15-NODAL QUARTIC SURFACES. PART II: THE AUTOMORPHISM GROUP

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Abstract. We describe a set of generators and defining relations for the group of birational automorphisms of a general 15-nodal quartic surface in the complex projective 3-dimensional space.

1. Introduction

A quartic surface in $\mathbf{P}^3$ with 16 ordinary double points as its only singularities is classically known as a Kummer quartic surface, and has been intensively investigated since 19th century (see, for example, Hudson [9] or Baker [1]). For a positive integer $n \leq 16$, we denote by X_n a quartic surface in $\mathbf{P}^3$ with n ordinary double points (nodes) satisfying the following assumptions:

(i) The quartic surface X_n can be degenerated by acquiring additional $16 - n$ nodes and hence becomes isomorphic to a Kummer quartic surface.

(ii) The Picard lattice S_n of the minimal resolution Y_n of X_n is embeddable into the Picard lattice of a general Kummer surface by specialization, and it is generated over $\mathbb{Q}$ by the class $h_4 \in S_n$ of a plane section of X_n and the classes of the exceptional curves of the resolution $Y_n \to X_n$. In particular, the rank of S_n is $n+1$.

(iii) The only isometries of the transcendental lattice T_n of Y_n that preserve the subspace $H^{2,0}(Y_n)$ of $T_n \otimes \mathbb{C}$ are ± 1.

The group $\mathrm{Aut}(Y_{16})$ of birational automorphisms of a general Kummer quartic surface X_{16} was discribed by Kondo [14]. We discribe the automorphism group $\mathrm{Aut}(Y_{15})$ of the $K3$ surface Y_{15} by the embedding of lattices $S_{15} \hookrightarrow S_{16}$ induced by the specialization of X_{15} to X_{16}. As was proved in [7], the 15-nodal quartic surfaces satisfy Condition (i) and form an irreducible family. If we choose a general member of this family, then Conditions (ii) and (iii) are satisfied. We give a generating set of $\mathrm{Aut}(Y_{15})$ explicitly in Theorem 5.9, and describe the defining relations of $\mathrm{Aut}(Y_{15})$ with respect to this generating set in Theorem 5.10.

Our main tool is Borcherds' method ([2], [3]), which was also used in the calculation of $\mathrm{Aut}(Y_{16})$ by Kondo [14]. We also compute the defining relations from the tessellation of the nef-and-big cone calculated by Borcherds' method. Our method is heavily computational, and is based on machine-aided calculations carried out by `GAP` [26]. Explicit numerical data is available from the second author's webpage [23].

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We hope that, by generalizing X_{15} further to X_n with $n \le 14$ and looking at the embedding $S_n \hookrightarrow S_{15}$, we can calculate $\mathrm{Aut}(Y_n)$ for $n \le 14$. See [22] for an example of an analysis of the change that the automorphism group undergoes when a $K3$ surface is generalized/specialized.

This paper is organized as follows. In Section 2, we set up notation and terminology about lattices, and present some computational tools that are used throughout this paper. In Section 3, we review Borcherds' method, and a method to calculate the defining relations. In Section 4, we review the result of Kondo [14] and Ohashi [17] on the birational automorphism group $\mathrm{Aut}(Y_{16})$ of a general Kummer quartic surface. Then in Section 5, we calculate the automorphism group $\mathrm{Aut}(Y_{15})$. The results are given in Theorems 5.9 and 5.10.

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2. Preliminaries

2.1. **Lattice.** A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $\langle\, ,\, \rangle\colon L \times L \to \mathbb{Z}$. The *signature* of a lattice L is the signature of the real quadratic space $L \otimes \mathbb{R}$. We say that a lattice L of rank $n > 1$ is *hyperbolic* if the signature of L is $(1, n-1)$. A lattice L is *even* if $\langle v, v\rangle \in 2\mathbb{Z}$ holds for all $v \in L$. The cokernel of the natural embedding $L \hookrightarrow L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ induced by $\langle\, ,\, \rangle$ is called the *discriminant group* of L. We say that L is *unimodular* if the discriminant group of L is trivial. When L is even, the discriminant group $L^\vee/L$ of L is equipped with a natural quadratic form

$$q_L\colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z},$$

which is called the *discriminant form* of L. We refer to Nikulin [15] for the basic properties of discriminant forms.

The group of isometries of a lattice L is denoted by $\mathrm{O}(L)$. We let $\mathrm{O}(L)$ act on L from the *right*, and write the action as $v \mapsto v^g$ for $v \in L$ and $g \in \mathrm{O}(L)$. For a subset A of $L \otimes \mathbb{R}$, we denote by A^g the image of A by the action of g (*not* the fixed locus of g in A). A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r\rangle = -2$. A (-2)-vector $r \in L$ gives rise to an isometry $x \mapsto x + \langle x, r\rangle r$ of L, which is called the *reflection* with respect to r. Let $W(L)$ denote the subgroup of $\mathrm{O}(L)$ generated by the reflections with respect to all (-2)-vectors.

Let L be an even hyperbolic lattice. We fix a *positive cone* $\mathcal{P}$ of L, i.e. one of the two connected components of $\{\, x \in L \otimes \mathbb{R} \mid \langle x, x\rangle > 0 \,\}$. We put

$$\mathrm{O}(L)' = \{\, g \in \mathrm{O}(L) \mid g \text{ leaves } \mathcal{P} \text{ invariant} \,\}.$$

It is a subgroup of index 2 in $\mathrm{O}(L)$ that contains the subgroup $W(L)$. A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of the complement

$$\mathcal{P} \setminus \bigcup (r)^\perp$$

of the union of hyperplanes $(r)^\perp := \{\, x \in \mathcal{P} \mid \langle x, r\rangle = 0 \,\}$, where r runs through the set of all (-2)-vectors. Then $\mathcal{P}$ is tessellated by standard fundamental domains of $W(L)$, and $W(L)$ acts on the set of standard fundamental domains simply-transitively.

2.2. **Computational tools.** Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$. Suppose that $v_0 \in L \cap \mathcal{P}$. Then, for any integers a and b, we can calculate the finite set

$$\{\, v \in L \mid \langle v, v\rangle = a,\ \langle v, v_0\rangle = b \,\}.$$

Suppose that $v_1 \in L$ also belongs to $\mathcal{P}$. Then, for any negative integer a, we can calculate the finite set

$$\{\, v \in L \mid \langle v, v\rangle = a,\ \langle v, v_0\rangle > 0,\ \langle v, v_1\rangle < 0 \,\}.$$

In particular, we can calculate the set

$$\{\, r \in L \mid \langle r, r\rangle = -2,\ \langle r, v_0\rangle > 0,\ \langle r, v_1\rangle < 0 \,\}$$

of (-2)-vectors r such that the hyperplane $(r)^\perp$ *separates* v_0 and v_1. See [20] for the details of these algorithms.

2.3. **Geometric applications.** Let S_Y denote the Picard lattice of a $K3$ surface Y with the intersection form $\langle\ ,\ \rangle$. By abuse of notation, for divisors D, D' of Y, we write by $\langle D, D'\rangle$ the intersection number of the divisor classes of D and D'. Let $\alpha \in S_Y$ be an ample class of Y, and let $\mathcal{P}_Y$ denote the positive cone of $S_Y \otimes \mathbb{R}$ containing α.

2.3.1. *The cone* $\mathcal{N}_Y$. Let $\overline{\mathcal{N}}_Y$ be the nef cone in $\mathcal{P}_Y$, i.e. the set

$$\{\, x \in S_Y \otimes \mathbb{R} \mid \langle x, C\rangle \geq 0, \text{for any curve } C \,\}.$$

We put

$$\mathcal{N}_Y := \overline{\mathcal{N}}_Y \cap \mathcal{P}_Y.$$

We say that $h \in \mathcal{P}_Y \cap S_Y$ is *nef* if $h \in \mathcal{N}_Y$. It is well-known that $\mathcal{N}_Y$ is the standard fundamental domain of the action of $W(S_Y)$ on $\mathcal{P}_Y$ containing the ample class α in its interior. Therefore, for any $v \in S_Y \cap \mathcal{P}_Y$, we can determine whether v belongs to $\mathcal{N}_Y$ or not by calculating the set of (-2)-vectors r such that $(r)^\perp$ separates α and v.

2.3.2. *Smooth rational curves.* Let $r \in S_Y$ be a (-2)-vector with $d := \langle r, \alpha\rangle > 0$. By Riemann-Roch, r is an effective divisor class. It is the class of a smooth rational curve if and only if, for any smooth rational curve C' with $\langle C', \alpha\rangle < d$, we have $\langle r, C'\rangle \geq 0$. Hence, by induction on d, we can calculate the set of classes of all smooth rational curves whose degree with respect to α is $\leq d$. In particular, we can determine whether a given (-2)-vector is the class of a smooth rational curve or not.

2.3.3. *Double-plane involutions.* By abuse of notation, for $h \in S_Y$, we denote by $|h|$ the complete linear system of a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = h$. A *double-plane cover* is a generically finite morphism $f : Y \to \mathbf{P}^2$ of degree 2. Obviously, the class $h_2 := c_1(f^*(\mathcal{O}_{\mathbf{P}^2}(1))$ is nef and satisfies $\langle h_2, h_2\rangle = 2$.

Conversely, let $h_2 \in S_Y$ be a nef class with $\langle h_2, h_2\rangle = 2$. By [16, Proposition 0.1], we can determine whether $|h_2|$ is fixed-component free or not by calculating the set of all vectors $v \in S_Y$ such that $\langle v, v\rangle = 0$ and $\langle v, h_2\rangle = 1$. Suppose that $|h_2|$ is fixed-component free. Then $|h_2|$ is base-point free by [19, Corollary 3.2], and hence $|h_2|$ defines a double-plane cover $\Phi\colon Y \to \mathbf{P}^2$. We denote by $\iota(h_2) \in \mathrm{Aut}(Y)$ the involution induced by the deck-transformation of $\Phi\colon Y \to \mathbf{P}^2$, and call it the *double-plane involution* associated with h_2. We can calculate the classes of smooth rational curves contracted by $\Phi\colon Y \to \mathbf{P}^2$, which form an ADE-configuration of

(-2)-vectors. The ADE-type of this configuration is the ADE-type of the singular points of the branch curve of $\Phi\colon Y \to \mathbf{P}^2$. The matrix representation of the action $\iota(h_2)^*$ of $\iota(h_2)$ on S_Y is then calculated from this set of classes of smooth rational curves contracted by Φ. The details of these algorithms are given in [20].

3. Borcherds' method

3.1. **Terminology.** We fix terminology about Borcherds' method ([2], [3]). See Chapter 10 of [6] for the Leech lattice and the Golay code. See [21] for the computational details of Borcherds' method.

We put

$$\Omega := \mathbb{P}^1(\mathbb{F}_{23}) = \{\infty, 0, 1, \dots, 22\},$$

and let $\mathbb{F}_2^\Omega$ and $\mathbb{Z}^\Omega$ be the modules of $\mathbb{F}_2$-valued and $\mathbb{Z}$-valued functions on Ω, respectively. Then $\mathbb{F}_2^\Omega$ can be identified with the power set of Ω with the addition being the symmetric-difference of subsets. We have the Golay code $\mathcal{G}$ in $\mathbb{F}_2^\Omega$. Let $\mathcal{G}(8)$ denote the set of words of Hamming weight 8 in $\mathcal{G}$. Elements of $\mathcal{G}(8)$ are called *octads*. For a subset Σ of Ω, let ν_Σ denote the vector of $\mathbb{Z}^\Omega$ such that $\nu_\Sigma(s) = 1$ if $s \in \Sigma$ and $\nu_\Sigma(s) = 0$ otherwise. We equip $\mathbb{Z}^\Omega$ with the negative-definite inner-product by

$$(\mathbf{x}, \mathbf{y}) \mapsto -(x_\infty y_\infty + x_0 y_0 + x_1 y_1 + \cdots + x_{22} y_{22})/8.$$

Then the *negative-definite Leech lattice* Λ is generated in $\mathbb{Z}^\Omega$ by $2\nu_K$, where K runs through the set $\mathcal{G}(8)$ of octads, and $\nu_\Omega - 4\nu_{\{\infty\}}$. We consider the orthogonal direct sum

$$II_{1,25} := U \oplus \Lambda,$$

where U is the hyperbolic plane with a fixed basis such that the Gram matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $II_{1,25}$ is an even unimodular hyperbolic lattice of rank 26. By Milnor's Theorem, it is unique up to isomorphism. A vector of $II_{1,25}$ is written as (a, b, λ), where $(a, b) \in U$ and $\lambda \in \Lambda$. Let $\mathcal{P}(II_{1,25})$ be the positive cone containing $(1, 1, 0)$, and $\overline{\mathcal{P}}(II_{1,25})$ the closure of $\mathcal{P}(II_{1,25})$ in $II_{1,25} \otimes \mathbb{R}$. A vector $\mathbf{w}$ of $II_{1,25}$ is called a *Weyl vector* if $\mathbf{w}$ is non-zero, primitive in $II_{1,25}$, contained in $\overline{\mathcal{P}}(II_{1,25})$, satisfying $\langle \mathbf{w}, \mathbf{w} \rangle = 0$, and such that $(\mathbb{Z}\mathbf{w})^\perp/\mathbb{Z}\mathbf{w}$ is isomorphic to Λ, where $(\mathbb{Z}\mathbf{w})^\perp := \{\, v \in II_{1,25} \mid \langle v, \mathbf{w} \rangle = 0 \,\}$. A (-2)-vector r of $II_{1,25}$ is said to be a *Leech root with respect to* $\mathbf{w}$ if $\langle r, \mathbf{w} \rangle = 1$. We put

$$\mathcal{D}(\mathbf{w}) := \{\, x \in \mathcal{P}(II_{1,25}) \mid \langle x, r \rangle \geq 0 \text{ for all Leech roots } r \text{ with respect to } \mathbf{w} \,\},$$

and call it the *Conway chamber* associated with $\mathbf{w}$. Then we have the following:

Theorem 3.1 (Conway [5]). *The mapping $\mathbf{w} \mapsto \mathcal{D}(\mathbf{w})$ is a bijection from the set of Weyl vectors to the set of standard fundamental domains of the action of $W(II_{1,25})$ on $\mathcal{P}(II_{1,25})$.* □

Example 3.2. We consider the Weyl vector $\mathbf{w}_0 := (1, 0, 0)$. For $\lambda \in \Lambda$, we put

$$r_0(\lambda) := (-1 - |\lambda|^2/2, 1, \lambda),$$

where $|\lambda|^2$ is the square-norm of λ in Λ. Then the mapping $\lambda \mapsto r_0(\lambda)$ gives a bijection from Λ to the set of Leech roots with respect to $\mathbf{w}_0$.

Let S be an even hyperbolic lattice. Suppose that S admits a primitive embedding

$$\epsilon_S \colon S \hookrightarrow II_{1,25},$$

and let $\mathcal{P}(S)$ be the positive cone of S that is mapped into $\mathcal{P}(II_{1,25})$. We use the same symbol ϵ_S to denote the induced embedding $\mathcal{P}(S) \hookrightarrow \mathcal{P}(II_{1,25})$. A Conway chamber $\mathcal{D}(\mathbf{w})$ is said to be *non-degenerate* with respect to ϵ_S if $\epsilon_S^{-1}(\mathcal{D}(\mathbf{w}))$ contains a non-empty open subset of $\mathcal{P}(S)$, and when this is the case, we say that the closed subset $\epsilon_S^{-1}(\mathcal{D}(\mathbf{w}))$ of $\mathcal{P}(S)$ is an *induced chamber* or a *chamber induced by* $\mathbf{w}$. The tessellation of $\mathcal{P}(II_{1,25})$ by Conway chambers induces a tessellation of $\mathcal{P}(S)$ by induced chambers. It is obvious by definition that every standard fundamental domain of the action of $W(S)$ on $\mathcal{P}(S)$ is also tessellated by induced chambers.

Remark 3.3. In general, the induced chambers are not congruent to each other under the action of $\mathrm{O}(S)'$. See [21] for examples.

Let G be a subgroup of $\mathrm{O}(S)'$ such that every isometry $g \in G$ lifts to an isometry of $II_{1,25}$. Then the tessellation of $\mathcal{P}(S)$ by induced chambers is preserved by the action of G on $\mathcal{P}(S)$. We consider the case where G is

$$\mathrm{O}(S)^{\omega} := \rho^{-1}(\{\pm \mathrm{id}_{S^\vee/S}\}), \tag{3.1}$$

where

$$\rho : \mathrm{O}(S)' \to \mathrm{O}(S^\vee/S, q_S) \tag{3.2}$$

is defined by the natural action of $\mathrm{O}(S)'$ on $S^\vee/S$. This group will be important for our geometric application.

Lemma 3.4. *Every isometry of* $\mathrm{O}(S)^{\omega}$ *lifts to an isometry of* $II_{1,25}$.

Proof. Let R be the orthogonal complement of S in $II_{1,25}$. By Nikulin [15, Proposition 1.5.1], the even unimodular overlattice $II_{1,25}$ of the orthogonal direct sum $S \oplus R$ induces an anti-isomorphism $q_S \cong -q_R$ of discriminant forms. For $g \in \mathrm{O}(S)^{\omega}$, there exists an isometry g_R of R such that the action of g on $S^\vee/S$ is equal to the action of g_R on $R^\vee/R$ via this isomorphism $q_S \cong -q_R$. (We can take id_R or $-\mathrm{id}_R$ for g_R.) Then the action of $g \oplus g_R$ on $S \oplus R$ preserves the overlattice $II_{1,25}$, and hence induces an isometry of $II_{1,25}$ whose restriction to S is equal to g. □

For an induced chamber $D := \epsilon_S^{-1}(\mathcal{D}(\mathbf{w}))$, we put

$$\mathbf{w}_S := \mathrm{pr}_S(\mathbf{w}), \tag{3.3}$$

where $\mathrm{pr}_S \colon II_{1,25} \to S^\vee$ is the natural projection. For $v \in S \otimes \mathbb{Q}$ with $\langle v, v \rangle < 0$, let $(v)^\perp$ denote the real hyperplane of $\mathcal{P}(S)$ defined by $\langle x, v \rangle = 0$. We say that $D \cap (v)^\perp$ is a *wall* of D if $(v)^\perp$ is disjoint from the interior of D and $D \cap (v)^\perp$ contains a non-empty open subset of $(v)^\perp$. We say that a vector $v' \in S \otimes \mathbb{Q}$ *defines* a wall $D \cap (v)^\perp$ of D if $(v)^\perp = (v')^\perp$ and $\langle v', x \rangle \geq 0$ holds for all $x \in D$. We have a unique defining vector $v' \in S^\vee$ of a wall $D \cap (v)^\perp$ such that v' is primitive in $S^\vee$, which we call the *primitive defining vector* of the wall $D \cap (v)^\perp$. The values

$$n := \langle v', v' \rangle, \quad a := \langle v', \mathbf{w}_S \rangle, \tag{3.4}$$

where v' is the primitive defining vector of a wall $D \cap (v)^\perp$, are important numerical invariants of the wall.

Henceforth, we assume the following:

Condition 1: The orthogonal complement R of S in $II_{1,25}$ can not be embedded into the negative-definite Leech lattice Λ. (This condition is satisfied, for example, when R contains a (-2)-vector.)

It is proved in [21] that, under Condition 1, each induced chamber $D = \epsilon_S^{-1}(\mathcal{D}(\mathbf{w}))$ has only a finite number of walls, and the primitive defining vectors of these walls can be calculated from the Weyl vector $\mathbf{w}$ inducing D. Let D and D' be induced chambers. We put

$$\mathrm{Isom}(D, D') := \{\, g \in \mathrm{O}(S) \mid g \text{ maps } D \text{ to } D' \,\}, \quad \mathrm{O}(S, D) := \mathrm{Isom}(D, D).$$

Since we can calculate the set of walls of each induced chamber, we can calculate all elements of $\mathrm{Isom}(D, D')$ and $\mathrm{O}(S, D)$. For each wall $D \cap (v)^\perp$, there exists a unique induced chamber D' such that $D \neq D'$ and $D' \cap (v)^\perp = D \cap (v)^\perp$. This induced chamber D' is said to be *adjacent to D across the wall $D \cap (v)^\perp$*. A Weyl vector $\mathbf{w}' \in II_{1,25}$ that induces D' can be calculated from the Weyl vector $\mathbf{w}$ inducing D and the primitive defining vector $v \in S^\vee$ of the wall $D \cap (v)^\perp$. See [21] for the detail of these computations.

3.2. **Application to a $K3$ surface.** We apply the above procedure to the case where S is the Picard lattice S_Y of a $K3$ surface Y. Let $\mathcal{P}_Y$ be the positive cone containing an ample class, and suppose that the primitive embedding $S_Y \hookrightarrow II_{1,25}$ maps $\mathcal{P}_Y$ into $\mathcal{P}(II_{1,25})$. Then $\mathcal{P}_Y$ is tessellated by induced chambers, and the nef-and-big cone $\mathcal{N}_Y$ is also tessellated by induced chambers.

For simplicity, we assume the following:

Condition 2: The only isometries of the transcendental lattice T_Y of Y that preserve the subspace $H^{2,0}(Y)$ of $T_Y \otimes \mathbb{C}$ are ± 1, and the discriminant group $T_Y^\vee / T_Y$ is not 2-elementary.

By Torelli's theorem for algebraic $K3$ surfaces [18], the action of $\mathrm{Aut}(Y)$ on $H^2(S, \mathbb{Z})$ is faithful, and an isometry $g \in \mathrm{O}(S_Y)'$ is contained in the image of the natural homomorphism

$$\varphi_Y \colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y)'$$

if and only if g preserves $\mathcal{N}_Y$, and g extends to an isometry of the even unimodular overlattice $H^2(S, \mathbb{Z})$ of $S_Y \oplus T_Y$ that preserves $H^{2,0}(Y)$. Suppose that $g \in \mathrm{Aut}(Y)$ is in the kernel of φ_Y. Then g acts on $S_Y^\vee / S_Y$ as 1, and hence acts on $T_Y^\vee / T_Y$ as 1. On the other hand, since g preserves $H^{2,0}(Y)$, the action of g on T_Y is ± 1. The condition that $T_Y^\vee / T_Y$ is not 2-elementary implies that $-1 \neq 1$ in $\mathrm{O}(T_Y^\vee / T_Y)$. Hence $g = 1$. Thus we obtain:

Proposition 3.5. *Suppose that Condition 2 holds. Then the natural homomorphism φ_Y injective. Moreover, an isometry $g \in \mathrm{O}(S_Y)'$ is contained in the image of φ_Y if and only if g belongs to $\mathrm{O}(S)^\omega$ and preserves $\mathcal{N}_Y$, where $\mathrm{O}(S)^\omega$ is defined by* (3.1). □

From now on, we regard $\mathrm{Aut}(Y)$ as a subgroup of $\mathrm{O}(S)^\omega \subset \mathrm{O}(S_Y)'$ by φ_Y.

Let $\mathrm{C}\mathcal{N}_Y$ denote the set of induced chambers contained in $\mathcal{N}_Y$. Let D be an element of $\mathrm{C}\mathcal{N}_Y$. A wall $D \cap (v)^\perp$ of D is said to be *outer* if the hyperplane $(v)^\perp$ of $\mathcal{P}_Y$ is disjoint from the interior of $\mathcal{N}_Y$, and to be *inner* otherwise. By definition, a wall $D \cap (v)^\perp$ is inner if and only if the induced chamber D' adjacent to D across $D \cap (v)^\perp$ is contained in $\mathcal{N}_Y$. By definition again, a wall $D \cap (v)^\perp$ is outer if and only if v is a multiple of a (-2)-vector of S_Y, which is the class of a smooth rational curve on Y. We denote by $\mathrm{Inn}(D)$ and $\mathrm{Out}(D)$ the set of inner walls and of outer walls of D, respectively.

We further assume the following:

Condition 3: We have an induced chamber $D_0 \in \mathrm{C}\mathcal{N}_Y$ such that, for every inner wall $w = D_0 \cap (v)^\perp$ of D_0, there exists an isometry g_w in $\mathrm{O}(S)^\omega$ that maps D_0 to the induced chamber adjacent to D_0 across the wall w.

Under Condition 3, induced chambers are congruent to each other. Note that, since w is inner, the isometry $g_w \in \mathrm{O}(S)^\omega$ in Condition 3 preserves $\mathcal{N}_Y$, and hence $g_w \in \mathrm{Aut}(Y)$. We have the following theorem, which is a special case of a more general result given in [21].

Theorem 3.6. *Suppose that Conditions 1-3 hold. Then the subgroup* $\mathrm{Aut}(Y)$ *of* $\mathrm{O}(S_Y)^\omega$ *is generated by the finite subgroup*

$$\mathrm{Aut}(Y, D_0) := \mathrm{O}(S_Y, D_0) \cap \mathrm{O}(S)^\omega$$

and the isometries g_w*, where* w *runs through* $\mathrm{Inn}(D_0)$*.* □

3.3. **Defining relations.** We continue to assume that Conditions 1-3 hold. By the classical theory of Poincaré relations [27], we calculate the defining relations of $\mathrm{Aut}(Y)$ with respect to the generating set given in Theorem 3.6 assuming the following:

Condition 4: The group $\mathrm{Aut}(Y, D_0)$ is trivial.

By Condition 4, the group $\mathrm{Aut}(Y)$ acts on the set $\mathrm{C}\mathcal{N}_Y$ of induced chambers in $\mathcal{N}_Y$ simply-transitively.

Definition 3.7. For $D \in \mathrm{C}\mathcal{N}_Y$, let $\tau_D \in \mathrm{Aut}(Y)$ denote the unique isometry such that

$$D = {D_0}^{\tau_D}.$$

Let w be an inner wall of D_0, and $D \in C\mathcal{N}_Y$ the adjacent chamber across w. We denote the isometry τ_D by g_w.

By Theorem 3.6, the group $\mathrm{Aut}(Y)$ is generated by

$$\mathrm{Gen} := \{\, g_w \mid w \in \mathrm{Inn}(D_0)\}.$$

Let $\langle\mathtt{Gen}\rangle$ denote the group freely generated by the alphabet

$$\mathtt{Gen} := \{\, \mathtt{g}_w \mid w \in \mathrm{Inn}(D_0)\}$$

equipped with a bijection $\mathtt{g}_w \mapsto g_w$ $(w \in \mathrm{Inn}(D_0))$ with Gen. Then the mapping $\mathtt{g}_w \mapsto g_w$ induces a surjective homomorphism

$$\psi \colon \langle\mathtt{Gen}\rangle \to \mathrm{Aut}(Y).$$

For a subset $\mathcal{R}$ of $\langle\mathtt{Gen}\rangle$, we denote by $\langle\langle\mathcal{R}\rangle\rangle$ the minimal normal subgroup of $\langle\mathtt{Gen}\rangle$ containing $\mathcal{R}$. Our goal is to find a subset $\mathcal{R}$ of $\langle\mathtt{Gen}\rangle$ such that that $\mathrm{Ker}\,\psi = \langle\langle\mathcal{R}\rangle\rangle$. In the following, an element of the free group $\langle\mathtt{Gen}\rangle$ is written as a sequence

$$(\mathtt{g}_{w_1}^{\pm 1}, \cdots, \mathtt{g}_{w_m}^{\pm 1})$$

of letters $\mathtt{g}_w^{\pm 1}$.

Suppose that two induced chambers $D, D' \in \mathrm{C}\mathcal{N}_Y$ are adjacent, and let $w_{D,D'} = w_{D',D}$ be the wall between them. Then

$$w_{D' \leftarrow D} := (w_{D',D})^{\tau_D^{-1}}$$

is an inner wall of D_0 such that $D'_0 := (D')^{\tau_D^{-1}}$ is the induced chamber adjacent to D_0 across $w_{D' \leftarrow D}$. Therefore, putting

$$g_{D' \leftarrow D} := g_{w_{D' \leftarrow D}} \in \text{Gen},$$

we have $D'_0 = D_0{}^{g_{D' \leftarrow D}}$ and we obtain

$$\tau_{D'} = g_{D' \leftarrow D} \cdot \tau_D.$$

In the same way, we have $\tau_D = g_{D \leftarrow D'} \cdot \tau_{D'}$, and hence

$$g_{D' \leftarrow D} \cdot g_{D \leftarrow D'} = 1, \tag{3.5}$$

from which we deduce that

(3.6) the generating set Gen is invariant under $g \mapsto g^{-1}$.

We put

$$\mathcal{R}_1 := \{ (\mathtt{g}_w, \mathtt{g}_{w'}) \mid g_w g_{w'} = 1 \}. \tag{3.7}$$

We obviously have $\langle\langle \mathcal{R}_1 \rangle\rangle \subset \operatorname{Ker} \psi$.

A *chamber path in* $\mathrm{C}\mathcal{N}_Y$ is a sequence

$$\mathbb{D} = (D^{(1)}, \dots, D^{(m)})$$

of induced chambers contained in $\mathcal{N}_Y$ such that $D^{(i-1)}$ and $D^{(i)}$ are adjacent for $i = 2, \dots, m$. (For a chamber path $\mathbb{D}$, we have $D^{(i-1)} \neq D^{(i)}$ for $i = 2, \dots, m$, but in general, we may have $D^{(i)} = D^{(j)}$ for some i, j with $|i - j| > 1$.) When $D^{(1)} = D^{(m)}$, the chamber path $\mathbb{D}$ is called a *chamber loop in* $\mathrm{C}\mathcal{N}_Y$. For a chamber loop

$$\mathbb{D} = (D_0, \dots, D_m)$$

in $\mathrm{C}\mathcal{N}_Y$ starting with the fixed induced chamber $D_0 = D_m$, we define a relation $R(\mathbb{D}) \in \operatorname{Ker} \psi$ associated with $\mathbb{D}$ as follows. We put

$$g_i := g_{D_i \leftarrow D_{i-1}} \in \text{Gen}$$

for $i = 1, \dots, m$, and let $\mathtt{g}_i \in \mathtt{Gen}$ be the letter corresponding to g_i. Then we have

$$\tau_{D_i} = g_i \cdots g_1$$

for $i = 1, \dots, m$. Since $D_0 = D_m$, we have $g_m \cdots g_1 = 1$ and the sequence

$$R(\mathbb{D}) := (\mathtt{g}_1^{-1}, \dots, \mathtt{g}_m^{-1})$$

belongs to the kernel of $\operatorname{Ker} \psi$.

Let D be an element of $\mathrm{C}\mathcal{N}_Y$. A non-empty closed subset f of D is said to be a *face* of D if f is an intersection of walls of D. The *dimension* $\dim f$ of a face f of D is the dimension of the minimal linear subspace of $S_Y \otimes \mathbb{R}$ containing f. The *codimension* of f is defined to be $\dim(S_Y \otimes \mathbb{R}) - \dim f$. The faces of D with codimension 1 are exactly the walls of D. If f is a face of D with codimension 2, then there exist exactly two walls of D containing f, and f is equal to the intersection of these two walls. A face of D is said to be *inner* if a general point of f belongs to the interior of $\mathcal{N}_Y$.

Remark 3.8. Suppose that a face f of D with codimension 2 is written as $f = w \cap w'$, where w and w' are walls of D. Even if w and w' are inner, the face f may fail to be inner. See Figure 3.1, in which the black dot is a face $f = w \cap w'$ of D with w and w' being inner, and the thick line is bounding $\mathcal{N}_Y$.

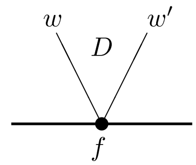

FIGURE 3.1. f is not an inner face

Let f be an inner face of $D \in \mathrm{C}\mathcal{N}_Y$ with codimension 2. A *simple chamber loop around* (f, D) is a chamber loop $(D^{(0)}, \dots, D^{(m)})$ in $\mathrm{C}\mathcal{N}_Y$ with $D^{(0)} = D^{(m)} = D$ such that each $D^{(i)}$ contains f as a face and that $D^{(i)} \neq D^{(j)}$ unless $i = j$ or $\{i, j\} = \{0, m\}$. Note that, for a fixed (f, D), there exist exactly two simple chamber loops around (f, D), which have opposite orientations.

From each inner face f of D_0 with codimension 2, we choose a simple chamber loop $\mathbb{D}(f, D_0)$ around (f, D_0), and make the set

$$\mathcal{R}_2 := \{\, R(\mathbb{D}(f, D_0)) \mid f \text{ is an inner face of } D_0 \text{ with codimension } 2 \,\}. \tag{3.8}$$

Its elements are called the *Poincaré relations*. By [27, Chapter 2, Section 1.3, Theorem 1.3], we have the following:

Theorem 3.9. *Suppose that Conditions 1-4 hold. Then the kernel of the surjective homomorphism* $\psi\colon \langle \mathtt{Gen} \rangle \to \mathrm{Aut}(Y)$ *is equal to* $\langle\langle \mathcal{R}_1 \cup \mathcal{R}_2 \rangle\rangle$. □

3.4. **Making the list of all faces of D_0 with codimension 2.** Let $w = D_0 \cap (v)^\perp$ be a wall of D_0. Then the list of faces of D_0 with codimension 2 contained in w can be calculated as follows. Let $\langle w \rangle_{\mathbb{Q}}$ be the minimal subspace of $S_Y \otimes \mathbb{Q}$ containing w. Since the intersection form of $S_Y \otimes \mathbb{Q}$ restricted to $\langle w \rangle_{\mathbb{Q}}$ is non-degenerate, we have the orthogonal projection

$$\mathrm{pr}_w\colon S_Y \otimes \mathbb{Q} \to \langle w \rangle_{\mathbb{Q}}.$$

Let $\{D_0 \cap (v_k)^\perp\}$ be the list of walls of D_0 with v_k defining the wall $D_0 \cap (v_k)^\perp$, i.e., $\langle x, v_k \rangle \geq 0$ for all $x \in D_0$. We construct a set $\mathcal{U}_w$ of vectors of $\langle w \rangle_{\mathbb{Q}}$ with the following properties:

(a) If $\mathrm{pr}_w(v_k) \in \langle w \rangle_{\mathbb{Q}}$ has a negative square-norm, then $\mathrm{pr}_w(v_k)$ is a positive-rational multiple of an element $u \in \mathcal{U}_w$.
(b) Every element u of $\mathcal{U}_w$ is a positive-rational multiple of $\mathrm{pr}_w(v_k)$ for some v_k with $\langle \mathrm{pr}_w(v_k), \mathrm{pr}_w(v_k) \rangle < 0$.
(c) No two vectors of $\mathcal{U}_w$ are linearly dependent.

Then w is defined in $\langle w \rangle_{\mathbb{Q}} \otimes \mathbb{R}$ by

$$w = \{\, x \in \langle w \rangle_{\mathbb{Q}} \otimes \mathbb{R} \mid \langle x, u \rangle \geq 0 \ \text{ for all } \ u \in \mathcal{U}_w \,\}.$$

Let $w' = D_0 \cap (v')^\perp$ be another wall of D_0 defined by v'. Then $w \cap w'$ is a face of D_0 with codimension 2 if and only if the following hold:

- The square-norm of $\mathrm{pr}_w(v')$ is < 0.
- Let u' be the unique vector of $\mathcal{U}_w$ that is a rational multiple of $\mathrm{pr}_w(v')$. Then the problem

  "minimize $\langle u', x \rangle$ under constraints $\langle u'', x \rangle \geq 0$ for all $u'' \in \mathcal{U}_w \setminus \{u'\}$"

  of linear programming on $\langle w \rangle_{\mathbb{Q}}$ is unbounded to $-\infty$.

Remark 3.10. This algorithm is easily generalized to an algorithm of making the list of faces of higher codimensions. See [24] for an application of this algorithm to the classification of Enriques involutions on a $K3$ surface.

3.5. **An algorithm to calculate the relation.** Let f be a face of D_0 with codimension 2. We present an algorithm to determine whether f is inner or not, and when f is inner, to calculate one of the two simple chamber loop $(D_0, \dots, D_m)$ around f from D_0 to $D_m = D_0$ and the relation $g_m \cdots g_1 = 1$ obtained from $(D_0, \dots, D_m)$, where $(g_m, \dots, g_1)$ are the sequence of elements of Gen such that $g_i \cdots g_1 = \tau_{D_i}$ for $i = 1, \dots, m$.

We put $i = 0$ and $\tau_0 = 1$. In the calculation below, we have that D_i is an induced chamber containing f as a face, and $\tau_i \in \mathrm{Aut}(Y)$ the unique isometry that maps D_0 to D_i.

(i) Let f_i be the image of f by τ_i^{-1}. Since f is a face of D_i, we see that f_i is a face of D_0.
(ii) Find the two walls w_i' and w_i'' of D_0 such that $f_i = w_i' \cap w_i''$.
(iii) If w_i' or w_i'' is outer, then f is outer, and we quit.
(iv) Suppose that w_i' and w_i'' are inner.
  (a) When $i = 0$, we put $g_1 := g_{w_0'}$, $D_1 := {D_0}^{g_1}$, $\tau_1 := g_1$. (If we interchange w_i' and w_i'', we obtain the opposite simple chamber loop around (f, D_0).)
  (b) Suppose that $i > 0$. We put $D_{i+1}' := {D_0}^{g_{w_i'} g_i \cdots g_1}$ and $D_{i+1}'' := {D_0}^{g_{w_i''} g_i \cdots g_1}$. Then either $D_{i+1}' = D_{i-1}$ or $D_{i+1}'' = D_{i-1}$ holds. In the former case, we put
$$g_{i+1} := g_{w_i''}, \quad D_{i+1} := D_{i+1}'',$$
  and in the latter case, we put
$$g_{i+1} := g_{w_i'}, \quad D_{i+1} := D_{i+1}'.$$
  We then put $\tau_{i+1} := g_{i+1} \tau_i$.
(v) If $\tau_{i+1} = 1$, then we stop and return $(D_0, \dots, D_{i+1})$. If $\tau_{i+1} \neq 1$, we increment i and repeat the process from (i) again.

4. The birational automorphism group of a 16-nodal quartic surface

From now on, let X_n be a n-nodal quartic surface satisfying the assumptions (i)-(iii) at the beginning of Introduction. Let S_n be the Picard lattice of the minimal resolution Y_n of X_n, and let $\mathcal{P}_n$ be the positive cone of S_n containing an ample class. Let $\mathcal{N}_n = \overline{\mathcal{N}}_n \cap \mathcal{P}_n$ be the intersection of the nef cone of Y_n with $\mathcal{P}_n$, and let $h_4 \in S_n$ denote the class of a plane section of $X_n \subset \mathbf{P}^3$. The specialization of X_n to X_{16} gives an embedding of lattices $S_n \hookrightarrow S_{16}$ that maps $h_4 \in S_n$ to $h_4 \in S_{16}$.

The minimal resolution Y_{16} of a general Kummer quartic surface X_{16} is the Kummer surface associated with the Jacobian variety $\mathrm{Jac}(C_0)$ of a general genus 2 curve C_0. A finite generating set of the automorphism group of Y_{16} was calculated by Kondo [14] by Borcherds' method, and Ohashi [17] supplemented this result with another set of generators. We review their results briefly.

Let $P_1, \dots, P_6$ be the Weierstrass points of C_0. We have a quotient morphism $\mathrm{Jac}(C_0) \to X_{16}$ by the action of $\{\pm 1\}$ on $\mathrm{Jac}(C_0)$. The 16 nodes of X_{16} correspond to the points of
$$\mathrm{Jac}(C_0)_2 := \{\, x \in \mathrm{Jac}(C_0) \mid 2x = 0 \,\} = \{[0]\} \cup \{\, [P_i - P_j] \mid 1 \le i < j \le 6 \,\}.$$

Let N_0 and N_{ij} be the smooth rational curves on Y_{16} corresponding to the points $[0]$ and $[P_i - P_j]$ of $\mathrm{Jac}(C_0)_2$, respectively, which we call *nodal curves* on Y_{16}. Let θ be a theta characteristic of the curve C_0, that is, θ is a divisor class of C_0 of degree 1 such that 2θ is linearly equivalent to the canonical divisor of C_0. Then the image of the embedding $x \mapsto [x - \theta]$ of C_0 into $\mathrm{Jac}(C_0)$ yields a *trope-conic* of X_{16}. Let T_i and T_{ij} be the smooth rational curves on Y_{16} corresponding to the trope-conics obtained from the theta characteristic $[P_i]$ $(1 \le i \le 6)$ and $[P_i + P_j - P_6]$ $(1 \le i < j \le 5)$, respectively.

It is known ([14, Lemma 3.1]) that the classes of these $16+16$ curves N_0, N_{ij}, T_j, T_{ij} generate the Picard lattice S_{16} of Y_{16}. The discriminant group $S_{16}^{\vee}/S_{16}$ of S_{16} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4\mathbb{Z})$. By [13, Theorem 4.1], Condition 2 in Section 3.2 and Condition (iii) from Introduction is satisfied.

It is also well-known that X_{16} is self-dual, that is, the dual surface $X'_{16} \subset \mathbf{P}^3$ is isomorphic to X_{16}. The minimal resolution $Y_{16} \to X'_{16}$ contracts the 16 curves T_i, T_{ij} to the nodes, and maps the other 16 curves N_0, N_{ij} to conics. The Gauss map $X_{16} \dashrightarrow X'_{16}$ induces an involution of Y_{16} that interchanges the 16 curves N_0, N_{ij} and the 16 curves T_i, T_{ij}. This involution is called a *switch*. Let $h'_4 \in S_{16}$ be the class of a plane section of the dual quartic surface $X'_{16} \subset \mathbf{P}^3$. Then we have

$$h'_4 = 3h_4 - N_0 - \sum N_{ij}, \quad h_4 = 3h'_4 - \sum T_i - \sum T_{ij}. \tag{4.1}$$

Kondo [14] embedded S_{16} into $II_{1,25}$ primitively. Recall that, for each octad $K \in \mathcal{G}(8)$, we have a vector $2\nu_K$ of the Leech lattice Λ and a Leech root $r_0(2\nu_K)$ of $II_{1,25} = U \oplus \Lambda$ with respect to the Weyl vector $\mathbf{w}_0 = (1,0,0) \in II_{1,25}$. (See Example 3.2.) With each smooth rational curve E_k in the $16+16$ curves N_0, N_{ij}, T_i, T_{ij}, an octad K_k is associated as in Table 4.1, and hence a Leech root $r_k := r_0(2\nu_{K_k})$ is also associated. Then the intersection number of E_k and $E_{k'}$ on Y_{16} is equal to the intersection number of r_k and $r_{k'}$ in $II_{1,25}$ for all $k, k' = 1, \dots, 32$, and thus we obtain an embedding

$$\epsilon_{16} \colon S_{16} \hookrightarrow II_{1,25},$$

which turns out to be primitive. This primitive embedding ϵ_{16} has the following properties.

(1) Let R_{16} denote the orthogonal complement of S_{16} in $II_{1,25}$. Then R_{16} is negative-definite of rank 9 and contains (-2)-vectors that form the Dynkin diagram of type $6A_1 + A_3$. In particular, Condition 1 is satisfied.

(2) The Conway chamber associated with the Weyl Vector $\mathbf{w}_0 = (1,0,0)$ is nondegenerate with respect to ϵ_{16}, and the induced chamber $D_{16} := \epsilon_{16}^{-1}(\mathcal{D}(\mathbf{w}_0))$ is contained in the nef-and-big cone $\mathcal{N}_{16}$. The vector

$$\alpha_{16} := \mathrm{pr}_S(\mathbf{w}_0) \in S_{16}^{\vee}$$

is in fact a vector of S_{16}, sits in the interior of D_{16}, and is a very ample class of degree 8 that embeds Y_{16} into $\mathbf{P}^5$ as a $(2,2,2)$-complete intersection

$$\begin{aligned} x_1^2 + \quad x_2^2 + \quad x_3^2 + \quad x_4^2 + \quad x_5^2 + \quad x_6^2 &= 0, \\ \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 x_5^2 + \lambda_6 x_6^2 &= 0, \\ \lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \lambda_3^2 x_3^2 + \lambda_4^2 x_4^2 + \lambda_5^2 x_5^2 + \lambda_6^2 x_6^2 &= 0, \end{aligned} \tag{4.2}$$

where $\lambda_1, \dots, \lambda_6$ are complex numbers such that the genus 2 curve C_0 is defined by

$$w^2 = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_6).$$

N_0	:	$\{\infty, 0, 1, 7, 12, 13, 14, 20\}$,	N_{12}	:	$\{\infty, 0, 1, 13, 15, 17, 18, 19\}$,
N_{13}	:	$\{\infty, 0, 1, 6, 11, 14, 15, 16\}$,	N_{14}	:	$\{\infty, 0, 1, 5, 10, 12, 16, 19\}$,
N_{15}	:	$\{\infty, 0, 1, 7, 10, 11, 17, 22\}$,	N_{16}	:	$\{\infty, 0, 1, 5, 6, 18, 20, 22\}$,
N_{23}	:	$\{\infty, 0, 1, 8, 16, 17, 20, 21\}$,	N_{24}	:	$\{\infty, 0, 1, 3, 7, 9, 16, 18\}$,
N_{25}	:	$\{\infty, 0, 1, 8, 9, 14, 19, 22\}$,	N_{26}	:	$\{\infty, 0, 1, 3, 12, 15, 21, 22\}$,
N_{34}	:	$\{\infty, 0, 1, 5, 9, 11, 13, 21\}$,	N_{35}	:	$\{\infty, 0, 1, 4, 6, 9, 12, 17\}$,
N_{36}	:	$\{\infty, 0, 1, 4, 5, 7, 8, 15\}$,	N_{45}	:	$\{\infty, 0, 1, 3, 4, 11, 19, 20\}$,
N_{46}	:	$\{\infty, 0, 1, 4, 10, 14, 18, 21\}$,	N_{56}	:	$\{\infty, 0, 1, 3, 6, 8, 10, 13\}$.
T_1	:	$\{\infty, 0, 2, 3, 4, 8, 9, 21\}$,	T_2	:	$\{\infty, 0, 2, 4, 5, 6, 10, 11\}$,
T_3	:	$\{\infty, 0, 2, 3, 10, 18, 19, 22\}$,	T_4	:	$\{\infty, 0, 2, 6, 8, 15, 17, 22\}$,
T_5	:	$\{\infty, 0, 2, 5, 15, 16, 18, 21\}$,	T_6	:	$\{\infty, 0, 2, 9, 11, 16, 17, 19\}$,
T_{12}	:	$\{\infty, 0, 2, 7, 8, 10, 14, 16\}$,	T_{13}	:	$\{\infty, 0, 2, 10, 12, 13, 17, 21\}$,
T_{14}	:	$\{\infty, 0, 2, 3, 7, 11, 13, 15\}$,	T_{15}	:	$\{\infty, 0, 2, 4, 12, 14, 15, 19\}$,
T_{23}	:	$\{\infty, 0, 2, 6, 9, 13, 14, 18\}$,	T_{24}	:	$\{\infty, 0, 2, 5, 8, 13, 19, 20\}$,
T_{25}	:	$\{\infty, 0, 2, 4, 7, 17, 18, 20\}$,	T_{34}	:	$\{\infty, 0, 2, 3, 6, 12, 16, 20\}$,
T_{35}	:	$\{\infty, 0, 2, 11, 14, 20, 21, 22\}$,	T_{45}	:	$\{\infty, 0, 2, 5, 7, 9, 12, 22\}$.

TABLE 4.1. $16 + 16$ Octads

(See Baker [1, Chapter 7], Hudson [9, §31] and modern expositions in Shioda [25] or Dolgachev [8, 10.3.3]). The $16 + 16$ curves N_0, N_{ij}, T_i, T_{ij} are mapped to the lines of this $(2, 2, 2)$-complete intersection. The group $\mathrm{O}(S_{16}, D_{16})$ is equal to the stabilizer subgroup of α_{16} in $\mathrm{O}(S_{16})'$ and, by [14, Lemma 4.5], we have

$$\mathrm{O}(S_{16}, D_{16}) \cong (\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_6. \tag{4.3}$$

The group $\mathrm{O}(S_{16}, D_{16}) \cap \mathrm{O}(S_{16})^\omega$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$, where $\mathrm{O}(S_{16})^\omega$ is defined by (3.1). Note that $\mathrm{Aut}(Y_{16}, D_{16}) = \mathrm{O}(S_{16}, D_{16}) \cap \mathrm{O}(S_{16})^\omega$ is equal to the projective automorphism group $\mathrm{Aut}(Y_{16}, \alpha_{16}) \cong (\mathbb{Z}/2\mathbb{Z})^5$ of the $(2, 2, 2)$-complete intersection $Y_{16} \subset \mathbf{P}^5$ that consists of switching signs of the coordinates. Note that the six involutions corresponding to the switch of the sign at one of the coordinates are the involutions arising from one of six realizations of the Kummer surface as the focal surface of a congruence of lines of bidegree $(2, 2)$. One of them acts on N_0, N_{ij}, T_i, T_{ij} as follows:

$$N_0 \leftrightarrow T_6, \quad N_{i6} \leftrightarrow T_i, \quad N_{ij} \leftrightarrow T_{ij} \quad (1 \le i < j < 6).$$

(3) The walls of D_{16} are as in Table 4.2. The 32 outer walls are defined by the classes of the curves N_0, N_{ij}, T_i, T_{ij}. The action of $\mathrm{O}(S_{16}, D_{16})$ decomposes the set of walls into four orbits, and each orbit is further decomposed into smaller orbits by the action of the subgroup $\mathrm{Aut}(Y_{16}, \alpha_{16}) \cong (\mathbb{Z}/2\mathbb{Z})^5$ of $\mathrm{O}(S_{16}, D_{16})$ as indicated in the third column of Table 4.2. (For example, the large orbit No. 2 of size 60 is decomposed into 15 small orbits, each of which is of size 4.) For each wall, the values $n = \langle v, v\rangle$, $a = \langle v, \alpha_{16}\rangle$ and $d = \langle \alpha_{16}, \mathbf{w}'_S\rangle$ are also given in Table 4.2, where v is the primitive defining vector of the wall $D_{16} \cap (v)^\perp$, D' is the induced chamber adjacent to D_{16} across the wall, $\mathbf{w}'$ is a Weyl vector inducing D', and $\mathbf{w}'_S = \mathrm{pr}_S(\mathbf{w}')$ is defined in (3.3).

(4) For each inner wall w, there exists an involution g_w that maps D_{16} to the induced chamber adjacent to D_{16} across the wall w. For an inner wall in the orbit No. 2, this involution is the Hutchinson-Göpel involution, which is an Enriques involution. For an inner wall in the orbit No. 3, this involution is obtained by the projection $X_{16} \dashrightarrow \mathbf{P}^2$ from a node of X_{16} or by the projection $X'_{16} \dashrightarrow \mathbf{P}^2$ from a node

(4.4)

No.	type		size	n	a	d
1	outer	(-2)-curve	1×32	-2	1	9
2	inner	Göpel	15×4	-1	2	16
3	inner	Projection	1×32	-1	3	26
4	inner	Weber	6×32	$-3/4$	3	32

TABLE 4.2. Walls of D_{16}

of X'_{16}. For an inner wall in the orbit No. 4, this involution is the Hutchinson-Weber involution, which is again an Enriques involution. (See Hutchinson [10], [11], [12] for the Hutchinson-Göpel and the Hutchinson-Weber involutions.)

By these results and Theorem 3.6, we obtain the following:

Theorem 4.1 (Kondo [14], Ohashi [17]). *The automorphism group* $\mathrm{Aut}(Y_{16})$ *of a general Jacobian Kummer surface* Y_{16} *is generated by the projective automorphism* $\mathrm{Aut}(Y_{16}, \alpha_{16})$ *of the* $(2,2,2)$*-complete intersection model, the involutions obtained from the projections with the center being the nodes of the quartic surface model* X_{16} *or its dual* X'_{16}*, the Hutchinson-Göpel involutions, and the Hutchinson-Weber involutions.* □

Remark 4.2. Kondo [14] used Keum's automorphisms [13] as a part of the generating set of $\mathrm{Aut}(Y_{16})$. Ohashi [17] showed that Keum's automorphisms can be replaced by Hutchinson-Weber involutions.

Remark 4.3. By the method in Section 2.3.2, we calculate the sets of classes of all smooth rational curves C on Y_{16} with $\langle C, \alpha_{16}\rangle = d$ for $d = 1, \dots, 14$. The sizes of these sets are as follows:

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
size	32	0	0	0	480	0	320	0	15264	0	1920	0	120992	0 .

5. The birational automorphism group of a 15-nodal quartic surface

Let X_{15} be a 15-nodal quartic surface. It is proven in [7] that it is isomorphic to a hyperplane section of the fixed hypersurface in $\mathbf{P}^4$ isomorphic to the Castelnuovo-Richmond-Igusa quartic hypersurface CR_4 with 15 double lines. When the hyperplane specializes to a tangent hyperplane, the section acquires an additional node p_0 and becomes isomorphic to X_{16}. This proves that any X_{15} is obtained by smoothing one node of X_{16}, and hence it satisfies condition (i) from Introduction.

Fix one of the six possible realizations of X_{15} as the focal surface of a congruence of lines in $\mathbf{P}^3$ of order 2 and class 3. Each such realization comes with an involution $\sigma^{(i)}$ of Y_{15} whose quotient is a quintic del Pezzo surface D. When X_{15} specializes to X_{16}, the involution acquires the new node as its fixed point, and the quotient of Y_{16} by the lift of this involution becomes isomorphic to a quartic del Pezzo surface, the blow-up of one point on D. Following [7], we can take one of the involutions σ among the involutions $\sigma^{(i)}$ in such a way that the nodes of X_{15} are indexed by 2-elements subsets of $[1,6] = \{1, \dots, 6\}$ with nodal curves E_{ij} such that $\sigma(E_{ij})$ are proper transforms of trope-conics for $i, j \neq 6$ and $\sigma(E_{i6})$ is the proper transform of a trope-quartic curve. The embedding

$$\epsilon_{15,16} \colon S_{15} \hookrightarrow S_{16}$$

is defined by mapping

$$\begin{array}{rcl}\epsilon_{15,16}(E_{ij}) & = & N_{ij} \quad (1 \le i < j \le 6), \\ \epsilon_{15,16}(\sigma(E_{ij})) & = & T_{ij} \quad (1 \le i < j \le 5), \\ \epsilon_{15,16}(\sigma(E_{i6})) & = & T_i + N_0 + T_6, \\ \epsilon_{15,16}(h_4) & = & h_4.\end{array}$$

In view of this notation, it is natural to identify σ with $\sigma^{(6)}$. Other involutions $\sigma^{(\nu)}$ and the corresponding embedding $\epsilon_{15,16}^{(\nu)}$ are defined by applying a permutation from $\mathfrak{S}_6$ that sends 6 to ν. It was proved in [7] that the embedding $\epsilon_{15,16}$ is primitive and

$$\epsilon_{15,16}(S_{15}) = (\mathbb{Z}N_0)^{\perp}.$$

The minimal resolutions of 15-nodal quartics form an open subset of the coarse moduli space of $K3$ surfaces lattice-polarized by S_{15}. If Y_{15} is chosen generally in this moduli space, then Y_{15} satisfies Conditions (ii) and (iii) at the beginning of Introduction.

Henceforth we regard S_{15} as a primitive sublattice of S_{16} embedded by $\epsilon_{15,16}$. The discriminant group of S_{15} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5 \oplus (\mathbb{Z}/4\mathbb{Z})$, which, combined with Condition (iii), implies that Condition 2 in Section 3.2 is satisfied. The positive cone $\mathcal{P}_{15}$ of S_{15} containing h_4 is equal to the real hyperplane $(N_0)^{\perp}$ of $\mathcal{P}_{16}$:

$$\mathcal{P}_{15} = (S_{15} \otimes \mathbb{R}) \cap \mathcal{P}_{16} = (N_0)^{\perp}.$$

Under the specialization of Y_{15} to Y_{16}, a smooth rational curve on Y_{15} becomes a union of smooth rational curves on Y_{16}. Hence we have

$$\mathcal{N}_{15} \supset \mathcal{P}_{15} \cap \mathcal{N}_{16}. \tag{5.1}$$

Composing the embedding $\epsilon_{15,16} \colon S_{15} \hookrightarrow S_{16}$ with Kondo's primitive embedding $\epsilon_{16} \colon S_{16} \hookrightarrow II_{1,25}$, we obtain a primitive embedding

$$\epsilon_{15} \colon S_{15} \hookrightarrow II_{1,25}.$$

The orthogonal complement R_{15} of S_{15} in $II_{1,25}$ contains (-2)-vectors that form the Dynkin diagram of type $7A_1 + A_3$, and hence $\epsilon_{15} \colon S_{15} \hookrightarrow II_{1,25}$ satisfies Condition 1 in Section 3.1. The induced chamber

$$D_{15} := \epsilon_{15}^{-1}(\mathcal{D}(\mathbf{w}_0)) = \epsilon_{15,16}^{-1}(D_{16})$$

is equal to the outer wall $D_{16} \cap (N_0)^{\perp}$ of D_{16}, and hence is contained in the nef-and-big cone $\mathcal{N}_{15}$ by (5.1). The walls are given in Table 5.1. The natural homomorphism from $\mathrm{O}(S_{15})$ to the automorphism group of the discriminant form of S_{15} restricted to $\mathrm{O}(S_{15}, D_{15})$ is injective, and hence we have

$$\mathrm{Aut}(Y_{15}, D_{15}) := \mathrm{O}(S_{15}, D_{15}) \cap \mathrm{O}(S_{15})^{\omega} = \{1\}. \tag{5.2}$$

The vector

$$\alpha_{15} := \mathrm{pr}_S(\mathbf{w}_0) = \alpha_{16} + \frac{1}{2}N_0 \in S_{15}^{\vee} \tag{5.3}$$

satisfies $2\alpha_{15} \in S_{15}$, is of square norm $\langle \alpha_{15}, \alpha_{15} \rangle = 17/2$, sits in the interior of D_{15}, and is invariant under the action of $\mathrm{O}(S_{15}, D_{15})$. In particular, the class $2\alpha_{15}$ is an ample class of Y_{15} with square-norm 34. This ample class can be used in the geometric algorithms given in Section 2.3.

No.	type	Root lattice	size	up	n	a	d
1	outer	$A_3 \oplus A_1^{\oplus 8}$	10	1	-2	1	19/2
2	outer	$A_3 \oplus A_1^{\oplus 8}$	15	1	-2	1	19/2
3	outer	$D_4 \oplus A_3 \oplus A_1^{\oplus 4}$	15	2	$-1/2$	5/2	67/2
4	outer	$D_5 \oplus A_1^{\oplus 6}$	10	3	$-1/2$	7/2	115/2
5	inner	$A_3 \oplus A_2 \oplus A_1^{\oplus 6}$	6	1	$-3/2$	3/2	23/2
6	inner	$A_3^{\oplus 2} \oplus A_1^{\oplus 5}$	45	2	-1	2	33/2
7	inner	$D_4 \oplus A_1^{\oplus 7}$	6	3	-1	3	53/2
8	inner	$D_4 \oplus A_1^{\oplus 7}$	15	3	-1	3	53/2
9	inner	$A_5 \oplus A_1^{\oplus 6}$	120	4	$-3/4$	3	65/2
10	inner	$D_6 \oplus A_1^{\oplus 5}$	72	4	$-1/4$	7/2	213/2

TABLE 5.1. Walls of D_{15}

The group $\mathrm{O}(S_{15}, D_{15})$ is equal to the stabilizer subgroup of α_{15} in $\mathrm{O}(S_{15})'$. It is immediate to see that $\mathrm{O}(S_{15}, D_{15})$ is equal to the subgroup of $\mathrm{O}(S_{16}, D_{16})$, and that

$$\mathrm{O}(S_{15}, D_{15}) \cong \mathfrak{S}_6.$$

The group $\mathrm{O}(S_{15}, D_{15}) \cong \mathfrak{S}_6$ decomposes the walls of D_{15} as in Table 5.1. In this Table the column "up" indicates that, for example, each wall of D_{15} in the orbit No. 4 is equal to the face $(N_0)^\perp \cap (v)^\perp$ of D_{16} with codimension 2, where $D_{16} \cap (v)^\perp$ is a wall of D_{16} contained in the orbit No. 3 in Table 4.2. The values $n = \langle v, v\rangle$, $a = \langle v, \alpha_{15}\rangle$ and $d = \langle \alpha_{15}, \mathbf{w}'_S\rangle$ are also given in Table 5.1, where v is the primitive defining vector of the wall $D_{15} \cap (v)^\perp$, D' is the induced chamber adjacent to D_{15} across the wall, $\mathbf{w}'$ is a Weyl vector inducing D', and $\mathbf{w}'_S$ is defined by (3.3). The third column gives a root sublattice of $II_{1,25}$ whose orthogonal complement defines the corresponding wall.

In the following, let O_i denote the orbit of walls of D_{15} under the action of $\mathrm{O}(S_{15}, D_{15}) \cong \mathfrak{S}_6$ given in the ith row of Table 5.1.

5.1. **Outer walls of D_{15}.** The outer walls of D_{15} are as follows.

(i) The 10 outer walls in O_1 are defined by the classes of the strict transforms $\sigma(E_{ij})$ of the trope-conics on X_{15}, where $1 \le i < j \le 5$.

(ii) The 15 outer walls in O_2 are defined by the classes of the nodal curves E_{ij} over the nodes of X_{15}, where $1 \le i < j \le 6$.

(iii) Each of the 15 outer walls in O_3 is defined by the class of a smooth rational quartic curve $C_{ij,kl,mn} \in |h_4 - E_{ij} - E_{kl} - E_{mn}|$ through three nodes of X_{15}. This curve remains irreducible under the specialization to Y_{16}, and becomes a smooth rational curve on Y_{16} whose degree with respect to α_{16} is 5. (See Remark 4.3.)

(iv) Each of the 10 outer walls in O_4 is defined by the class of a smooth rational octic curve $C_{ijk} \in |2h_4 - \sum_{t\ne i,j,k}(E_{it} + E_{jt} + E_{kt})|$. This curve remains irreducible under the specialization to Y_{16}, and becomes a smooth rational curve on Y_{16} whose degree with respect to α_{16} is 7. (See Remark 4.3.)

We describe these outer walls combinatorially.

Definition 5.1. For distinct elements $i_1, \dots, i_k$ of $[1, 6] = \{1, \dots, 6\}$, we write by $(i_1 \dots i_k)$ the subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, 6\}$. Recall that, following Sylvester, a *duad* is a subset (ij) of size 2 and a *syntheme* is a non-ordered triple $(ij)(kl)(mn) = \{(ij),(kl),(lm)\}$ of duads whose union is $[1,6]$. We call a *trio* a subset (ijk) of size 3 and a *double trio* a non-ordered pair $(abc)(def)$ of two complementary trios (abc) and (def).

We say that a syntheme τ is *incident* to a double trio θ if $|\delta \cap t| = 1$ holds for any duad δ in τ and any trio t in θ.

In view of these terminology, we have the following indexing of outer walls, which is compatible with the action of $\mathrm{O}(S_{15}, D_{15}) \cong \mathfrak{S}_6$ on the set of walls and on $[1,6]$.

(i) The wall in O_1 defined by the curve $\sigma(E_{ij})$ with $1 \le i < j \le 5$ is indexed by the double trio $\theta = (ij6)(klm)$ containing the trio $(ij6)$. In the following, we write $\sigma(E_\theta)$ for $\sigma(E_{ij})$. Note that we have

$$\sigma(E_\theta) \sim \frac{1}{2}(h_4 - (E_{ij} + E_{i6} + E_{j6}) - (E_{kl} + E_{lm} + E_{mk})).$$

(ii) The wall in O_2 defined by the nodal curve E_{ij} with $1 \le i < j \le 6$ is indexed by the duad $\delta = (ij)$. In the following, we write E_δ for E_{ij}.

(iii) The wall in O_3 defined by the smooth rational quartic curve $C_{ij,kl,mn}$ is indexed by the syntheme $\tau = (ij)(kl)(mn)$.

(iv) The wall in O_4 defined by the smooth octic rational curve C_{ijk} is indexed by the double trio $(ijk)(lmn)$. Note that we have $C_{ijk} = C_{lmn}$.

In the following, let r_1 be a (-2)-vector defining a wall in O_1 corresponding to a double trio $\theta(r_1) = (ijk)(lmn)$, and let r_2 be a (-2)-vector defining a wall in O_2 corresponding to a duad $\delta(r_2) = (i'j')$. Then we have

$$\langle r_1, r_2 \rangle = \begin{cases} 1 & \text{if } \delta(r_2) \text{ is a subset of one of the two trios in } \theta(r_1), \\ 0 & \text{otherwise.} \end{cases}$$

Since S_{15} is generated by the $10 + 15$ vectors r_1 and r_2, a vector $v \in S_{15} \otimes \mathbb{Q}$ is specified by $10 + 15$ numbers $\langle r_1, v \rangle$ and $\langle r_2, v \rangle$.

(iii) The wall $w_3 = D_{15} \cap (v)^\perp$ in O_3 indexed by a syntheme τ is defined by the primitive vector $v \in S_{15}^\vee$ that satisfies

$$\langle r_1, v \rangle = \begin{cases} 1 & \text{if } \theta(r_1) \text{ and } \tau \text{ are incident,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle r_2, v \rangle = \begin{cases} 1 & \text{if } \delta(r_2) \text{ is one of the three duads in } \tau, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) The wall $w_4 = D_{15} \cap (v)^\perp$ in O_4 indexed by a double trio θ is defined by the primitive vector $v \in S_{15}^\vee$ that satisfies

$$\langle r_1, v \rangle = \begin{cases} 2 & \text{if } \theta(r_1) = \theta, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle r_2, v \rangle = \begin{cases} 0 & \text{if } \delta(r_2) \text{ is a subset of one of the two trios in } \theta, \\ 1 & \text{otherwise.} \end{cases}$$

5.2. **Indexings of graphs.** For later use, we generalize the notions of duads, synthemes, trios, double trios to the *indexings* of graphs with 6 vertices.

Definition 5.2. Let Γ be a simple graph such that the set $V(\Gamma)$ of vertices is of size 6. We denote by $\mathrm{Indx}(\Gamma)$ the set of bijections from $V(\Gamma)$ to $\{1, \dots, 6\}$ modulo the natural action of the symmetry group $\mathrm{Sym}(\Gamma)$ of the graph Γ on $V(\Gamma)$.

Remark 5.3. Let $E(\Gamma)$ be the set of edges of Γ. For a fixed indexing $t \in \mathrm{Indx}(\Gamma)$, each edge $\{a, b\}$ $(a, b \in V(\Gamma))$ of Γ gives a duad $t(\{a, b\}) := \{t(a), t(b)\}$. Thus t is considered as a map on $E(\Gamma)$, and $t(E(\Gamma))$ can be regarded as a set of nodes of X_{15}, or as a set of nodal curves on Y_{15}. When t runs through the set $\mathrm{Indx}(\Gamma)$, these sets $t(E(\Gamma))$ form an $\mathfrak{S}_6$-orbit of sets of nodes of X_{15}.

Example 5.4. Consider the graphs in Figure 5.1. The set of duads is naturally identified with $\mathrm{Indx}(\Gamma_\delta)$. The set of synthemes (resp. double trios) is naturally identified with $\mathrm{Indx}(\Gamma_\tau)$ (resp. with $\mathrm{Indx}(\Gamma_\theta)$).

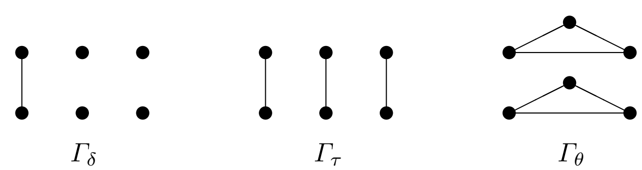

FIGURE 5.1. Graphs for duads, synthemes, and double trios

5.3. **Involutions of Y_{15}.** In order to exhibit a generating set of $\mathrm{Aut}(Y_{15})$ in Theorem 5.9, we present some involutions in $\mathrm{Aut}(Y_{15})$, and calculate their actions on S_{15}. These involutions are obtained as double-plane involutions (see Section 2.3.3). Recall that we have an interior point α_{15} of D_{15} defined by (5.3). The α_{15}-*degree* of an involution $\iota \in \mathrm{Aut}(Y_{15})$ is defined to be

$$\deg(\iota) := \langle \alpha_{15}, \alpha_{15}^{\iota} \rangle.$$

5.3.1. *The six involutions $\sigma^{(i)}$.* Recall that a quintic del Pezzo surface D is isomorphic to the blow-up of $\mathbf{P}^2$ at 4 points no three of which are collinear, and contains ten (-1)-curves that form the Petersen graph as the dual graph.

As we have already observed, D can be realized as the quotient of Y_{15} by one of six involutions $\sigma^{(1)}, \dots, \sigma^{(6)}$ coming from a realization of X_{15} as the focal surface of a congruence of bi-degree $(2, 3)$ isomorphic to D.

Let ν be an element of $\{1, \dots, 6\}$. Then there exist ten duads $\delta_1, \dots, \delta_{10}$ not containing ν. For $k = 1, \dots, 10$, let θ_k be the double trio containing the trio $\{\nu\} \cup \delta_k$. We denote by N_k the nodal curve E_{δ_k} corresponding to the duad δ_k, and by T_k the trope-conic $\sigma(E_{\theta_k})$ corresponding the double trio θ_k. Note that we have $\langle N_k, T_k \rangle = 1$. Then we have an involution $\sigma^{(\nu)}$ that interchanges N_k and T_k and maps the class h_4 of a plane section of $X_{15} \subset \mathbf{P}^3$ to

$$4h_4 - \sum_{j \neq \nu} E_{(\nu j)} - 2 \sum_{k=1}^{10} N_k.$$

(See Sections 4.1 and 4.2 of [7].) Since the classes of $10 + 10$ curves N_k and T_k together with h_4 span $S_{15} \otimes \mathbb{Q}$, the action of $\sigma^{(\nu)}$ on S_{15} is determined uniquely by these conditions. There exist five ways to blow down D to $\mathbf{P}^2$, and they correspond to the five choices of 4-tuples of disjoint (-1)-curves. The composition of these blowings down with the double cover $Y_{15} \to \mathsf{D}$ with deck transformation $\sigma^{(\nu)}$ gives five double-plane covers $Y_{15} \to \mathbf{P}^2$. The branch curve of each of these double-plane covers is a 4-cuspidal plane sextic. The ten nodal curves N_k and their corresponding trope-conics T_k are mapped to (-1)-curves on D which are tangent to the proper transform of the branch curve on D. The α_{15}-degree of the involution $\sigma^{(\nu)}$ is $23/2$.

5.3.2. *Reye involutions.* See Section 6.4 of [7] on the geometric definition of *Reye involution* τ_{Rey}. Let ν be an element of $\{1,\dots,6\}$. Then we obtain 10 nodal curves $N_1,\dots,N_{10}$ as in Section 5.3.1. We put

$$\mathfrak{r} := 2h_4 - \sum_{k=1}^{10} N_k,$$

which is a vector of square-norm -4. The reflection

$$s_{\mathfrak{r}} \colon v \mapsto v + \frac{\langle v, \mathfrak{r}\rangle}{2}\mathfrak{r}$$

is in fact an isometry of S_{15}, and gives the action on S_{15} of the Reye involution $\tau_{\mathrm{Rey}}^{(\nu)}$ indexed by ν.

5.3.3. *Double-plane involution associated with a multi-set of nodes.* We consider a class of the form

$$h_2 := mh_4 - \sum_{\delta} a_\delta E_\delta,$$

where δ runs through the set of duads, m is a positive integer, and a_δ are non-negative integers such that

$$\langle h_2, h_2\rangle = 4m^2 - 2\sum_{\delta} a_\delta^2 = 2.$$

Then $|h_2|$ is the linear system cut out on X_{15} by surfaces of degree m passing through each node p_δ corresponding to the nodal curve E_δ with multiplicity a_δ. We determine whether h_2 is nef or not, and if h_2 is nef, determine whether $|h_2|$ is fixed-component free or not, and if $|h_2|$ is fixed-component free, calculate the matrix representation of the double-plane involution associated with the rational double covering $Y_{15} \to \mathbf{P}^2$ induced by $|h_2|$.

Example 5.5. The class $h_4 - E_\delta$ of degree 2 gives the involution obtained from the projection $X_{15} \dashrightarrow \mathbf{P}^2$ with the center being the node p_δ. The ADE-type of the singularities of the branch curve is $14A_1$. The α_{15}-degree of this involution is $53/2$.

Example 5.6. Consider the graph Γ_7 with 6 vertices in Figure 5.2. The set $\mathrm{Indx}(\Gamma_7)$ is of size 360. As was explained in Remark 5.3, this graph Γ_7 defines an $\mathfrak{S}_6$-orbit of sets of 7 nodal curves. Let $\{N_1,\dots,N_7\}$ be an element of this orbit. Then the class

$$h_2 := 2h_4 - (N_1 + \cdots + N_7)$$

of degree 2 is nef, and $|h_2|$ defines a rational double covering $Y_{15} \to \mathbf{P}^2$, the branch curve of which has singularities of type $2A_1 + 3A_3$. The α_{15}-degree of the associated involution is $81/2$. The rational involution of X_{15} is known as the *Kantor involution* [4]. It is defined as follows. The net of quadrics through 7 nodes $x_1,\dots,x_7$ resolved

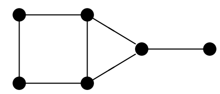

FIGURE 5.2. Graph Γ_7

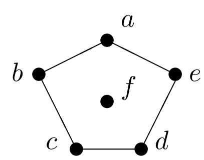

FIGURE 5.3. Pentagon

by the exceptional curves $N_1, \ldots, N_7$ has the additional base point x_0 (because three quadrics intersect at 8 points). For a general point $x \in X_{15}$ the quadrics from the net that vanish at x form a pencil with the base locus a quartic elliptic curve passing through $x_0, x_1, \ldots, x_7, x$. If we take x_0 as the origin in its group law, the involution sends x to the unique point x' on the curve such that $x + x' = x_0$ in the group law. The Kantor involution is defined when some generality condition is imposed on the seven points. It is determined by our choice of $N_1, \ldots, N_7$.

Example 5.7. Consider the graph Γ with 6 vertices in Figure 5.3, which we call a *pentagon*. The set $\mathrm{Indx}(\Gamma)$ is of size 72. This graph Γ defines an $\mathfrak{S}_6$-orbit of sets of 5 nodal curves. Let $\{N_1, \ldots, N_5\}$ be an element of this orbit. Then, for $\nu = 1, \ldots, 5$, the class

$$h_{2,\nu} := 3h_4 - 2(N_1 + \cdots + N_5) + N_\nu$$

of degree 2 is nef, and $|h_{2,\nu}|$ defines a rational double covering $Y_{15} \to \mathbf{P}^2$, the branch curve of which has singularities of type $6A_1 + 2D_4$. The associated involution does not depend on ν, and its α_{15}-degree is $213/2$. The involution is obtained from an admissible pentad $x_1, \ldots, x_5$ of nodes of type 3 from [7, Table 2]. The corresponding birational involution of X_{15} assigns to a general point $x \in X_{15}$ the unique remaining intersection point of a rational normal curve through the points $x_1, \ldots, x_5, x$ with the surface.

5.3.4. *Double-plane involution obtained from the model of degree* 6. In Section 6.2 of [7], it was shown that the complete linear system $|h_6|$ of the class

$$h_6 := 3h_4 - \sum_\delta E_\delta \tag{5.4}$$

of degree 6 gives a birational morphism $Y_{15} \to X_{15}^{(6)}$ to a $(2,3)$-complete intersection $X_{15}^{(6)}$ in $\mathbf{P}^4$. This morphism maps each nodal curve E_δ to a conic and contracts each trope-conic $\sigma(E_\theta)$ to a node of $X_{15}^{(6)}$. The image is the intersection of the Segre cubic primal with a quadric. We have

$$h_4 = 2h_6 - \sum_\theta \sigma(E_\theta). \tag{5.5}$$

Remark 5.8. This model $X_{15}^{(6)}$ can be regarded as the projective dual of $X_{15} \subset \mathbf{P}^3$ (see [7, Section 6.4]). Compare (5.4) and (5.5) with (4.1). Smooth rational curves defining the walls in the orbits O_3 and O_4 are of degree 6 with respect to h_6.

Let θ and θ' be distinct double trios. We consider the class

$$h_2(\theta,\theta') := h_6 - \sigma(E_\theta) - \sigma(E_{\theta'})$$

of degree 2. Then $h_2(\theta,\theta')$ is nef, and $|h_2(\theta,\theta')|$ defines a double cover $Y_{15} \to \mathbf{P}^2$, whose fiber lies on the intersection of $X_{15}^{(6)}$ and a plane in $\mathbf{P}^4$ passing through the two nodes of $X_{15}^{(6)}$, the images of $\sigma(E_\theta)$ and $\sigma(E_{\theta'})$ under the map from Y_{15} to $X_{15}^{(6)}$. The singularities of the branch curve of this double-plane covering is of type $4A_1 + 2A_3$. The α_{15}-degree of the associated involution is $33/2$.

5.4. **Inner walls and the associated extra-automorphisms.** We describe the inner walls of D_{15}. Note that, by (5.2), Condition 4 is satisfied. We show that Condition 3 is also satisfied by presenting the automorphism g_w explicitly for each inner wall w of D_{15}. We call $g_w \in \mathrm{Aut}(Y_{15})$ the *extra-automorphism* for $w \in \mathrm{Inn}(D_{15})$.

As in Section 5.1, let r_1 and r_2 be the defining (-2)-vectors of outer walls in O_1 and O_2, respectively, that is, r_1 is the class of the trope-conic $\sigma(E_{\theta(r_1)})$ indexed by a double trio $\theta(r_1)$, and r_2 is the class of the nodal curve $E_{\delta(r_2)}$ indexed by a duad $\delta(r_2)$. As was remarked in Section 5.1, a vector $v \in S_{15} \otimes \mathbb{Q}$ is uniquely characterized by the $10+15$ numbers $\langle r_1, v\rangle$ and $\langle r_2, v\rangle$.

5.4.5. *The orbit* O_5. Each inner wall $w_5 = D_{15} \cap (v)^\perp$ in O_5 is indexed by a number $\nu \in \{1,\dots,6\}$ in such a way that the primitive defining vector v of w_5 is characterized by

$$\langle r_1, v\rangle = 0,$$

$$\langle r_2, v\rangle = \begin{cases} 1 & \text{if } \nu \in \delta(r_2), \\ 0 & \text{otherwise.} \end{cases}$$

The extra-automorphism for w_5 is the involution $\sigma^{(\nu)}$ defined in Section 5.3.1. We denote this involution by $\gamma_5(\nu)$.

5.4.6. *The orbit* O_6. Each inner wall $w_6 = D_{15} \cap (v)^\perp$ in O_6 is indexed by a non-ordered pair $\{\theta_1, \theta_2\}$ of distinct double trios in the following way. Let $\{i_1, i_2\}$ and $\{j_1, j_2\}$ be the two duads of the form $\tau_1 \cap \tau_2$, where τ_1 is a trio in θ_1 and τ_2 is a trio in θ_2. Then we obtain four duads $\{i_1, j_1\}$, $\{i_1, j_2\}$, $\{i_2, j_1\}$, $\{i_2, j_2\}$. The primitive defining vector v of w_6 is characterized by the following:

$$\langle r_1, v\rangle = \begin{cases} 1 & \text{if } \theta(r_1) = \theta_1 \text{ or } \theta(r_1) = \theta_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle r_2, v\rangle = \begin{cases} 1 & \text{if } \delta(r_2) \text{ is one of the 4 duads } \{i_a, j_b\}\ (a, b = 1, 2), \\ 0 & \text{otherwise.} \end{cases}$$

The extra-automorphism for w_6 is equal to the involution obtained from θ_1, θ_2 by the procedure described in Section 5.3.4, which we will denote by $\gamma_6(\{\theta_1, \theta_2\})$.

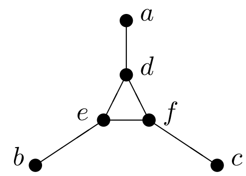

FIGURE 5.4. Tripod

5.4.7. *The orbit* O_7. Each inner wall $w_7 = D_{15} \cap (v)^\perp$ in O_7 is indexed by a number $\nu \in \{1, \dots, 6\}$ in such a way that the primitive defining vector v of w_7 is characterized by

$$\langle r_1, v\rangle = 0,$$
$$\langle r_2, v\rangle = \begin{cases} 0 & \text{if } \nu \in \delta(r_2), \\ 1 & \text{otherwise.} \end{cases}$$

The extra-automorphism for w_7 is equal to the Reye involution $\tau^{(\nu)}_{\mathrm{Rey}}$ given in Section 5.3.2, which we will denote by $\gamma_7(\nu)$.

5.4.8. *The orbit* O_8. Each inner wall $w_8 = D_{15} \cap (v)^\perp$ in O_8 is indexed by a duad δ_v in such a way that

$$\langle r_1, v\rangle = \begin{cases} 0 & \text{if } \delta_v \text{ is a subset of one of the two trios in } \theta(r_1), \\ 1 & \text{otherwise,} \end{cases}$$
$$\langle r_2, v\rangle = \begin{cases} 2 & \text{if } \delta_v = \delta(r_2), \\ 0 & \text{otherwise.} \end{cases}$$

Let p_{δ_v} be the node of X_{15} corresponding to the nodal curve E_{δ_v} indexed by the duad δ_v. The extra-automorphism for w_8 is the involution obtained from the projection $X_{15} \dashrightarrow \mathbf{P}^2$ with the center p_{δ_v}, which we denote by $\gamma_8(\delta_v)$.

5.4.9. *The orbit* O_9. Consider the graph $\mathit{\Gamma}_{\mathrm{tripod}}$ given in Figure 5.4. Then the set $\mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$ is of size 120. Each inner wall $w_9 = D_{15} \cap (v)^\perp$ in O_9 is indexed by an indexing $t_v \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$ as follows. For $t \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$, let $\Delta_9(t)$ be the set $t(E(\mathit{\Gamma}_{\mathrm{tripod}}))$ of duads obtained from the six edges of $\mathit{\Gamma}_{\mathrm{tripod}}$, and let $\Theta_9(t)$ be the set of three double trios obtained by applying t to the following:

$$\{\{a, e, f\}, \{d, b, c\}\},\ \ \{\{b, d, f\}, \{e, a, c\}\},\ \ \{\{c, d, e\}, \{f, a, b\}\}. \tag{5.6}$$

Then the primitive defining vector v of w_9 is characterized by

$$\langle r_1, v\rangle = \begin{cases} 1 & \text{if } \theta(r_1) \in \Theta_9(t_v), \\ 0 & \text{otherwise,} \end{cases}$$
$$\langle r_2, v\rangle = \begin{cases} 1 & \text{if } \delta(r_2) \in \Delta_9(t_v), \\ 0 & \text{otherwise.} \end{cases}$$

The extra-automorphism for w_9 is of infinite order, and can be written as a product of two involutions as follows. Let $\Theta_9(t_v)$ be $\{\theta_1, \theta_2, \theta_3\}$, and for $\theta_i \in \Theta_9(t_v)$, let $\{\theta_j, \theta_k\}$ be the complement of $\{\theta_i\}$ in $\Theta_9(t_v)$. Let τ_1 and τ_2 be the complementary trios such that $\theta_i = \{\tau_1, \tau_2\}$. Interchanging τ_1 and τ_2 if necessary, we can assume that $\tau_1 \cap \{t_v(a), t_v(b), t_v(c)\}$ is a duad δ_i. Connecting the two vertices of $\mathit{\Gamma}_{\mathrm{tripod}}$

that are mapped by t_v to δ_i, we obtain the graph $\mathit{\Gamma}_7$ in Figure 5.2 with an indexing $\tilde{t}_v \in \mathrm{Indx}(\mathit{\Gamma}_7)$ induced by $t_v \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$. Then, by Example 5.6, we obtain a double-plane involution, which will be denoted by $\gamma_9'(t_v, \theta_i)$. It turns out that

$$\gamma_9(t_v) = \gamma_6(\{\theta_j, \theta_k\})\gamma_9'(t_v, \theta_i)$$

for $i = 1, 2, 3$, where $\gamma_6(\{\theta_j, \theta_k\})$ is defined in Section 5.4.6, is independent of i and is equal to the extra-automorphism for w_9.

5.4.10. *The orbit O_{10}.* Consider the graph $\mathit{\Gamma}_{\mathrm{penta}}$ given in Figure 5.3. Each inner wall $w_{10} = D_{15} \cap (v)^{\perp}$ in O_{10} is indexed by an indexing $p_v \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{penta}})$ as follows. For $p \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{penta}})$, let $\Delta_{10}(p)$ be the set $t(E(\mathit{\Gamma}_{\mathrm{penta}}))$ of duads obtained from the five edges of $\mathit{\Gamma}_{\mathrm{penta}}$, and let $\Theta_{10}(p)$ be the set of five double trios obtained by applying p to the following:

$$\begin{aligned} &\{\{a, c, d\}, \{b, e, f\}\}, \quad \{\{b, d, e\}, \{a, e, f\}\}, \quad \{\{c, e, a\}, \{b, d, f\}\}, \\ &\{\{d, a, b\}, \{c, e, f\}\}, \quad \{\{e, b, c\}, \{d, a, f\}\}. \end{aligned}$$

Then the primitive defining vector v of w_{10} is characterized by

$$\langle r_1, v\rangle = \begin{cases} 1 & \text{if } \theta(r_1) \in \Theta_{10}(p_v), \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle r_2, v\rangle = \begin{cases} 1 & \text{if } \delta(r_2) \in \Delta_{10}(p_v), \\ 0 & \text{otherwise.} \end{cases}$$

The extra-automorphism for w_{10} is the involution given in Example 5.7, which we will denote by $\gamma_{10}(p_v)$.

5.5. **Main Theorems.** Summarizing the results in the previous section, we obtain the following:

Theorem 5.9. *The automorphism group* $\mathrm{Aut}(Y_{15})$ *is generated by the following elements.*

(v) *The six involutions $\gamma_5(\nu)$, where $\nu = 1, \dots, 6$, that make Y_{15} the focal surface of a congruence of bi-degree* $(2, 3)$.

(vi) *The* 45 *involutions $\gamma_6(\{\theta_1, \theta_2\})$, where $\{\theta_1, \theta_2\}$ is a non-ordered pair of double trios. Each $\gamma_6(\{\theta_1, \theta_2\})$ is obtained from the projection $X_{15}^{(6)} \dashrightarrow \mathbf{P}^2$, where $X_{15}^{(6)}$ is the* $(2, 3)$*-complete intersection model of Y_{15} given by the class* (5.4)*, and the center of the projection is the line passing through the two nodes of $X_{15}^{(6)}$ corresponding to θ_1 and θ_2.*

(vii) *The six Reye involutions $\gamma_7(\nu)$, where $\nu = 1, \dots, 6$.*

(viii) *The* 15 *involutions $\gamma_8(\delta)$ obtained from the projection of $X_{15} \dashrightarrow \mathbf{P}^2$ with the center being the node of X_{15} corresponding to the duad δ.*

(ix) *The* 120 *automorphisms*

$$\gamma_9(t) = \gamma_6(\{\theta_j, \theta_k\})\gamma_9'(t, \theta_i)$$

of infinite order, where $t \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$, $\Theta_9(t) = \{\theta_i, \theta_j, \theta_k\}$ is the set of three double trios obtained by applying t to (5.6)*, $\gamma_6(\{\theta_j, \theta_k\})$ is the involution defined in* (vi)*, and $\gamma_9'(t, \theta_i)$ is the involution given in Example 5.6, where the graph defining the* 7 *nodes in Figure 5.2 is obtained from $\mathit{\Gamma}_{\mathrm{tripod}}$ with indexing t by connecting the vertices that are the intersection of $\{t(a), t(b), t(c)\}$ and one of the two trios in θ_i.*

(x) *The* 72 *involutions* $\gamma_{10}(p)$ *given in Example 5.7, where* $p \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{penta}})$.

□

We then describe the defining relations of $\mathrm{Aut}(Y_{15})$ with respect to the generating set

$$\gamma_5(\nu),\ \ \gamma_6(\{\theta_1,\theta_2\}),\ \ \gamma_7(\nu),\ \ \gamma_8(\delta),\ \ \gamma_9(t),\ \ \gamma_{10}(p), \tag{5.7}$$

given in Theorem 5.9. By Theorem 3.9, it is enough to calculate the relations $\mathcal{R}_1$ and $\mathcal{R}_2$ defined by (3.7) and (3.8), respectively. The method to calculate $\mathcal{R}_2$ is explained in Sections 3.4 and 3.5.

Theorem 5.10. *The relations in* $\mathcal{R}_1$ *are as follows:*

$$\gamma_5(\nu)^2=1,\ \ \gamma_6(\{\theta_1,\theta_2\})^2=1,\ \ \gamma_7(\nu)^2=1,\ \ \gamma_8(\delta)^2=1,\ \ \gamma_{10}(p)^2=1,$$

and

$$\gamma_9(t)\gamma_9(t')=1,$$

where $t, t' \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$ *are pairs of distinct indexings such that* $\Theta_9(t)=\Theta_9(t')$.

There exist exactly 5235 *inner faces of* D_{15} *with codimension* 2*, and they are decomposed into* 19 *orbits* $F_1,\dots,F_{19}$ *under the action of* $\mathrm{O}(S_{15},D_{15})\cong\mathfrak{S}_6$*. In the table below, the* size *is the number of faces in the orbit* F_i*,* f *is an element of* F_i*, and* $R_f=(g_m,\dots,g_1)$ *is the relation* $g_m\cdots g_1=1$ *associated with the simple chamber loop* $(D_0,\dots,D_m)$ *around* (f,D_0)*, where the starting chamber* D_0 *is our induced chamber* D_{15}*.*

F_1: size $=180$
$f=w_5(1)\cap w_6(\{(123),(124)\})$
$R_f=(\gamma_6(\{(123),(124)\}),\gamma_5(2),\gamma_6(\{(125),(126)\}),\gamma_5(1))$

F_2: size $=30$
$f=w_5(1)\cap w_7(2)$
$R_f=(\gamma_7(2),\gamma_5(1),\gamma_8((12)),\gamma_5(1))$

F_3: size $=30$
$f=w_5(1)\cap w_8((12))$
$R_f=(\gamma_8((12)),\gamma_5(1),\gamma_7(2),\gamma_5(1))$

F_4: size $=360$
$f=w_5(1)\cap w_9([234156])$
$R_f=(\gamma_9([156234]),\gamma_5(2),\gamma_9([256143]),\gamma_5(1))$

F_5: size $=45$
$f=w_6(\{(123),(124)\})\cap w_6(\{(135),(145)\})$
$R_f=(\gamma_6(\{(135),(145)\}),\gamma_6(\{(123),(124)\}),\gamma_6(\{(135),(145)\}),\gamma_6(\{(123),(124)\}))$

F_6: size $=180$
$f=w_6(\{(123),(124)\})\cap w_6(\{(123),(125)\})$
$R_f=(\gamma_6(\{(123),(125)\}),\gamma_6(\{(125),(126)\}),\gamma_6(\{(124),(126)\}),\gamma_6(\{(123),(124)\}))$

F_7: size $=180$
$f=w_6(\{(123),(124)\})\cap w_6(\{(123),(135)\})$

$$R_f = (\gamma_6(\{(123),(135)\}), \gamma_6(\{(124),(135)\}), \gamma_6(\{(123),(124)\}),$$
$$\gamma_6(\{(123),(135)\}), \gamma_6(\{(124),(135)\}), \gamma_6(\{(123),(124)\}))$$

F_8: size $= 360$

$f = w_6(\{(123),(124)\}) \cap w_6(\{(125),(135)\})$

$$R_f = (\gamma_6(\{(125),(135)\}), \gamma_6(\{(124),(146)\}), \gamma_6(\{(125),(126)\}),$$
$$\gamma_6(\{(123),(146)\}), \gamma_6(\{(126),(135)\}), \gamma_6(\{(123),(124)\}))$$

F_9: size $= 90$

$f = w_6(\{(123),(124)\}) \cap w_7(3)$

$R_f = (\gamma_7(3), \gamma_6(\{(123),(124)\}), \gamma_7(4), \gamma_6(\{(123),(124)\}))$

F_{10}: size $= 180$

$f = w_6(\{(123),(124)\}) \cap w_8((15))$

$R_f = (\gamma_8((15)), \gamma_6(\{(123),(124)\}), \gamma_8((26)), \gamma_6(\{(123),(124)\}))$

F_{11}: size $= 360$

$f = w_6(\{(123),(124)\}) \cap w_9([135642])$

$R_f = (\gamma_9([246531]), \gamma_6(\{(123),(124)\}), \gamma_9([246531]), \gamma_6(\{(123),(124)\}))$

F_{12}: size $= 360$

$f = w_6(\{(123),(124)\}) \cap w_9([134256])$

$$R_f = (\gamma_9([256134]), \gamma_9([134265]), \gamma_6(\{(123),(124)\}),$$
$$\gamma_9([156243]), \gamma_9([234156]), \gamma_6(\{(123),(124)\}))$$

F_{13}: size $= 360$

$f = w_6(\{(123),(124)\}) \cap w_9([123564])$

$$R_f = (\gamma_9([456312]), \gamma_9([124563]), \gamma_6(\{(123),(124)\}),$$
$$\gamma_9([456312]), \gamma_9([124563]), \gamma_6(\{(123),(124)\}))$$

F_{14}: size $= 720$

$f = w_6(\{(123),(124)\}) \cap w_9([134526])$

$R_f = (\gamma_9([256314]), \gamma_6(\{(125),(126)\}), \gamma_9([234651]), \gamma_6(\{(123),(124)\}))$

F_{15}: size $= 720$

$f = w_6(\{(123),(124)\}) \cap w_9([135462])$

$$R_f = (\gamma_9([246513]), \gamma_9([235461]), \gamma_6(\{(123),(124)\}),$$
$$\gamma_9([235614]), \gamma_9([246351]), \gamma_6(\{(123),(124)\}))$$

F_{16}: size $= 360$

$f = w_6(\{(123),(124)\}) \cap w_{10}([13526])$

$R_f = (\gamma_{10}([13526]), \gamma_6(\{(123),(124)\}), \gamma_{10}([15246]), \gamma_6(\{(123),(124)\}))$

F_{17}: size $= 180$

$f = w_9([123456]) \cap w_9([123465])$

$$R_f = (\gamma_9([456123]), \gamma_6(\{(145),(146)\}), \gamma_9([234561]),$$
$$\gamma_9([156432]), \gamma_6(\{(145),(146)\}), \gamma_9([123465]))$$

F_{18}: size $= 180$
$f = w_9([123456]) \cap w_9([126453])$
$R_f = (\gamma_9([456123]), \gamma_6(\{(123),(126)\}), \gamma_9([126453]),$
$\gamma_9([456123]), \gamma_6(\{(123),(126)\}), \gamma_9([126453]))$

F_{19}: size $= 360$
$f = w_9([123456]) \cap w_9([124356])$
$R_f = (\gamma_9([356124]), \gamma_6(\{(126),(134)\}), \gamma_9([356241]),$
$\gamma_9([123645]), \gamma_6(\{(126),(134)\}), \gamma_9([123456]))$

□

In the table in Theorem 5.10, a wall $w_i(K)$ is the wall in the orbit O_i indexed by K, where

- K is the number $\nu \in \{1,\dots,6\}$ that indexes $w_i(K)$ when $i = 5$ or 7,
- K is the pair of trios $(i_1j_1k_1) \in \theta_1$ and $(i_2j_2k_2) \in \theta_2$ when $i = 6$ and $w_i(K)$ is indexed by $\{\theta_1, \theta_2\}$ (the other trio of θ_ν is $\{1,\dots,6\} \setminus (i_\nu j_\nu k_\nu)$),
- K is the duad $\delta = (ij)$ that indexes $w_i(K)$ when $i = 8$,
- K is $[t(a)t(b)t(c)t(d)t(e)t(f)]$ when $i = 9$ and $w_i(K)$ is indexed by $t \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{tripod}})$, and
- K is $[p(a)p(b)p(c)p(d)p(e)]$ when $i = 10$ and $w_i(K)$ is indexed by $p \in \mathrm{Indx}(\mathit{\Gamma}_{\mathrm{penta}})$.

The automorphism $\gamma_i(K)$ is the extra-automorphism associated with the wall $w_i(K)$.

Example 5.11. By the relations associated with the faces F_2 and F_3, we obtain the following relations

$$\gamma_7(\nu) = \gamma_5(\mu)\gamma_8(\{\nu,\mu\})\gamma_5(\mu)$$

for $\mu, \nu \in [1,6]$ with $\mu \neq \nu$ between the Reye involution $\gamma_7(\nu)$, the involution $\gamma_8(\{\nu,\mu\})$ induced by the projection $X_{15} \dashrightarrow \mathbb{P}^2$ from the node $p_{\{\nu,\mu\}}$, and the involution $\gamma_5(\mu)$ obtained from the double covering of a quintic del Pezzo surface D in the Grassmannian.

5.6. **Writing an automorphism as a product of the generators.** Suppose that an automorphism $g \in \mathrm{Aut}(Y_{15})$ is given. Then g is expressed as a product of the generators (5.7). To obtain such an expression, we can use the following method. We choose a vector $\alpha' \in S_{15}$ from the interior of D_{15}. By making the choice random enough, we can assume that the line segment ℓ in $\mathcal{P}_{15}$ connecting α' and α'^g does not intersect any face of induced chambers with codimension ≥ 2. Let D' be the induced chamber containing α'^g in its interior. We calculate the set of (-2)-vectors $r_1,\dots,r_N$ of $II_{1,25}$ such that the hyperplane $(r_i)^\perp$ of $\mathcal{P}(II_{1,25})$ separates the points $\epsilon_{15}(\alpha')$ and $\epsilon_{15}(\alpha'^g)$. Calculating the vector $\mathrm{pr}_S(r_i)$, where $\mathrm{pr}_S\colon II_{1,25} \to S_{15}^\vee$ is the natural projection, we obtain the list of walls of induced chambers that the line segment ℓ intersects. Thus we obtain a chamber path $(D^{(0)},\dots,D^{(m)})$ from $D^{(0)} = D_{15}$ to $D^{(m)} = D'$ such that ℓ intersects every $D^{(i)}$. By the method described in Section 3.3, we calculate the sequence $g_1,\dots,g_m \in \mathrm{Gen}$ such that

$$D^{(i)} = {D_{15}}^{g_i\cdots g_1}$$

for $i = 1,\dots,m$. Thus we obtain an expression $g = g_m \cdots g_1$.

type $= 1$, nodes $= \{(12),(13),(14),(15),(16)\}$: $\gamma_5(1), \gamma_7(1), \gamma_5(1)$
type $= 2$, nodes $= \{(12),(13),(24),(34),(56)\}$: $\gamma_6(\{(145),(146)\}), \gamma_8((56)), \gamma_6(\{(145),(146)\})$
type $= 3$, nodes $= \{(12),(13),(24),(35),(45)\}$: $\gamma_{10}([12453])$
type $= 4$, nodes $= \{(12),(13),(14),(23),(24)\}$: $\gamma_6(\{(125),(126)\}), \gamma_8((12)), \gamma_6(\{(125),(126)\})$
type $= 5$, nodes $= \{(12),(13),(14),(15),(26)\}$: $\gamma_5(1), \gamma_5(6), \gamma_8((12)), \gamma_5(6), \gamma_5(1)$
type $= 6$, nodes $= \{(12),(13),(14),(15),(23)\}$: $\gamma_5(1), \gamma_5(6), \gamma_8((45)), \gamma_5(6), \gamma_5(1)$
type $= 7$, nodes $= \{(12),(13),(14),(25),(35)\}$: $\gamma_6(\{(145),(156)\}), \gamma_8((56)), \gamma_6(\{(145),(156)\})$
type $= 8$, nodes $= \{(12),(13),(14),(23),(25)\}$: $\gamma_9([123456]), \gamma_8((45)), \gamma_9([456123])$
type $= 9$, nodes $= \{(12),(13),(14),(25),(36)\}$: $\gamma_9([123456]), \gamma_8((14)), \gamma_9([456123])$

Table 5.2. Admissible pentads

Example 5.12. We apply this method to the involutions obtained from the admissible pentads (see [7, Section 9]). Under the action of $\mathfrak{S}_6$, the admissible pentads are decomposed into 9 orbits as in [7, Table 2]. In Table 5.2, we choose an admissible pentad from each orbit, and write the associated involution as a product of generators. The 5 nodes are given by corresponding duads. The involution is the product of generators in the second line.

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