, the existence of w_* in the claim follows from Banach's fixed point theorem. Since $w_* = T_\lambda[w_*] \ge \lambda h/M_\Omega(\cdot, \xi)$, w_* is positive on Ω. □

Claim 4. *Let w_* be as in Claim 3. Then $w_*(\cdot, t)$ converges uniformly on Ω as $t \to \infty$. Moreover, the limit function, say w_∞, satisfies*

$$w_\infty(x) = \lambda + \frac{1}{M_\Omega(x,\xi)} \int_\Omega G_\Omega(x,y) f(y, w_\infty(y)M_\Omega(y,\xi))\,dy \quad \textit{for all } x \in \Omega. \tag{3.14}$$

Proof. Note that

$$w_*(x,t) = T_\lambda[w_*](x,t) = \frac{\lambda h(x,t) + v(x,t)}{M_\Omega(x,\xi)} + \Gamma[w_*](x,t) \quad \text{for } (x,t) \in \Omega \times (0,\infty). \tag{3.15}$$

By Lemmas 2.7, 2.10 and Dini's theorem,

$$\lim_{t\to\infty} \frac{h(x,t)}{M_\Omega(x,\xi)} = 1 \quad \text{uniformly for } x \in \overline{\Omega}. \tag{3.16}$$

Also, since

$$g(y)M_\Omega(y,\xi) \le C\left(\frac{g(y)}{g(b_{y\xi})}\right)^2 \|y-\xi\|^{2-n} \le C\|y-\xi\|^{2-n} \quad \text{for all } y \in \Omega \text{ and } b_{y\xi} \in \mathcal{B}_e(y,\xi)$$

by Lemmas 2.2 and 2.3, we have $\int_\Omega g(y)M_\Omega(y,\xi)\,dy < \infty$, and so by (2.8)

$$0 \le \sup_{x\in\Omega} \frac{v(x,t)}{M_\Omega(x,\xi)} \le Ce^{-C_2 t} \int_\Omega g(y)u_0(y)\,dy \to 0 \quad \text{as } t \to \infty. \tag{3.17}$$

Claim 2 and Ascoli–Arzelá's theorem imply that $\Gamma[w_*](\cdot,t)$ converges uniformly on $\overline{\Omega}$ as $t \to \infty$. All of the above conclude that $w_*(\cdot,t)$ converges uniformly on Ω as $t \to \infty$.

Moreover, using Lebesgue's dominated convergence theorem and the continuity of $f(y,\cdot)$, we get

$$\lim_{t\to\infty} \Gamma[w_*](x,t) = \frac{1}{M_\Omega(x,\xi)} \int_\Omega G_\Omega(x,y) f(y, w_\infty(y)M_\Omega(y,\xi))\,dy \quad \text{for } x \in \Omega.$$

This, together with (3.15)–(3.17), yields (3.14). $\square$

Proof of Theorem 1.1. Let $0 < \lambda \le \lambda_0$ and let w_* be as in Claim 3. Define

$$u(x,t) = M_\Omega(x,\xi)w_*(x,t).$$

Then u is a positive continuous solution of (3.4), and so of (1.2), satisfying $u(x,t) \le 3\lambda M_\Omega(x,\xi)$ for all $(x,t) \in \Omega \times (0,\infty)$. Moreover, we obtain from (3.2), (3.8), (3.15) and Lemma 2.10 that for each $t > 0$,

$$\lim_{x\to\xi} \frac{u(x,t)}{M_\Omega(x,\xi)} = \lim_{x\to\xi} w_*(x,t) = \lambda \lim_{x\to\xi} \frac{h(x,t)}{M_\Omega(x,\xi)} = \lambda.$$

Thus the first assertion of Theorem 1.1 is proved.

To show the second assertion, we define $u_\infty(x) = M_\Omega(x,\xi)w_\infty(x)$ with w_∞ being as in Claim 4. Then $u_\infty(x) \le 3\lambda M_\Omega(x,\xi)$ for all $x \in \Omega$ and u_∞ is continuous on $\overline{\Omega} \setminus \{\xi\}$, vanishes on $\partial\Omega \setminus \{\xi\}$, and satisfies

$$u_\infty(x) = \lambda M_\Omega(x,\xi) + \int_\Omega G_\Omega(x,y) f(y, u_\infty(y))\,dy \quad \text{for all } x \in \Omega$$

by Claim 4. The last equality gives $u_\infty(x) \ge \lambda M_\Omega(x,\xi)$ for all $x \in \Omega$, and so (1.4) holds. Also, (1.6) follows from Claim 4. Lemma 2.18 implies that $\int_\Omega G_\Omega(x,y)f(y,u_\infty(y))\,dy/M_\Omega(x,\xi) \to 0$ as $x \to \xi$, and so (1.5) holds. Hence u_∞ is a positive solution of (1.3) with the required properties. This completes the proof. $\square$

4 Examples

Lemma 4.1. *Let $\beta \le 1 \le \alpha$ be as in* (2.1)*, and let $p(y)$ and $\gamma(y)$ be Borel measurable functions on Ω such that $p(y) \ge 1$ and*

$$\operatorname*{esssup}_{y\in\Omega}\{p(y)(n-2+\alpha) + \alpha(\gamma(y)-1)\} \vee \operatorname*{esssup}_{y\in\Omega}\{p(y)(n-2+\beta) + \beta(\gamma(y)-1)\} < n.$$

Then $g(y)^{-\gamma(y)}M_\Omega(y,\xi)^{p(y)-1} \in \mathcal{K}_\xi(\Omega)$.

Proof. We give a proof for $n \geq 3$, because the case $n = 2$ is simpler. Write

$$\varphi(y) = g(y)^{-\gamma(y)} M_\Omega(y,\xi)^{p(y)-1}.$$

Before a proof, let us remark the following. If $\gamma_1(y) \leq \gamma_2(y)$, then $g(y)^{-\gamma_1(y)} \leq g(y)^{-\gamma_2(y)}$, so that $g(y)^{-\gamma_2(y)} M_\Omega(y,\xi)^{p(y)-1} \in \mathcal{K}_\xi(\Omega)$ implies $g(y)^{-\gamma_1(y)} M_\Omega(y,\xi)^{p(y)-1} \in \mathcal{K}_\xi(\Omega)$. Moreover,

$$p(y) < \frac{n}{n-2} \quad \text{is equivalent to} \quad 1 - p(y) < \frac{n+\alpha-p(y)(n-2+\alpha)}{\alpha}. \tag{4.1}$$

Therefore, by considering

$$\gamma_0(y) = \begin{cases} \gamma(y) \vee (1-p(y)) & \text{if } p(y) < \dfrac{n}{n-2}, \\ \gamma(y) & \text{if } p(y) \geq \dfrac{n}{n-2}, \end{cases}$$

we may assume that $\gamma(y) \geq 1 - p(y)$ on the set where $p(y) < n/(n-2)$. Note from (4.1) that for all $y \in \Omega$ satisfying $p(y) \geq n/(n-2)$,

$$\gamma(y) < \frac{n+\beta-p(y)(n-2+\beta)}{\beta} \leq \frac{n+\alpha-p(y)(n-2+\alpha)}{\alpha} \leq 1 - p(y) \leq 0,$$

and that $\gamma(y) < 2/\alpha \leq 2$ for all $y \in \Omega$ because of $p(y) \geq 1$. These will be used tacitly below. Let $x, y \in \Omega$ and take $b_{x\xi} \in \mathcal{B}_e(x,\xi)$, $b_{y\xi} \in \mathcal{B}_e(y,\xi)$ and $b_{xy} \in \mathcal{B}_e(x,y)$. By Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \frac{M_\Omega(y,\xi) G_\Omega(x,y)}{M_\Omega(x,\xi)} \varphi(y) &= \frac{G_\Omega(x,y) M_\Omega(y,\xi)^{p(y)}}{M_\Omega(x,\xi) g(y)^{\gamma(y)}} \\ &\leq C \frac{g(y)^{p(y)+1-\gamma(y)} g(b_{x\xi})^2}{g(b_{y\xi})^{2p(y)} g(b_{xy})^2} \left(\frac{\|x-\xi\|}{\|x-y\| \|y-\xi\|^{p(y)}} \right)^{n-2}. \end{aligned} \tag{4.2}$$

Now, we fix $x \in \Omega$ and $r > 0$. Put

$$\begin{aligned} E_1 &= \Omega \cap B(x,r) \setminus B(\xi, \frac{1}{2}\|x-\xi\|), \\ E_2 &= \Omega \cap B(x,r) \cap B(\xi, \frac{1}{2}\|x-\xi\|). \end{aligned}$$

Let $y \in E_1$. We first show that

$$g(b_{x\xi}) \leq C g(b_{y\xi}). \tag{4.3}$$

If $\|x-\xi\|/2 \leq \|y-\xi\| \leq 2C_1\|x-\xi\|$ with C_1 being the constant in the definition of $\mathcal{B}_e$, then

$$\begin{aligned} \|b_{x\xi} - b_{y\xi}\| &\leq \|b_{x\xi} - \xi\| + \|\xi - b_{y\xi}\| \leq C_1(\|x-\xi\| + \|y-\xi\|) \\ &\leq C(\|x-\xi\| \wedge \|y-\xi\|) \leq C(\delta_\Omega(b_{x\xi}) \wedge \delta_\Omega(b_{y\xi})), \end{aligned}$$

and so $g(b_{x\xi}) \leq C g(b_{y\xi})$ by Lemma 2.2. If $\|y-\xi\| \geq 2C_1\|x-\xi\|$, then

$$\begin{aligned} \|b_{x\xi} - y\| &\leq \|b_{x\xi} - \xi\| + \|\xi - y\| \leq C_1\|x-\xi\| + \|\xi - y\| \leq 2\|y-\xi\|, \\ \|b_{x\xi} - y\| &\geq \|y-\xi\| - \|b_{x\xi} - \xi\| \geq \|y-\xi\| - C_1\|x-\xi\| \geq \frac{1}{2}\|y-\xi\|. \end{aligned}$$

Therefore, for $b \in \mathcal{B}_e(b_{x\xi}, y)$, we have

$$\begin{aligned} \|b - b_{y\xi}\| &\leq \|b-y\| + \|y - b_{y\xi}\| \leq C_1(\|b_{x\xi} - y\| + \|y-\xi\|) \\ &\leq C(\|b_{x\xi} - y\| \wedge \|y-\xi\|) \leq C(\delta_\Omega(b) \wedge \delta_\Omega(b_{y\xi})), \end{aligned}$$

and so $g(b_{x\xi}) \le Cg(b) \le Cg(b_{y\xi})$ by Lemma 2.2. Hence (4.3) holds.

Also, $g(y) \le C(g(b_{xy}) \wedge g(b_{y\xi}))$ and $\|x-y\| \le \|x-\xi\| + \|\xi - y\| \le 3\|y-\xi\|$. Therefore, if $\gamma(y) \ge 0$ (this does not occur for y satisfying $p(y) \ge n/(n-2)$), then we use $g(b_{x\xi})^2 \le Cg(b_{y\xi})^2$, $g(y)^{p(y)-1} \le Cg(b_{y\xi})^{p(y)-1}$ and $g(y)^{2-\gamma(y)} \le Cg(b_{xy})^{2-\gamma(y)}$ to get

$$\frac{g(y)^{p(y)+1-\gamma(y)}g(b_{x\xi})^2}{g(b_{y\xi})^{2p(y)}g(b_{xy})^2} \le \frac{C}{g(b_{y\xi})^{p(y)-1}g(b_{xy})^{\gamma(y)}} \le \frac{C}{\|x-y\|^{\alpha(p(y)-1+\gamma(y))}};$$

if $\gamma(y) < 0$, then we use $g(y)^{p(y)-1-\gamma(y)} \le Cg(b_{y\xi})^{p(y)-1-\gamma(y)}$, $g(y)^2 \le Cg(b_{xy})^2$ and $g(b_{x\xi})^2 \le Cg(b_{y\xi})^2$ to get

$$\frac{g(y)^{p(y)+1-\gamma(y)}g(b_{x\xi})^2}{g(b_{y\xi})^{2p(y)}g(b_{xy})^2} \le \frac{C}{g(b_{y\xi})^{p(y)-1+\gamma(y)}} \le \begin{cases} \dfrac{C}{\|x-y\|^{\alpha(p(y)-1+\gamma(y))}} & \text{if } p(y) < \dfrac{n}{n-2}, \\ \dfrac{C}{\|y-\xi\|^{\beta(p(y)-1+\gamma(y))}} & \text{if } p(y) \ge \dfrac{n}{n-2}. \end{cases}$$

Here the last inequality in the case $p(y) \ge n/(n-2)$ follows by $g(b_{y\xi}) \le C\delta_\Omega(b_{y\xi})^\beta \le C\|b_{y\xi} - \xi\|^\beta \le C\|y-\xi\|^\beta$. Hence, we obtain from (4.2) that for $y \in E_1$ satisfying $p(y) < n/(n-2)$,

$$\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y) \le \frac{C}{\|x-y\|^{p(y)(n-2+\alpha)+(\gamma(y)-1)\alpha}},$$

and for $y \in E_1$ satisfying $p(y) \ge n/(n-2)$,

$$\begin{aligned} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y) &\le \frac{C}{\|x-y\|^{n-2}\|y-\xi\|^{(p(y)-1)(n-2+\beta)+\beta\gamma(y)}} \\ &\le \begin{cases} \dfrac{C}{\|x-y\|^{p(y)(n-2+\beta)+(\gamma(y)-1)\beta}} & \text{if } (p(y)-1)(n-2+\beta)+\beta\gamma(y) > 0, \\ \dfrac{C}{\|x-y\|^{n-2}} & \text{if } (p(y)-1)(n-2+\beta)+\beta\gamma(y) \le 0. \end{cases} \end{aligned}$$

Let $y \in E_2$. Since $\|x-\xi\|/2 \le \|x-y\| \le 2\|x-\xi\|$, we have

$$\begin{aligned} \|b_{x\xi} - b_{xy}\| &\le \|b_{x\xi} - x\| + \|x - b_{xy}\| \le C_1(\|x-\xi\| + \|x-y\|) \\ &\le C(\|x-\xi\| \wedge \|x-y\|) \le C(\delta_\Omega(b_{x\xi}) \wedge \delta_\Omega(b_{xy})), \end{aligned}$$

and so $g(b_{x\xi}) \le Cg(b_{xy})$ by Lemma 2.2. Since $p(y)+1-\gamma(y) > 0$ and $g(y) \le Cg(b_{y\xi})$, it follows from (4.2) that

$$\begin{aligned} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y) &\le \frac{C}{g(b_{y\xi})^{p(y)-1+\gamma(y)}\|y-\xi\|^{p(y)(n-2)}} \\ &\le \begin{cases} \dfrac{C}{\|y-\xi\|^{p(y)(n-2+\alpha)+(\gamma(y)-1)\alpha}} & \text{if } p(y) < \dfrac{n}{n-2}, \\ \dfrac{C}{\|y-\xi\|^{p(y)(n-2+\beta)+(\gamma(y)-1)\beta}} & \text{if } p(y) \ge \dfrac{n}{n-2}. \end{cases} \end{aligned}$$

Here we used $\|y-\xi\|^\alpha/C \le \delta_\Omega(b_{y\xi})^\alpha/C \le g(b_{y\xi}) \le C\delta_\Omega(b_{y\xi})^\beta \le C\|y-\xi\|^\beta$, which follows from (2.1) and the definition of $\mathcal{B}_e$. Note that $E_2 \ne \emptyset$ implies that $E_2 \subset B(\xi, r)$. Using estimates above, we easily see that φ satisfies (2.14). Also, (2.15) is shown by using (2.14). Indeed, since we get from

(4.2), $g(b_{xy}) \geq \|x-y\|^\alpha/C$, $g(b_{x\xi}) \leq C\|x-\xi\|^\beta$ and $\|y-\xi\|^\alpha/C \leq g(b_{y\xi}) \leq C\|y-\xi\|^\beta$ that

$$\begin{aligned}\frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y) &\leq \frac{C}{g(b_{y\xi})^{p(y)-1+\gamma(y)}}\frac{g(b_{x\xi})^2}{g(b_{xy})^2}\left(\frac{\|x-\xi\|}{\|x-y\|\|y-\xi\|^{p(y)}}\right)^{n-2} \\ &\leq \begin{cases} \dfrac{C\|x-\xi\|^{n-2+2\beta}}{\|x-y\|^{n-2+2\alpha}\|y-\xi\|^{p(y)(n-2+\alpha)+(\gamma(y)-1)\alpha}} & \text{if } p(y) < \dfrac{n}{n-2}, \\ \dfrac{C\|x-\xi\|^{n-2+2\beta}}{\|x-y\|^{n-2+2\alpha}\|y-\xi\|^{p(y)(n-2+\beta)+(\gamma(y)-1)\beta}} & \text{if } p(y) \geq \dfrac{n}{n-2}, \end{cases}\end{aligned}$$

it follows that for sufficiently small $\delta > 0$,

$$\begin{aligned}\int_{\Omega\cap B(\xi,r)} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y)\,dy &\leq \varepsilon + \int_{\Omega\cap B(\xi,r)\setminus B(x,\delta)} \frac{M_\Omega(y,\xi)G_\Omega(x,y)}{M_\Omega(x,\xi)}\varphi(y)\,dy \\ &\leq \varepsilon + \frac{C}{\delta^{n-2+2\alpha}} \times \begin{cases} r^{n-m(\alpha)} & \text{if } p(y) < \dfrac{n}{n-2}, \\ r^{n-m(\beta)} & \text{if } p(y) \geq \dfrac{n}{n-2}, \end{cases}\end{aligned}$$

where $m(\alpha) = \operatorname{esssup}_{y\in\Omega}\{p(y)(n-2+\alpha)+(\gamma(y)-1)\alpha\}$. Hence $\varphi \in \mathcal{K}_\xi(\Omega)$. □

For a Borel measurable function $p(x)$ on Ω, we let

$$p^+ = \operatorname*{esssup}_{x\in\Omega} p(x).$$

Example 4.2. Let $\beta \leq 1 \leq \alpha$ be as in (2.1), and let $p(x)$, $q(x)$ and $\gamma(x)$ be Borel measurable functions on Ω such that all of the p^+, q^+, γ^+ are finite and

$$\operatorname*{esssup}_{x\in\Omega}\{p(x)(n-2+\alpha)+\alpha(\gamma(x)-1)\} \vee \operatorname*{esssup}_{x\in\Omega}\{p(x)(n-2+\beta)+\beta(\gamma(x)-1)\} < n.$$

Moreover, we assume one of the following:

(i) $p(x) > 1$ and $q(x) \geq 0$,

(ii) $p(x) \geq 1$ and $q(x) > 0$.

If $V(x)$ is a nonnegative Borel measurable function on Ω such that

$$V(x) \leq Cg(x)^{-\gamma(x)}(\log(1+M_\Omega(x,\xi)))^{-q(x)}$$

for almost every $x \in \Omega$ and some constant $C > 0$, then $f(x,u) = V(x)u^{p(x)}(\log(1+u))^{q(x)}$ satisfies (A1) and (A2).

Proof. Because (A1) is trivial, we have only to check (A2). Let

$$\psi(x,u) = (p^+ + q^+)V(x)u^{p(x)-1}(\log(1+u))^{q(x)}.$$

Then ψ is nonnegative and Borel measurable on $\Omega\times[0,\infty)$. Also, it is easy to see that for each $x \in \Omega$, $\psi(x,\cdot)$ is increasing on $[0,\infty)$ and $\lim_{u\to 0+}\psi(x,u) = 0$. Observe from Lemma 4.1 that $\psi(\cdot, M_\Omega(\cdot,\xi)) \in \mathcal{K}_\xi(\Omega)$. To show the Lipschitz type continuity, we note that

$$0 \leq f_u(x,u) = V(x)\left(p(x)u^{p(x)-1}(\log(1+u))^{q(x)} + q(x)u^{p(x)}\frac{(\log(1+u))^{q(x)-1}}{1+u}\right) \leq \psi(x,u),$$

since $(1+u)\log(1+u) \geq u$. Let $0 \leq u_1 \leq u_2$ and $x \in \Omega$. Then, by the mean value theorem, we find $\theta \in [u_1,u_2]$ such that

$$|f(x,u_1) - f(x,u_2)| = f_u(x,\theta)|u_1-u_2| \leq \psi(x,u_2)|u_1-u_2|.$$

Hence f satisfies (A2). □

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