AN ALGORITHM TO COMPUTE AUTOMORPHISM GROUPS OF $K3$ SURFACES AND AN APPLICATION TO SINGULAR $K3$ SURFACES

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ABSTRACT. Let X be a complex algebraic $K3$ surface or a supersingular $K3$ surface in odd characteristic. We present an algorithm by which, under certain assumptions on X, we can calculate a finite set of generators of the image of the natural homomorphism from the automorphism group of X to the orthogonal group of the Néron-Severi lattice of X. We then apply this algorithm to certain complex $K3$ surfaces, among which are singular $K3$ surfaces whose transcendental lattices are of small discriminants.

1. INTRODUCTION

The automorphism group $\mathrm{Aut}(X)$ of an algebraic $K3$ surface X is an important and interesting object. Suppose that X is defined over the complex number field $\mathbb{C}$, or is supersingular in odd characteristic. Then, thanks to the Torelli-type theorem due to Piatetski-Shapiro and Shafarevich [23] and Ogus [21], [22], we can study $\mathrm{Aut}(X)$ by the Néron-Severi lattice S_X of X. We denote by $\mathrm{O}(S_X)$ the orthogonal group of S_X. Then we have a natural homomorphism

$$\varphi_X\colon\ \mathrm{Aut}(X) \to \mathrm{O}(S_X).$$

It is known that this homomorphism has only a finite kernel. Using the reduction theory for arithmetic subgroups of $\mathrm{O}(S_X)$, Sterk [36] and Lieblich and Maulik [16] proved that $\mathrm{Aut}(X)$ is finitely generated. The Néron-Severi lattices for which $\mathrm{Aut}(X)$ are finite were classified by Nikulin [18], [19] and Vinberg [40]. On the other hand, when $\mathrm{Aut}(X)$ is infinite, it is in general a difficult problem to give a set of generators.

We also have the following related problem. Let $\mathrm{Nef}(X)$ denote the nef cone of X; that is, the cone of $S_X \otimes \mathbb{R}$ consisting of vectors $x \in S_X \otimes \mathbb{R}$ such that $\langle x, C\rangle \geq 0$ holds for any curve C on X, where $\langle\ ,\ \rangle$ is the intersection form on S_X. In order to classify various geometric objects on X (for example, smooth rational curves, Jacobian fibrations, or polarizations of a fixed degree) *modulo* $\mathrm{Aut}(X)$, it is useful to describe explicitly a fundamental domain of the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)$.

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Several authors have studied these problems by using the idea of Borcherds [4], [5] to embed S_X into an even unimodular hyperbolic lattice of rank 26. In these works, however, they required that S_X should satisfy a certain strong condition (see Section 1.1 below for the details), and hence the range of applications is limited.

The purpose of this paper is to present an algorithm (Algorithm 6.1) that calculates, under assumptions on S_X milder than the preceding works, a finite set of generators of the image of φ_X and a closed domain F of $\mathrm{Nef}(X)$ with the following properties:

(i) For any $v \in \mathrm{Nef}(X)$, there exists an element $g \in \mathrm{Aut}(X)$ such that $v^g \in F$.

(ii) The domain F is tiled by a finite number of convex cones, which we call chambers. Each chamber is bounded by a finite number of hyperplanes and its stabilizer subgroup in $\mathrm{Aut}(X)$ is finite.

See Remark 6.5 for the relation of F with a fundamental domain of the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)$. The detailed description of the assumptions we impose on S_X will be given in Section 8.

The algorithm can be applied to a wide class of $K3$ surfaces. We give two examples.

Example 1.1. As an example of a $K3$ surface with small Picard number and an infinite automorphism group, we consider a complex $K3$ surface X whose Néron-Severi lattice S_X has a Gram matrix

$$M := \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -24 \end{bmatrix}$$

with respect to a certain basis f_ϕ, z_ϕ, v such that f_ϕ and z_ϕ are the classes of a fiber and the zero section of a Jacobian fibration $\phi : X \to \mathbb{P}^1$. Then the Mordell-Weil group MW_ϕ of ϕ is isomorphic to $\mathbb{Z}$. Hence $\mathrm{Aut}(X)$ contains an infinite subgroup $\mathrm{MW}_\phi \rtimes \mathbb{Z}/2\mathbb{Z}$ (see Section 9). We assume that the period ω_X of X is generic in $T_X \otimes \mathbb{C}$, where T_X is the transcendental lattice of X. We let $\mathrm{O}(S_X)$ act on S_X from the right so that

$$\mathrm{O}(S_X) = \{\, g \in \mathrm{GL}_3(\mathbb{Z}) \mid g\, M\, {}^t g = M \,\}.$$

Then φ_X is injective, and its image is generated by the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & -1 \\ 24 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 37 & 12 & -5 \\ 36 & 13 & -5 \\ 360 & 120 & -49 \end{bmatrix}, \quad \begin{bmatrix} 97 & 48 & -14 \\ 0 & 1 & 0 \\ 672 & 336 & -97 \end{bmatrix}.$$

The first two elements of these four matrices are the images of the involutions that generate $\mathrm{MW}_\phi \rtimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. We need two more automorphisms to generate the full automorphism group $\mathrm{Aut}(X)$. Moreover we can show that $\mathrm{Aut}(X)$ acts on the set of smooth rational curves on X transitively. In Figure 9.4, we give a tessellation of $\mathrm{Nef}(X)$ by a fundamental domain of $\mathrm{Aut}(X)$. See Section 9 or the author's web-page [31] for more examples of this type.

Example 1.2. A complex algebraic $K3$ surface is said to be *singular* if its Picard number attains the possible maximum 20. By the result of Shioda and Inose [34], the isomorphism class of a singular $K3$ surface X is determined by its oriented transcendental lattice T_X, and $\mathrm{Aut}(X)$ is always infinite. See [33] for the standard Gram matrices

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

of oriented transcendental lattices of singular $K3$ surfaces. Let $\operatorname{disc} T_X$ denote the discriminant of T_X. Since T_X is an even lattice, we see that $\operatorname{disc} T_X = ac - b^2$ is congruent to 0 or 3 mod 4. We consider singular $K3$ surfaces X with small $\operatorname{disc} T_X$. The automorphism groups of the singular $K3$ surfaces X with $\operatorname{disc} T_X = 3$ and 4 were determined by Vinberg [39], and $\mathrm{Aut}(X)$ of the singular $K3$ surface X with $\operatorname{disc} T_X = 7$ was determined by Ujikawa [37]. The case where $\operatorname{disc} T_X = 8$ can be handled by *usual* Borcherds' method (see Section 10). Therefore the next example to be considered is the singular $K3$ surface X whose transcendental lattice is given by a Gram matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}.$$

In this case, φ_X is injective. We find that $\operatorname{Im} \varphi_X$ is generated by 767 elements of $\mathrm{O}(S_X)$, and, as in Example 1.1, we can present them explicitly as 20×20 matrices with respect to a certain basis of S_X. These matrices and related computational data are given in the author's web-page [31]. From these data, we see that the number of smooth rational curves on X is at most 347 modulo $\mathrm{Aut}(X)$. (Note that the number 767 of generators of $\mathrm{Aut}(X)$ and the upper-bound 347 of the number of smooth rational curves modulo $\mathrm{Aut}(X)$ are not minimal.) See Section 10 for more examples of this type.

Our main algorithm (Algorithm 6.1) can be applied to a wide class of $K3$ surfaces theoretically. The most crucial assumption that S_X be embedded primitively into an even unimodular hyperbolic lattice of rank 10, 18 or 26 is always satisfied when X is defined over $\mathbb{C}$ (see Proposition 8.1). However, experiments show that the computational complexity of Algorithm 6.1 grows rapidly as the rank and the discriminant of S_X become large.

To the best knowledge of the author, the automorphism group of the complex Fermat quartic surface X is still unknown. This surface is a singular $K3$ surface with $\operatorname{disc} T_X = 64$, and we can apply our algorithm to this surface. It turns out, however, that the computation is very heavy, and we could not finish the calculation (see Remark 10.3).

Even when the computation is too heavy to be completed in a reasonable time, our algorithm yields many interesting automorphisms and projective models of the given $K3$ surface on the way of computation. We have applied this method to the supersingular $K3$ surface in characteristic 5 with Artin invariant 1 in [12].

It is a totally different problem to give geometrically an automorphism g of X such that $\varphi_X(g)$ is equal to a given matrix in $\mathrm{O}(S_X)$. In [32], we discuss an algorithmic approach to this problem, and apply it to certain singular $K3$ surfaces.

1.1. **The difference of our algorithm from the preceding works.** We briefly review Borcherds' method [4], [5], and its applications to $K3$ surfaces. A lattice of rank $n > 1$ is said to be *hyperbolic* if its signature is $(1, n-1)$. Let S be an even hyperbolic lattice of rank < 26. A *positive cone* $\mathcal{P}_S$ of S is one of the two connected components of $\{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$. We denote by $\mathrm{O}^+(S)$ the stabilizer subgroup of $\mathcal{P}_S$ in $\mathrm{O}(S)$. Let $W(S)$ denote the subgroup of $\mathrm{O}^+(S)$ generated by the reflections in the hyperplanes $(r)^\perp$ perpendicular to r, where r runs through the set $\mathcal{R}_S := \{r \in S \mid r^2 = -2\}$. The mirrors $(r)^\perp$ decompose $\mathcal{P}_S$ into the union of standard fundamental domains of the action of $W(S)$ on $\mathcal{P}_S$. We call each of these standard fundamental domains an $\mathcal{R}_S^*$*-chamber*. Let G be a subgroup of $\mathrm{O}^+(S)$ with finite index. Let N be an $\mathcal{R}_S^*$-chamber, and let $Aut_G(N)$ denote the stabilizer subgroup in G of the $\mathcal{R}_S^*$-chamber N. In the application to a complex $K3$ surface X, S is the Néron-Severi lattice S_X, $\mathcal{P}_S$ is the connected component containing an ample class, the group G consists of elements $g \in \mathrm{O}^+(S)$ liftable to isometries of $H^2(X, \mathbb{Z})$ that preserve the period of X, and N is the intersection of $\mathrm{Nef}(X)$ with $\mathcal{P}_S$.

Borcherds gave a method to calculate a finite set of generators of $Aut_G(N)$. Suppose that S is primitively embedded into an even unimodular hyperbolic lattice $\mathbf{L} := \mathrm{II}_{1,25}$ of rank 26 in such a way that every $g \in G$ lifts to an isometry of $\mathbf{L}$. We also assume that the orthogonal complement R of S in $\mathbf{L}$ cannot be embedded into the (negative-definite) Leech lattice. Let $\mathcal{P}_{\mathbf{L}} \subset \mathbf{L} \otimes \mathbb{R}$ be the positive cone of $\mathbf{L}$ containing $\mathcal{P}_S$. The structure of $\mathcal{R}_{\mathbf{L}}^*$-chambers is well-understood by Conway [7, Chapter 27]. Then the tessellation of $\mathcal{P}_{\mathbf{L}}$ by the $\mathcal{R}_{\mathbf{L}}^*$-chambers induces a tessellation of $\mathcal{P}_S$, which is invariant under the action of G. We call the tiles constituting this induced tessellation $\mathcal{R}_{\mathbf{L}|S}^*$*-chambers*. Note that the $\mathcal{R}_S^*$-chamber N is a union of $\mathcal{R}_{\mathbf{L}|S}^*$-chambers. Two $\mathcal{R}_{\mathbf{L}|S}^*$-chambers D and D' are said to be *G-congruent* if there exists an element $g \in G$ that maps D to D'. By the reduction theory for arithmetic subgroups of $\mathrm{O}(S)$, we see that the number of G-congruence classes of $\mathcal{R}_{\mathbf{L}|S}^*$-chambers is finite. Let

$$\mathbb{D} := \{D_0, \dots, D_{m-1}\}$$

be a complete set of representatives of G-congruence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers such that each D_i is contained in N. From the list $\mathbb{D}$, we can obtain a finite set of generators of $Aut_G(N)$.

Kondo [14] applied this method to the Néron-Severi lattice of a generic Jacobian Kummer surface, and described its automorphism group. Since then, automorphism groups of the following $K3$ surfaces have been determined by this method;

(a) the supersingular $K3$ surface in characteristic 2 with Artin invariant 1 by Dolgachev and Kondo [9],
(b) complex Kummer surfaces of product type by Keum and Kondo [13],
(c) the Hessian quartic surface by Dolgachev and Keum [10],
(d) the singular $K3$ surface X with $\operatorname{disc} T_X = 7$ by Ujikawa [37],
(e) the supersingular $K3$ surface in characteristic 3 with Artin invariant 1 by Kondo and Shimada [15].

The classical two examples of Vinberg [39] can also be treated by this method.

In all these examples, the number $|\mathbb{D}|$ of G-congruence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers is 1. Borcherds [4] studied the case where the orthogonal complement R of S in $\mathbf{L}$ contains a root sublattice of finite index, and gave in [4, Lemma 5.1] a sufficient condition for any two $\mathcal{R}^*_{\mathbf{L}|S}$-chambers to be $\mathrm{O}^+(S)$-congruent. In particular, the method employed in the above examples is limited to $K3$ surfaces X such that S_X can be embedded primitively in $\mathbf{L}$ with R containing a root lattice as a sublattice of finite index. Note that, for example, only a finite number of isomorphism classes of singular $K3$ surfaces satisfy this condition.

We extend Borcherds' method to the situation in which $|\mathbb{D}|$ is not necessarily 1. In fact, we have $|\mathbb{D}| = 46$ in Example 1.1 and $|\mathbb{D}| = 1098$ in Example 1.2 above. Starting from an initial $\mathcal{R}^*_{\mathbf{L}|S}$-chamber, we compute $\mathcal{R}^*_{\mathbf{L}|S}$-chambers adjacent to the $\mathcal{R}^*_{\mathbf{L}|S}$-chambers obtained so far successively until no new G-congruence classes appear. Each $\mathcal{R}^*_{\mathbf{L}|S}$-chamber is expressed by its Weyl vector (see Definition 5.5). Thus our main algorithm contains sub-algorithms that calculate the set of walls of a given $\mathcal{R}^*_{\mathbf{L}|S}$-chamber (Algorithm 5.11), compute the Weyl vector of the adjacent $\mathcal{R}^*_{\mathbf{L}|S}$-chamber across a given wall (Algorithm 5.14), and determine whether an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber is G-congruent to another $\mathcal{R}^*_{\mathbf{L}|S}$-chamber (Algorithm 3.19). In these algorithms, we use methods given in our previous paper [30]. In the calculation of the set of walls, we also employ the standard algorithm of linear programming (Algorithm 3.17).

1.2. **The plan of the paper.** In Section 2, we fix notions and notation about lattices and hyperbolic spaces. In Section 3, we introduce the notion of chamber decomposition of a positive cone of a hyperbolic lattice, and present some algorithms about chambers. In Section 4, we review Vinberg-Conway theory ([38] and [7, Chapter 27]) on the structure of $\mathcal{R}^*_{\mathbf{L}}$-chambers in the even unimodular hyperbolic lattice $\mathbf{L} = \mathrm{II}_{1,n-1}$ of rank $n = 10, 18$ and 26. Then, in Section 5, we introduce the notion of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers associated with a primitive embedding $S \hookrightarrow \mathbf{L}$ of an even hyperbolic lattice S, and present some algorithms. In Section 6, we present our main algorithm (Algorithm 6.1), and prove its correctness. In Section 7, we review the Torelli-type theorem for $K3$ surfaces, and show how to calculate the automorphism group from the Néron-Severi lattice. In Section 8, we explain how to apply Algorithm 6.1 to the study of $K3$ surfaces. In particular, we describe in detail what geometric data of a $K3$ surface X must be calculated before we apply Algorithm 6.1 to X. In Sections 9 and 10, we demonstrate Algorithm 6.1 on some $K3$ surfaces with Picard number 3, and some singular $K3$ surfaces.

In this paper, *Aut* denotes the automorphism group of a lattice theoretic object, whereas Aut denotes the geometric automorphism group of a $K3$ surface.

For the actual computation, we used the `C` library `gmp` [11].

2. Preliminaries on lattices and hyperbolic spaces

Let L be a free $\mathbb{Z}$-module of finite rank. We say that a submodule M of L is *primitive* if L/M is torsion-free, and that $v \in L$ is *primitive* if so is the submodule $\langle v \rangle := \mathbb{Z}v \subset L$.

Let L be a lattice; that is, a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle_L \colon L \times L \to \mathbb{Z}$. The subscript L in $\langle\ ,\ \rangle_L$ is omitted if no confusion will occur. The symmetric bilinear form on $L \otimes \mathbb{R}$ obtained from $\langle\ ,\ \rangle$ is also denoted by $\langle\ ,\ \rangle$. We denote by $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ the *dual lattice of* L, which is naturally embedded in $L \otimes \mathbb{Q}$ by $\langle\ ,\ \rangle$. We say that L is *unimodular* if $L = L^\vee$ holds. The *norm* $\langle v, v \rangle$ of $v \in L \otimes \mathbb{R}$ is denoted by v^2. We let the *orthogonal group* $\mathrm{O}(L)$ of L act on L from the right, and write the action of $g \in \mathrm{O}(L)$ on $v \in L \otimes \mathbb{R}$ by $v \mapsto v^g$.

A lattice L is said to be *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$. For an even lattice L, we put

$$\mathcal{R}_L := \{\ r \in L \ \mid\ r^2 = -2\ \}.$$

Elements of $\mathcal{R}_L$ are called (-2)*-vectors*. Each $r \in \mathcal{R}_L$ defines the *reflection*

$$s_r \colon x \mapsto x + \langle x, r \rangle r,$$

which is an element of $\mathrm{O}(L)$. We denote by $W(L)$ the subgroup of $\mathrm{O}(L)$ generated by all reflections s_r with respect to the (-2)-vectors $r \in \mathcal{R}_L$, and call it the *Weyl group*.

A lattice L of rank $n > 0$ is said to be *negative-definite* if the signature of the real quadratic space $L \otimes \mathbb{R}$ is $(0, n)$. A negative-definite lattice L is said to be a *root lattice* if L is generated by $\mathcal{R}_L$.

A lattice L of rank $n > 1$ is said to be *hyperbolic* if the signature of $L \otimes \mathbb{R}$ is $(1, n-1)$. Let L be a hyperbolic lattice. Then a *positive cone of* L is one of the two connected components of $\{x \in L \otimes \mathbb{R} \mid x^2 > 0\}$. The closure of a positive cone

$\mathcal{P}_L$ in $L \otimes \mathbb{R}$ is denoted by $\overline{\mathcal{P}}_L$. We denote by $\overline{\mathcal{P}}_L^{\mathbb{Q}}$ the convex hull of $\overline{\mathcal{P}}_L \cap (L \otimes \mathbb{Q})$. The stabilizer subgroup in $\mathrm{O}(L)$ of a positive cone is denoted by $\mathrm{O}^+(L)$. We have $\mathrm{O}(L) = \mathrm{O}^+(L) \times \{\pm 1\}$. Note that $W(L)$ is contained in $\mathrm{O}^+(L)$.

The following algorithms will be used frequently in this paper.

Algorithm 2.1. Let Q be a positive-definite symmetric matrix of size n with rational entries, ℓ a column vector of length n with rational entries, and c a rational number. Then we can calculate the list of all row vectors $x \in \mathbb{Z}^n$ satisfying

$$x\, Q\, {}^t x + 2\, x\, \ell + c = 0$$

by the method described in [30, Section 3.1].

Algorithm 2.2. Let L be a hyperbolic lattice, let v be a vector of $L \otimes \mathbb{Q}$ with $v^2 > 0$, let α be a rational number, and let d be an integer. Then the finite set

$$\{\ x \in L \ \mid\ \langle x, v \rangle = \alpha,\ \ \langle x, x \rangle = d\ \}$$

can be calculated by the method described in [30, Sections 3.2].

Algorithm 2.3. Let L be a hyperbolic lattice, let v, h be vectors of $L \otimes \mathbb{Q}$ such that

$$\langle v, h \rangle > 0,\ \ \langle h, h \rangle > 0,\ \ \langle v, v \rangle > 0,$$

and let d be a negative integer. Then the finite set

$$\{\ x \in L \ \mid\ \langle v, x \rangle < 0,\ \ \langle h, x \rangle > 0,\ \ \langle x, x \rangle = d\ \}$$

can be calculated by the method described in [30, Sections 3.3].

Let L be a hyperbolic lattice of rank $m+1$, and let $\mathcal{P}_L$ be a positive cone of L. The multiplicative group $\mathbb{R}_{>0}$ of positive real numbers acts on $\overline{\mathcal{P}}_L \setminus \{0\}$ by scalar multiplication. We put

$$\begin{aligned} &\overline{\mathbb{H}}_L := (\overline{\mathcal{P}}_L \setminus \{0\})/\mathbb{R}_{>0}, \quad \overline{\mathbb{H}}_L^{\mathbb{Q}} := (\overline{\mathcal{P}}_L^{\mathbb{Q}} \setminus \{0\})/\mathbb{R}_{>0}, \quad \mathbb{H}_L := \mathcal{P}_L/\mathbb{R}_{>0}, \\ &\partial\, \overline{\mathbb{H}}_L := \overline{\mathbb{H}}_L \setminus \mathbb{H}_L, \quad \partial\, \overline{\mathbb{H}}_L^{\mathbb{Q}} := \overline{\mathbb{H}}_L^{\mathbb{Q}} \setminus \mathbb{H}_L, \end{aligned}$$

and denote by $\pi_L \colon \overline{\mathcal{P}}_L \setminus \{0\} \to \overline{\mathbb{H}}_L$ the natural projection. The space $\mathbb{H}_L$ is naturally endowed with a structure of the hyperbolic m-space. A point of $\partial\, \overline{\mathbb{H}}_L^{\mathbb{Q}}$ is called a *rational boundary point*. For simplicity, when we are given a subset T of $\overline{\mathcal{P}}_L$, we denote by $\pi_L(T) \subset \overline{\mathbb{H}}_L$ the image of $T \setminus \{0\}$ by π_L. For example, we have $\partial\, \overline{\mathbb{H}}_L^{\mathbb{Q}} = \pi_L(\overline{\mathcal{P}}_L \cap L)$. A subset K of $\mathbb{H}_L$ is said to be a *linear subspace of* $\mathbb{H}_L$ if there exists a linear subspace $\tilde{K}$ of $L \otimes \mathbb{R}$ such that $K = \pi_L(\mathcal{P}_L \cap \tilde{K})$ holds. Let b be a point of $\partial\, \overline{\mathbb{H}}_L$. A linear subspace K of $\mathbb{H}_L$ is said to *pass through b at infinity* if the linear subspace $\tilde{K}$ of $L \otimes \mathbb{R}$ that satisfies $K = \pi_L(\mathcal{P}_L \cap \tilde{K})$ contains a non-zero vector v such that $b = \pi_L(v)$.

We denote by $\mathcal{H}_L$ the set of hyperplanes of $\mathcal{P}_L$. We put

$$\mathcal{N}_L := \{\ v \in L \otimes \mathbb{R}\ \mid\ v^2 < 0\ \},$$

and, for $v \in L \otimes \mathbb{R}$, we put

$$[v]^{\perp} := \{\, x \in L \otimes \mathbb{R} \;\mid\; \langle x, v\rangle = 0 \,\} \quad \text{and} \quad (v)^{\perp} := [v]^{\perp} \cap \mathcal{P}_L.$$

Since $[v]^{\perp}$ intersects $\mathcal{P}_L$ if and only if $v^2 < 0$, the map $v \mapsto (v)^{\perp}$ induces a bijection from $\mathcal{N}_L/\mathbb{R}^{\times}$ to $\mathcal{H}_L$. For $v \in \mathcal{N}_L$, we obtain a hyperplane $\pi_L((v)^{\perp})$ of the hyperbolic space $\mathbb{H}_L$.

We recall some properties of *horospheres*. For the details, see Ratcliffe [24, Chapter 4]. Let b be a point of $\partial\,\overline{\mathbb{H}}_L$. Consider the upper halfspace model

$$\{\, (z_1, \dots, z_m) \in \mathbb{R}^m \;\mid\; z_1 > 0 \,\} \tag{2.1}$$

of $\mathbb{H}_L$ such that b corresponds to the infinite point given by $z_1 = \infty$. Then every horosphere HS_b with the base b is defined by $z_1 = \gamma$ with some positive real constant γ. Therefore, as a Riemannian submanifold of $\mathbb{H}_L$, every horosphere is isomorphic to a Euclidean affine space. If K is a linear subspace of $\mathbb{H}_L$ that passes through b at infinity, then $K \cap HS_b$ is an affine subspace of any horosphere HS_b with the base b. We say that a horosphere defined by $z_1 = \gamma$ is *smaller* than a horosphere defined by $z_1 = \gamma'$ if $\gamma > \gamma'$.

The definition of horospheres can be restated as follows. Let f be a non-zero vector in $\overline{\mathcal{P}}_L$ with $f^2 = 0$ such that $\pi_L(f) = b$. Then the function

$$h_f \colon v \mapsto \langle v, f\rangle^2/v^2$$

on $\mathcal{N}_L \cup \mathcal{P}_L \cup (-\mathcal{P}_L)$ is invariant under the scaling of v by multiplicative constants, and hence its restriction to $\mathcal{P}_L$ induces a function

$$\bar{h}_f \colon \mathbb{H}_L \to \mathbb{R}_{>0}.$$

The horospheres with the base b are exactly the level sets of the function $\bar{h}_f$. Note that the horosphere defined by $\bar{h}_f = \alpha$ is smaller than the horosphere defined by $\bar{h}_f = \alpha'$ if and only if $\alpha < \alpha'$.

The following lemma should be well-known, but we could not find appropriate references. Let f and b be as above. The function $-h_f(v) = -\langle v, f\rangle^2/v^2$ restricted to $\mathcal{N}_L$ measures how far the hyperplane $\pi_L((v)^{\perp})$ of $\mathbb{H}_L$ is from the boundary point b. For a horosphere HS_b with the base b, we put

$$c(f, HS_b) := \sup \{\, -h_f(v) \;\mid\; v \in \mathcal{N}_L,\;\; \pi_L((v)^{\perp}) \cap HS_b \neq \emptyset \,\}.$$

Lemma 2.4. *Suppose that HS_b is defined by $\bar{h}_f = \alpha$. Then we have $c(f, HS_b) \le \alpha$.*

Proof. We choose linear coordinates $(x_0, x_1, \dots, x_m)$ of $L \otimes \mathbb{R}$ such that the quadratic form $x \mapsto x^2$ on $L \otimes \mathbb{R}$ is given by

$$(x_0, x_1, \dots, x_m) \mapsto x_0^2 - x_1^2 - x_2^2 - \cdots - x_m^2,$$

such that $\mathcal{P}_L$ is contained in the halfspace $x_0 > 0$, and such that $f = (1, 1, 0, \dots, 0)$. Let $v = (v_0, v_1, \dots, v_m)$ be a vector in $\mathcal{N}_L$. Suppose that there exists a vector

$x = (x_0, x_1, \dots, x_m)$ in $\mathcal{P}_L$ such that $x \in (v)^\perp$ and that $\pi_L(x) \in HS_b$. It is enough to show that

$$-h_f(v) \le \alpha. \tag{2.2}$$

Rescaling x by a positive real constant, we can assume that $x^2 = 1$. If $\langle v, f\rangle = 0$, then (2.2) holds. Suppose that $\langle v, f\rangle \ne 0$. Replacing v by $-v$ if necessary, we can assume that $\langle v, f\rangle = v_0 - v_1$ is positive. Since $\pi_L(x) \in HS_b$, we have $h_f(x) = \alpha$. Since $x \in \mathcal{P}_L$, we have $\langle x, f\rangle = x_0 - x_1 > 0$. Combining these, we get

$$x_0 - x_1 = \sqrt{\alpha}.$$

Using $\langle v, x\rangle = v_0x_0 - v_1x_1 - \cdots - v_mx_m = 0$, we obtain

$$x_0 = \frac{(v_2x_2 + \cdots + v_mx_m) - v_1\sqrt{\alpha}}{v_0 - v_1}, \quad x_1 = \frac{(v_2x_2 + \cdots + v_mx_m) - v_0\sqrt{\alpha}}{v_0 - v_1}.$$

Combining these with $x^2 = 1$ and using $v_0 \ne v_1$, we get

$$g_2(x_2) + \cdots + g_m(x_m) + \alpha(v_0 + v_1) + (v_0 - v_1) = 0, \tag{2.3}$$

where

$$g_i(t) = (v_0 - v_1)\, t^2 - 2\sqrt{\alpha}\, v_i t.$$

Since $\langle v, f\rangle = v_0 - v_1 > 0$, we have $g_i(t) \ge -\alpha v_i^2/(v_0 - v_1)$ for any $t \in \mathbb{R}$. Thus we obtain $\alpha v^2 + \langle v, f\rangle^2 \le 0$ from (2.3). Since $v^2 < 0$, we get (2.2). $\square$

3. Chamber decomposition

Let L be an even hyperbolic lattice with a fixed positive cone $\mathcal{P}_L$. For a subset Δ of $\mathcal{N}_L = \{v \in L \otimes \mathbb{R} \mid v^2 < 0\}$, we define a cone $\Sigma_L(\Delta)$ in $L \otimes \mathbb{R}$ by

$$\Sigma_L(\Delta) := \{\ x \in L \otimes \mathbb{R} \ \mid\ \langle x, v\rangle \ge 0 \ \text{ for all }\ v \in \Delta\ \}. \tag{3.1}$$

Definition 3.1. A closed subset D of $\mathcal{P}_L$ is called a *chamber* if its interior D° is non-empty and there exists a subset Δ of $\mathcal{N}_L$ such that $D = \Sigma_L(\Delta) \cap \mathcal{P}_L$ holds. Let D be a chamber. A hyperplane $(v)^\perp$ of $\mathcal{P}_L$ is called a *wall of D* if $(v)^\perp \cap D^\circ$ is empty and $(v)^\perp \cap D$ contains a non-empty open subset of $(v)^\perp$.

Definition 3.2. Let $\mathcal{F} \subset \mathcal{H}_L$ be a locally finite family of hyperplanes in $\mathcal{P}_L$. Then the closure in $\mathcal{P}_L$ of each connected component of

$$\mathcal{P}_L \setminus \bigcup_{(v)^\perp \in \mathcal{F}} (v)^\perp$$

is a chamber, which we call an *$\mathcal{F}$-chamber*.

By definition, every wall of an $\mathcal{F}$-chamber is an element of $\mathcal{F}$. If D and D' are distinct $\mathcal{F}$-chambers, then $D^\circ \cap D' = \emptyset$ holds.

Definition 3.3. Let D be an $\mathcal{F}$-chamber, and let $(v)^\perp \in \mathcal{F}$ be a wall of D. Then there exists a unique $\mathcal{F}$-chamber D' such that $D' \neq D$ and that $D \cap D' \cap (v)^\perp$ contains a non-empty open subset of $(v)^\perp$. We say that D' is *adjacent to D across the wall $(v)^\perp$*.

We fix a subgroup G of $\mathrm{O}^+(L)$ with finite index. We assume that G satisfies the following condition of the existence of a membership algorithm:

[G] There exists an algorithm by which we can determine, for a given $g \in \mathrm{O}^+(L)$, whether $g \in G$ or not.

Let $\mathcal{V}$ be a subset of $\mathcal{N}_L \cap L^\vee = \{v \in L^\vee \mid v^2 < 0\}$, and consider the family of hyperplanes

$$\mathcal{V}^* := \{ \ (v)^\perp \ \mid \ v \in \mathcal{V} \ \}$$

in $\mathcal{P}_L$. We assume that $\mathcal{V}$ has the following properties:

[V1] There exists a positive real number $c \in \mathbb{R}$ such that, for any $v \in \mathcal{V}$, we have $-v^2 < c$.

[V2] The set $\mathcal{V}$ is invariant under the action of G on $\mathcal{N}_L \cap L^\vee$.

Then we can consider $\mathcal{V}^*$-chambers by the following:

Lemma 3.4. *The family $\mathcal{V}^*$ of hyperplanes is locally finite in $\mathcal{P}_L$.*

Proof. Since $\{v^2 \mid v \in L^\vee\}$ is discrete in $\mathbb{R}$, the subset $\{v^2 \mid v \in \mathcal{V}\}$ of the interval $(-c, 0) \subset \mathbb{R}$ is finite by [V1] and $\mathcal{V} \subset \mathcal{N}_L \cap L^\vee$. For a negative real number a and a compact subset J of $\mathcal{P}_L$, the locus

$$\{ \ x \in L \otimes \mathbb{R} \ \mid \ x^2 = a, \ (x)^\perp \cap J \neq \emptyset \ \}$$

is compact. Since $L^\vee$ is discrete in $L \otimes \mathbb{R}$, we obtain the proof of Lemma 3.4. □

Definition 3.5. Let D be a $\mathcal{V}^*$-chamber. A subset Δ of $\mathcal{N}_L \cap L^\vee$ is called a *defining set of D* if $D = \Sigma_L(\Delta) \cap \mathcal{P}_L$ holds. A defining set Δ of D is said to be *minimal* if the following hold:

- For any $v \in \Delta$, the hyperplane $(v)^\perp$ is a wall of D, and
- if v and v' are distinct vectors of Δ, then $(v)^\perp \neq (v')^\perp$.

We define two types of minimal defining sets, each of which is unique for a given $\mathcal{V}^*$-chamber D. The one is called the *$\mathcal{V}$-minimal defining set*, denoted by $\Delta_{\mathcal{V}}(D)$, and characterized by the following property: $\Delta_{\mathcal{V}}(D)$ is a subset of $\mathcal{V}$, and if $v \in \Delta_{\mathcal{V}}(D)$, then $\alpha v \notin \mathcal{V}$ for any $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. The other is called the *primitively minimal defining set*, denoted by $\Delta_{L^\vee}(D)$, and characterized by the following property: Every $v \in \Delta_{L^\vee}(D)$ is primitive in $L^\vee$. (Note that $\Delta_{L^\vee}(D)$ may not be contained in $\mathcal{V}$, because elements of $\mathcal{V}$ need not be primitive in $L^\vee$.)

Let D be a $\mathcal{V}^*$-chamber. Then D^g is also a $\mathcal{V}^*$-chamber for any $g \in G$ by the property [V2]. For a $\mathcal{V}^*$-chamber D, we put

$$Aut_G(D) := \{ \ g \in G \ \mid \ D^g = D \ \}.$$

Let D and D' be $\mathcal{V}^*$-chambers. We say that D and D' are *G-congruent* if there exists an element $g \in G$ such that $D' = D^g$. The following simple observation is the key point of our method:

$$\text{If } g \in G \text{ satisfies } D^g \cap D'^\circ \neq \emptyset, \text{ then } D^g = D'. \tag{3.2}$$

In particular, if $g \in G$ satisfies $D^g \cap D^\circ \neq \emptyset$, then $g \in Aut_G(D)$.

Example 3.6. The subset $\mathcal{R}_L$ of $\mathcal{N}_L \cap L^\vee$ has the properties [V1] and [V2] for $G = \mathrm{O}^+(L)$. Let D be an $\mathcal{R}_L^*$-chamber. Then D is a fundamental domain of the action of the Weyl group $W(L)$ on $\mathcal{P}_L$, and $W(L)$ is generated by the reflections s_r, where r runs through the $\mathcal{R}_L$-minimal defining set $\Delta_{\mathcal{R}_L}(D)$ of D. Moreover, any two $\mathcal{R}_L^*$-chambers are $\mathrm{O}^+(L)$-congruent, and $\mathrm{O}^+(L)$ is isomorphic to the semi-direct product $W(L) \rtimes Aut_{\mathrm{O}^+(L)}(D)$.

We consider the following properties of $\mathcal{V}$:

[V3] Any $\mathcal{V}^*$-chamber has a *finite* defining set.

[V4] For any $\mathcal{V}^*$-chamber D, the set $\pi_L(\overline{D}) \cap \partial\,\overline{\mathbb{H}}_L$ is contained in $\partial\,\overline{\mathbb{H}}_L^{\mathbb{Q}}$, where $\overline{D}$ is the closure of D in $\overline{\mathcal{P}}_L$.

The main results of this section are the following:

Theorem 3.7. *Suppose that $\mathcal{V}$ satisfies* [V1]-[V4]. *Then there exist only a finite number of G-congruence classes of $\mathcal{V}^*$-chambers.*

Theorem 3.8. *Suppose that $\mathcal{V}$ satisfies* [V1]-[V4]. *Then the group $Aut_G(D)$ is finite for any $\mathcal{V}^*$-chamber D.*

3.1. **Proof of Theorem 3.7.** We assume that $\mathcal{V}$ satisfies [V1]-[V4].

Definition 3.9. A subset Π of $\overline{\mathcal{P}}_L^{\mathbb{Q}}$ is called a *rational polyhedral cone* if there exist a finite number of non-zero vectors $v_1, \dots, v_n \in \overline{\mathcal{P}}_L \cap L$ such that

$$\Pi = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n.$$

Recall that G is assumed to be of finite index in $\mathrm{O}^+(L)$. We have the following result from the reduction theory of arithmetic subgroups of $\mathrm{O}(L)$ (see Ash et al. [2, Chapter II, Section 4] and Sterk [36]).

Theorem 3.10. *There exist a finite number of rational polyhedral cones $\Pi_1, \dots, \Pi_N$ in $\overline{\mathcal{P}}_L^{\mathbb{Q}}$ such that $\mathcal{P}_L$ is equal to*

$$\bigcup_{g \in G} \bigcup_{i=1}^{N} (\Pi_i^g \cap \mathcal{P}_L).$$

Therefore Theorem 3.7 follows from the following:

Proposition 3.11. *Let Π be a rational polyhedral cone in $\overline{\mathcal{P}}_L^{\mathbb{Q}}$. Then the number of $\mathcal{V}^*$-chambers that intersect $\Pi \cap \mathcal{P}_L$ is finite.*

For the proof of Proposition 3.11, we need two corollaries of Lemma 2.4. Let b be a point of $\partial\,\overline{\mathbb{H}}_L$. A *closed horoball* HB_b with the base b is a subset of $\mathbb{H}_L$ defined by $z_1 \geq \gamma$ with some positive real constant γ in the upper halfspace model (2.1) with b at $z_1 = \infty$. Let ∂HB_b be the horosphere defined by $z_1 = \gamma$. The map $\rho_b \colon HB_b \to \partial HB_b$ defined by

$$\rho_b(z_1, z_2, \dots, z_m) := (\gamma, z_2, \dots, z_m)$$

is called the *natural projection*. Let $b = \pi_L(f) \in \partial\,\overline{\mathbb{H}}_L^{\mathbb{Q}}$ be a rational boundary point, where f is a non-zero vector in $\overline{\mathcal{P}}_L \cap L$ with $f^2 = 0$. We put

$$\mathcal{V}_b := \{\ v \in \mathcal{V} \ \mid\ \langle v, f \rangle = 0\ \}.$$

If $v \in \mathcal{V}$ satisfies $v \notin \mathcal{V}_b$, then we have $\langle v, f\rangle^2 \geq 1$ because $f \in L$ and $\mathcal{V} \subset L^{\vee}$. By the property [V1], there exists a positive real number δ_b such that

$$\delta_b < -\langle v, f\rangle^2/v^2 \quad \text{for any } v \in \mathcal{V} \setminus \mathcal{V}_b.$$

Therefore we obtain the following corollary of Lemma 2.4:

Corollary 3.12. *Let b be a rational boundary point. If we choose a sufficiently small closed horoball HB_b with the base b, then the following hold.*

(1) *Let v be an element of $\mathcal{V}$. Then the hyperplane $\pi_L((v)^{\perp})$ of $\mathbb{H}_L$ intersects HB_b if and only if $\pi_L((v)^{\perp})$ passes through b at infinity.*

(2) *Let D be a $\mathcal{V}^*$-chamber. If $b \notin \pi_L(\overline{D})$, then $\pi_L(D) \cap HB_b$ is empty, whereas if $b \in \pi_L(\overline{D})$, then $\pi_L(D) \cap HB_b = \rho_b^{-1}(\pi_L(D) \cap \partial HB_b)$ holds.*

We regard ∂HB_b as a Euclidean affine space. Then the family of affine hyperplanes

$$\{\ \pi_L((v)^{\perp}) \cap \partial HB_b \ \mid\ v \in \mathcal{V}_b\ \}$$

of ∂HB_b is locally finite, because $\mathcal{V}^*$ is locally finite in $\mathcal{P}_L$ and $\pi_L^{-1}(\partial HB_b) \subset \mathcal{P}_L$. Therefore we obtain the following:

Corollary 3.13. *Let b and HB_b be as in Corollary 3.12, and let J be a compact subset of ∂HB_b. Then the number of $\mathcal{V}^*$-chambers that intersect the subset $\pi_L^{-1}(\rho_b^{-1}(J))$ of $\mathcal{P}_L$ is finite.*

Proof of Proposition 3.11. Let $v_1, \dots, v_n \in \overline{\mathcal{P}}_L \cap L$ be non-zero vectors such that rational polyhedral cone Π is equal to $\mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_n$. We number $v_1, \dots, v_n$ in such a way that

$$v_1^2 = \dots = v_k^2 = 0 \quad \text{and} \quad v_{k+1}^2 > 0,\ \dots,\ v_n^2 > 0.$$

Let b_i be the rational boundary point $\pi_L(v_i)$ for $i = 1, \dots, k$, and let HB_i be a sufficiently small closed horoball with the base b_i. The natural projection is denoted by $\rho_i : HB_i \to \partial HB_i$. We put $HB_i^\circ := HB_i \setminus \partial HB_i$. Since $\pi_L(\Pi) \cap \partial \overline{\mathbb{H}}_L = \{b_1, \dots, b_k\}$, we see that

$$\pi_L(\Pi)' := \pi_L(\Pi) \setminus \bigcup_{i=1}^{k} (HB_i^\circ \cap \pi_L(\Pi))$$

is a compact subset of $\mathbb{H}_L$. Therefore the number of $\mathcal{V}^*$-chambers D such that $\pi_L(D)$ intersects $\pi_L(\Pi)'$ is finite. For $j \neq i$, let $p_{i,j} \in \partial HB_i$ denote the intersection point of ∂HB_i and the geodesic line in $\mathbb{H}_L$ passing through b_i at infinity and passing through $\pi_L(v_j)$ possibly at infinity. Then the convex hull J_i of these points $p_{i,j}$ with $j \neq i$ in the Euclidean affine space ∂HB_i is compact and satisfies $HB_i \cap \pi_L(\Pi) = \rho_i^{-1}(J_i)$. Consequently, the number of $\mathcal{V}^*$-chambers D such that $\pi_L(D)$ intersects $HB_i \cap \pi_L(\Pi)$ is finite by Corollary 3.13. □

3.2. **Proof of Theorem 3.8.** We continue to assume that $\mathcal{V}$ satisfies [V1]-[V4].

Lemma 3.14. *Let Δ be a subset of $\mathcal{N}_L \cap L^\vee$ such that $D = \Sigma_L(\Delta) \cap \mathcal{P}_L$ is a $\mathcal{V}^*$-chamber. Then $\Sigma_L(\Delta)$ is contained in $\overline{\mathcal{P}}_L$.*

Proof. Note that $\Sigma_L(\Delta)$ is a closed convex subset of $L \otimes \mathbb{R}$. Suppose that there exists an element $x_0 \in \Sigma_L(\Delta)$ such that $x_0 \notin \overline{\mathcal{P}}_L$. Let U be a non-empty open subset of D°. Then, for any $y \in U$, the line segment $\overline{x_0 y}$ of $L \otimes \mathbb{R}$ connecting x_0 and y is contained in $\Sigma_L(\Delta)$, and $\overline{x_0 y} \cap \mathcal{P}_L$ is contained in D. Hence the intersection point $z(y)$ of $\overline{x_0 y}$ and $\partial \overline{\mathcal{P}}_L = \{x \in \overline{\mathcal{P}}_L \mid x^2 = 0\}$ belongs to the closure $\overline{D}$ of D in $\overline{\mathcal{P}}_L$. Since U is open, the subset $\pi_L(\{z(y) \mid y \in U\})$ of $\pi_L(\overline{D}) \cap \partial \overline{\mathbb{H}}_L$ has uncountably many points, which contradicts the property [V4] of $\mathcal{V}$. □

Lemma 3.15. *Let D be a $\mathcal{V}^*$-chamber. Then any defining set Δ of D spans $L \otimes \mathbb{R}$.*

Proof. Let V be the linear subspace of $L \otimes \mathbb{R}$ spanned by Δ. The orthogonal complement $V^\perp$ of V in $L \otimes \mathbb{R}$ is contained in $\Sigma_L(\Delta)$ by definition and hence $V^\perp \subset \overline{\mathcal{P}}_L$ by Lemma 3.14. This holds only when $V^\perp = 0$. □

Theorem 3.8 is now easy to prove by [V3]. We give, however, a proof based on an algorithm (Algorithm 3.18) to compute $Aut_G(D)$.

Lemma 3.16. *Suppose that a defining set Δ_1 of a $\mathcal{V}^*$-chamber D satisfies the following; if $v, v' \in \Delta_1$ are distinct, then $(v)^\perp \neq (v')^\perp$ holds. Let v be an element of Δ_1.* (1) *The hyperplane $(v)^\perp$ is a wall of D if and only if $\Sigma_L(\Delta_1) \neq \Sigma_L(\Delta_1 \setminus \{v\})$.* (2) *If $\Delta_1 \setminus \{v\}$ does not span $L \otimes \mathbb{R}$, then $(v)^\perp$ is a wall of D.*

Proof. The 'only if' part of (1) is obvious by the assumption on Δ_1. Conversely, suppose that $\Sigma_L(\Delta_1) \neq \Sigma_L(\Delta_1 \setminus \{v\})$. Then there exists a vector $x_0 \in L \otimes \mathbb{R}$ such that $\langle v, x_0 \rangle < 0$ and $x_0 \in \Sigma_L(\Delta_1 \setminus \{v\})$. Assume that $(v)^\perp$ is not a wall of D. Then D is equal to $\Sigma_L(\Delta_1 \setminus \{v\}) \cap \mathcal{P}_L$, and hence $\Sigma_L(\Delta_1 \setminus \{v\})$ is contained in $\overline{\mathcal{P}}_L$ by Lemma 3.14. In particular, we have $x_0 \in \overline{\mathcal{P}}_L$. Let y_0 be a point of D°. Then the intersection point z_0 of the line segment $\overline{x_0 y_0}$ and $(v)^\perp$ satisfies $z_0^2 > 0$, $\langle v, z_0 \rangle = 0$ and $\langle v', z_0 \rangle > 0$ for any v' in $\Delta_1 \setminus \{v\}$. Since $\mathcal{V}^*$ is locally finite in $\mathcal{P}_L$, these mean that a sufficiently small open neighborhood of z_0 in $(v)^\perp$ is contained in D. In anyway, $(v)^\perp$ is a wall of D. Thus (1) is proved. If $(v)^\perp$ is not a wall of D, then $\Delta_1 \setminus \{v\}$ is also a defining set of D. Hence (2) follows from Lemma 3.15. □

Algorithm 3.17. Let Δ be a *finite* defining set of a $\mathcal{V}^*$-chamber D. This algorithm calculates the primitively minimal defining set $\Delta_{L^\vee}(D)$ of D.

Step 0. We set $\Delta_1 := \{\}$ and $\Delta_2 := \{\}$.

Step 1. For each element $v \in \Delta$, we calculate the maximal positive integer a_v such that $v/a_v \in L^\vee$, and append v/a_v to the set Δ_1. Then we have $D = \Sigma_L(\Delta_1) \cap \mathcal{P}_L$, and moreover, if $v, v' \in \Delta_1$ are distinct, then $(v)^\perp \neq (v')^\perp$ holds.

Step 2. For each $v \in \Delta_1$, we carry out the following computation. Suppose that $\Delta_1 \setminus \{v\}$ does not span $L \otimes \mathbb{R}$. Then $(v)^\perp$ is a wall of D by Lemma 3.16, and we append v to Δ_2. Suppose that $\Delta_1 \setminus \{v\}$ spans $L \otimes \mathbb{R}$. Then we can solve the following problem of linear programming, in which the variable x ranges through the vector space $L \otimes \mathbb{Q}$:

$$\begin{cases} \text{minimize} & \langle v, x \rangle \\ \text{subject to} & \langle v', x \rangle \geq 0 \ \text{ for all } \ v' \in \Delta_1 \setminus \{v\}. \end{cases} \tag{3.3}$$

(See, for example, Chvátal [6] for the algorithms of linear programming.) Note that the solution is either 0 or unbounded to $-\infty$. If the solution is 0, then $(v)^\perp$ is not a wall of D by Lemma 3.16. Suppose that the solution is unbounded to $-\infty$. Then there exists a vector $x_0 \in L \otimes \mathbb{R}$ such that $\langle v, x_0 \rangle < 0$ and $x_0 \in \Sigma_L(\Delta_1 \setminus \{v\})$. Hence $(v)^\perp$ is a wall of D by Lemma 3.16, and we append v to the set Δ_2.

Step 3. We then output Δ_2 as $\Delta_{L^\vee}(D)$.

Remark that, for any $\mathcal{V}^*$-chamber D, the primitively minimal defining set $\Delta_{L^\vee}(D)$ is finite by the property [V3] of $\mathcal{V}$ and Algorithm 3.17. We use the obvious brute-force method based on the finiteness of $\Delta_{L^\vee}(D)$ in the following two algorithms.

Algorithm 3.18. Suppose that the primitively minimal defining set $\Delta_{L^\vee}(D)$ of a $\mathcal{V}^*$-chamber D is given. This algorithm calculates all elements of $Aut_G(D)$. Let $\Delta_{L^\vee}(D)^l$ denote the set of ordered l-tuples of distinct elements of $\Delta_{L^\vee}(D)$, where $l := \operatorname{rank} L$. By Lemma 3.15, there exists an l-tuple $[v_1, \dots, v_l] \in \Delta_{L^\vee}(D)^l$ that forms a basis of $L \otimes \mathbb{Q}$. We set $A := \{\}$. For each $[v'_1, \dots, v'_l] \in \Delta_{L^\vee}(D)^l$, we calculate the linear transformation g of $L \otimes \mathbb{Q}$ such that

$$v_i^g = v'_i \qquad (i = 1, \dots, l).$$

Recall that we can determine whether $g \in G$ or not by the assumption [G]. If g belongs to G and induces a permutation of $\Delta_{L^\vee}(D)$, then we append g to A. When this calculation is done for all $[v'_1, \dots, v'_l] \in \Delta_{L^\vee}(D)^l$, the set A is equal to $Aut_G(D)$.

Proof of Theorem 3.8. Since $\Delta_{L^\vee}(D)$ is finite, Algorithm 3.18 terminates in finite steps. □

Algorithm 3.19. Let D and D' be $\mathcal{V}^*$-chambers. Suppose that $\Delta_{L^\vee}(D)$ and $\Delta_{L^\vee}(D')$ are given. This algorithm determines whether D is G-congruent to D' or not. We fix an element $[v_1, \dots, v_l]$ of $\Delta_{L^\vee}(D)^l$ that forms a basis of $L \otimes \mathbb{Q}$. For each $[v_1'', \dots, v_l''] \in \Delta_{L^\vee}(D')^l$, we calculate the linear transformation g of $L \otimes \mathbb{Q}$ that satisfies $v_i^g = v_i''$ for $i = 1, \dots, l$. If g belongs to G and induces a bijection from $\Delta_{L^\vee}(D)$ to $\Delta_{L^\vee}(D')$, then D and D' are G-congruent. If no such $[v_1'', \dots, v_l'']$ are found, then D and D' are not G-congruent.

Remark 3.20. Suppose that p is a point in the interior of D. By (3.2), we see that an element $g \in G$ is contained in $Aut_G(D)$ if and only if $p^g \in D$, and $g \in G$ induces an isomorphism from D to D' if and only if $p^g \in D'$.

4. Vinberg-Conway theory

Let n be 10, 18 or 26. Throughout this section, we denote by $\mathbf{L}$ an even unimodular hyperbolic lattice $\mathrm{II}_{1,n-1}$ of rank n. Note that $\mathbf{L}$ exists and is unique up to isomorphism (see, for example, [27, Chapter V]). We fix a positive cone $\mathcal{P}_{\mathbf{L}}$ of $\mathbf{L}$. Vinberg [38] and Conway [7, Chapter 27] described the structure of $\mathcal{R}_{\mathbf{L}}^*$-chambers; that is, the standard fundamental domains of the action of $W(\mathbf{L})$ on $\mathcal{P}_{\mathbf{L}}$ (see Example 3.6).

Definition 4.1. Let $\mathcal{D}$ be an $\mathcal{R}_{\mathbf{L}}^*$-chamber. We say that a vector $w \in \mathbf{L}$ is a *Weyl vector of* $\mathcal{D}$ if the $\mathcal{R}_{\mathbf{L}}$-minimal defining set $\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D})$ of $\mathcal{D}$ is given by

$$\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D}) = \{ \, r \in \mathcal{R}_{\mathbf{L}} \;\mid\; \langle w, r \rangle = 1 \, \}.$$

If a Weyl vector of an $\mathcal{R}_{\mathbf{L}}^*$-chamber $\mathcal{D}$ exists, then it is unique, because, as will be shown below, $\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D})$ spans $\mathbf{L} \otimes \mathbb{R}$. If w is the Weyl vector of $\mathcal{D}$, then w^g is the Weyl vector of $\mathcal{D}^g$ for any $g \in \mathrm{O}^+(\mathbf{L})$. Since any two $\mathcal{R}_{\mathbf{L}}^*$-chambers are $\mathrm{O}^+(\mathbf{L})$-congruent, the Weyl vector of a single $\mathcal{R}_{\mathbf{L}}^*$-chamber gives Weyl vectors of all $\mathcal{R}_{\mathbf{L}}^*$-chambers via the action of $\mathrm{O}^+(\mathbf{L})$.

Theorem 4.2 (Conway, Chapter 27 of [7])**.** *For any $\mathcal{R}_{\mathbf{L}}^*$-chamber $\mathcal{D}$, there exists a Weyl vector $w \in \mathbf{L}$ of $\mathcal{D}$. We have*

$$w^2 = \begin{cases} 1240 & \textit{if } n = 10 \\ 620 & \textit{if } n = 18 \\ 0 & \textit{if } n = 26 \end{cases} \quad \textit{and} \quad |\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D})| = \begin{cases} 10 & \textit{if } n = 10 \\ 19 & \textit{if } n = 18 \\ \infty & \textit{if } n = 26. \end{cases}$$

Remark 4.3. The finite sets $\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D})$ for $n = 10$ and 18 had been calculated by Vinberg [38].

We give an explicit description of the Weyl vectors and the set $\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D})$, and prove the following:

Proposition 4.4. *If $\mathcal{D}$ is an $\mathcal{R}^*_{\mathbf{L}}$-chamber, then $\pi_{\mathbf{L}}(\overline{\mathcal{D}}) \cap \partial\,\overline{\mathbb{H}}_{\mathbf{L}}$ is contained in $\partial\,\overline{\mathbb{H}}^{\mathbb{Q}}_{\mathbf{L}}$.*

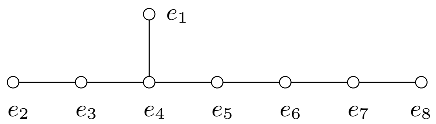
In the following, we denote by U the even hyperbolic lattice of rank 2 with a fixed basis f_U, z_U, with respect to which the Gram matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \tag{4.1}$$

and by $f_U^\vee, z_U^\vee$ the basis of U dual to f_U, z_U.

Remark 4.5. We choose this non-standard basis of U for geometric reasons; f_U will be the class of a fiber of a Jacobian fibration on a $K3$ surface and z_U will be the class of the zero section. See Sections 9 and 10.

Let E_8 denote the (negative-definite) root lattice of type E_8 with the standard basis $e_1, \dots, e_8$, whose Coxeter graph is



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We denote by $e_1^\vee, \dots, e_8^\vee$ the basis of E_8 dual to $e_1, \dots, e_8$. We put

$$\theta := 3e_1 + 2e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8.$$

4.1. **The case where $n = 10$.** We put $\mathbf{L} := U \oplus E_8$, and choose $\mathcal{P}_{\mathbf{L}}$ in such a way that $2f_U + z_U \in \mathcal{P}_{\mathbf{L}}$. Then the vector

$$w_0 := 30 f_U^\vee + z_U^\vee + e_1^\vee + \cdots + e_8^\vee$$

is the Weyl vector of an $\mathcal{R}^*_{\mathbf{L}}$-chamber $\mathcal{D}_0$. By Vinberg [38], we have

$$\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D}_0) = \{z_U, e_1, \dots, e_8, f_U - \theta\},$$

and

$$\pi_{\mathbf{L}}(\overline{\mathcal{D}}_0) \cap \partial\,\overline{\mathbb{H}}_{\mathbf{L}} = \{\pi_{\mathbf{L}}(f_U)\} \subset \partial\,\overline{\mathbb{H}}^{\mathbb{Q}}_{\mathbf{L}}.$$

4.2. **The case where $n = 18$.** We put $\mathbf{L} := U \oplus E_8 \oplus E_8$, and choose $\mathcal{P}_{\mathbf{L}}$ in such a way that $2f_U + z_U \in \mathcal{P}_{\mathbf{L}}$. Let $e'_1, \dots, e'_8$ be the basis of the second E_8 with the same Coxeter graph as $e_1, \dots, e_8$. Then the vector

$$w_0 := 30 f_U^\vee + z_U^\vee + e_1^\vee + \cdots + e_8^\vee + e_1'^\vee + \cdots + e_8'^\vee$$

is the Weyl vector of an $\mathcal{R}^*_{\mathbf{L}}$-chamber $\mathcal{D}_0$. By Vinberg [38], we have

$$\Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D}_0) = \{z_U, e_1, \dots, e_8, e'_1, \dots, e'_8, f_U - \theta, f_U - \theta'\},$$

where θ' is defined in the same way as θ with e_i replaced by e'_i. Moreover, we have

$$\pi_{\mathbf{L}}(\overline{\mathcal{D}}_0) \cap \partial\,\overline{\mathbb{H}}_{\mathbf{L}} = \{\pi_{\mathbf{L}}(f_U), \pi_{\mathbf{L}}(v_1)\} \subset \partial\,\overline{\mathbb{H}}^{\mathbb{Q}}_{\mathbf{L}},$$

where

$$v_1 := e_1 + e_3 + e'_1 + e'_3 + 2(z_U + (f_U - \theta) + (f_U - \theta') + e_4 + \cdots + e_8 + e'_4 + \cdots + e'_8).$$

4.3. **The case where $n = 26$.** We denote by Λ the *negative-definite* Leech lattice; that is, Λ is an even unimodular negative-definite lattice of rank 24 such that $\mathcal{R}_\Lambda = \emptyset$. We put $\mathbf{L} := U \oplus \Lambda$, and choose $\mathcal{P}_\mathbf{L}$ in such a way that $2f_U + z_U \in \mathcal{P}_\mathbf{L}$.

Theorem 4.6 (Conway and Sloane, Chapter 26 of [7])**.** *A vector $w \in \mathbf{L}$ is a Weyl vector of some $\mathcal{R}^*_\mathbf{L}$-chamber if and only if w is a non-zero primitive vector of norm 0 contained in $\overline{\mathcal{P}}_\mathbf{L}$ such that $\langle w\rangle^\perp/\langle w\rangle$ is isomorphic to Λ, where $\langle w\rangle^\perp$ is the orthogonal complement of the primitive submodule $\langle w\rangle$ of $\mathbf{L}$.*

Therefore $w_0 := f_U$ is the Weyl vector of an $\mathcal{R}^*_\mathbf{L}$-chamber $\mathcal{D}_0$. For simplicity, we denote vectors of $\mathbf{L} \otimes \mathbb{R} = (U \oplus \Lambda) \otimes \mathbb{R}$ by

$$[s, t, y] := sf_U + tz_U + y, \quad \text{where} \quad s, t \in \mathbb{R},\ \ y \in \Lambda \otimes \mathbb{R}.$$

Then we have

$$\Delta_{\mathcal{R}_\mathbf{L}}(\mathcal{D}_0) = \{\ r_\lambda \ \mid\ \lambda \in \Lambda\ \}, \quad \text{where} \quad r_\lambda := \left[-\lambda^2/2, 1, \lambda\right].$$

Conway, Parker and Sloane [8] ([7, Chapter 23]) proved that the covering radius of the Leech lattice is $\sqrt{2}$ (see also Borcherds [3]). A point c of $\Lambda \otimes \mathbb{R}$ is called a *deep hole* if $-(c-\lambda)^2 \geq 2$ holds for any $\lambda \in \Lambda$. In Lemma 4.4 of [4], Borcherds observed the following:

Lemma 4.7. *Let b be a point of $\pi_\mathbf{L}(\overline{\mathcal{D}}_0) \cap \partial\, \overline{\mathbb{H}}_\mathbf{L}$. Then either $b = \pi_\mathbf{L}(w_0)$ or there exists a deep hole c such that*

$$b = \pi_\mathbf{L}(v_c), \quad \textit{where} \quad v_c := \left[-c^2/2 + 1, 1, c\right].$$

In particular, the set $\pi_\mathbf{L}(\overline{\mathcal{D}}_0) \cap \partial\, \overline{\mathbb{H}}_\mathbf{L}$ is contained in $\partial\, \overline{\mathbb{H}}^{\mathbb{Q}}_\mathbf{L}$.

Proof. Note that $\overline{\mathcal{P}}_\mathbf{L}$ is contained in $\{x \in \mathbf{L} \otimes \mathbb{R} \mid \langle w_0, x\rangle \geq 0\}$, and that $x \in \overline{\mathcal{P}}_\mathbf{L}$ satisfies $\langle x, w_0\rangle = 0$ if and only if x belongs to the half-line $\mathbb{R}_{\geq 0} w_0$. Suppose that $b = \pi_\mathbf{L}(u)$, where $u = [s, t, y]$ is a non-zero vector of norm 0 in $\overline{\mathcal{D}}_0$. Since $u \in \overline{\mathcal{P}}_\mathbf{L}$, we have $t = \langle w_0, u\rangle \geq 0$, and $t = 0$ holds if and only if $\mathbb{R}_{\geq 0} w_0 = \mathbb{R}_{\geq 0} u$. Hence $t = 0$ implies $b = \pi_\mathbf{L}(w_0)$. Suppose that $t > 0$. We can assume that $t = 1$. Since $u^2 = 0$, we have $2s - 2 + y^2 = 0$. Since $u \in \overline{\mathcal{D}}_0$, we have

$$\langle u, r_\lambda\rangle = -\frac{(y-\lambda)^2}{2} - 1 \geq 0 \ \text{ for any }\ \lambda \in \Lambda.$$

Therefore $y \in \Lambda \otimes \mathbb{R}$ is a deep hole c and $u = v_c$ holds. Let $p_1, \dots, p_N$ be the points of Λ nearest to c; that is, $p_1, \dots, p_N$ are the points of Λ satisfying $(p_i - c)^2 = -2$. Then their differences $p_i - p_j$ span $\Lambda \otimes \mathbb{Q}$, and c is the intersection point of the bisectors of distinct two points of $p_1, \dots, p_N$. Hence c belongs to $\Lambda \otimes \mathbb{Q}$, and we have $\pi_\mathbf{L}(u) \in \partial\, \overline{\mathbb{H}}^{\mathbb{Q}}_\mathbf{L}$. □

Remark 4.8. The coordinates of deep holes of Λ are explicitly given in Conway, Parker and Sloane [8].

Remark 4.9. We have an isomorphism

$$\mathbf{L} \cong U \oplus E_8 \oplus E_8 \oplus E_8.$$

The vector $w_E \in U \oplus E_8^3$ given by $[30, 1, 1^8, 1^8, 1^8]$ in terms of the dual basis $f^\vee, z^\vee, e_1^\vee, \dots, e_8''^\vee$ is a Weyl vector of $\mathbf{L}$. Indeed, let $u \in U \oplus E_8^3$ be the vector given by $[1, -2, 0^8, 0^8, 0^8]$ in terms of the dual basis above. Since $w_E^2 = 0$ and $\langle w_0, u \rangle = 1$, the vectors w_E and u span an even hyperbolic unimodular lattice of rank 2, and its orthogonal complement in $U \oplus E_8^3$ is isomorphic to $\langle w_E \rangle^\perp / \langle w_E \rangle$. Calculating a Gram matrix of this orthogonal complement and using Algorithm 2.1, we confirm that $\langle w_E \rangle^\perp / \langle w_E \rangle$ has no (-2)-vectors, and therefore is isomorphic to Λ.

5. Generalized Borcherds' method

Suppose that S is an even hyperbolic lattice with a fixed positive cone $\mathcal{P}_S$, and let G be a subgroup of $\mathrm{O}^+(S)$ with finite index satisfying the existence of membership algorithm [G] in Section 3 with L replaced by S. We present an algorithm that calculates a set of generators of $Aut_G(N) = \{g \in G \mid N^g = N\}$ for a given $\mathcal{R}_S^*$-chamber N under the assumptions [SG1], [SG2] and [SG3] below.

We recall the definition of the discriminant form of an even lattice. See Nikulin [17] for the details. For an even lattice L, the *discriminant group* $A_L := L^\vee / L$ is equipped with a non-degenerate quadratic form

$$q_L \colon A_L \to \mathbb{Q}/2\mathbb{Z}, \qquad x \bmod L \mapsto x^2 \bmod 2\mathbb{Z},$$

which is called the *discriminant form of* L. Let $\mathrm{O}(q_L)$ denote the group of automorphisms of (A_L, q_L). We have a natural homomorphism

$$\eta_L \colon \mathrm{O}(L) \to \mathrm{O}(q_L).$$

For an isomorphism $\delta \colon (A_1, q_1) \xrightarrow{\sim} (A_2, q_2)$ of discriminant forms, we denote by

$$\delta_* \colon \mathrm{O}(q_1) \xrightarrow{\sim} \mathrm{O}(q_2)$$

the induced isomorphism on the automorphism groups.

Let n be 10, 18 or 26. As in Section 4, we denote by $\mathbf{L}$ an even unimodular hyperbolic lattice of rank n. We assume that S satisfies the following embeddability condition:

[SG1] S is primitively embedded into $\mathbf{L}$.

Let R denote the orthogonal complement of S in $\mathbf{L}$. Then R is an even negative-definite lattice. We assume that

[SG2] if $n = 26$, the lattice R cannot be embedded into the Leech lattice Λ.

For example, if $\mathcal{R}_R$ is non-empty, then [SG2] is satisfied, because $\mathcal{R}_\Lambda = \emptyset$. In fact, in all examples that are treated in this paper, the assumption [SG2] is verified by showing $\mathcal{R}_R \neq \emptyset$.

We denote by

$$x \mapsto x_S \quad \text{and} \quad x \mapsto x_R$$

the orthogonal projections from $\mathbf{L} \otimes \mathbb{R}$ to $S \otimes \mathbb{R}$ and $R \otimes \mathbb{R}$, respectively. Since $\mathbf{L}$ is contained in $S^\vee \oplus R^\vee$, the images of $\mathbf{L}$ by these projections are contained in $S^\vee$ and $R^\vee$, respectively. Since $\mathbf{L}$ is unimodular, the result of Nikulin [17, Proposition 1.6.1] implies that the subgroup $\mathbf{L}/(S \oplus R)$ of $A_S \oplus A_R$ is the graph of an isomorphism

$$\delta_{\mathbf{L}} \colon (A_S, q_S) \xrightarrow{\sim} (A_R, -q_R).$$

We assume that the subgroup G of $\mathrm{O}^+(S)$ satisfies the following liftability condition (see Proposition 5.2 below):

[SG3] $\quad \delta_{\mathbf{L}*}(\eta_S(G)) \subset \operatorname{Im} \eta_R, \quad$ where $\delta_{\mathbf{L}*} \colon \mathrm{O}(q_S) \xrightarrow{\sim} \mathrm{O}(q_R)$ is induced by $\delta_{\mathbf{L}}$.

For example, if G is contained in $\eta_S^{-1}(\{\pm 1\})$, then [SG3] is satisfied.

Let $\mathcal{P}_{\mathbf{L}}$ be the positive cone of $\mathbf{L}$ that contains the fixed positive cone $\mathcal{P}_S$ of S. Let r be an element of $\mathcal{R}_{\mathbf{L}}$. Then the hyperplane $(r)^\perp \in \mathcal{H}_{\mathbf{L}}$ of $\mathcal{P}_{\mathbf{L}}$ intersects $\mathcal{P}_S$ if and only if $r_S^2 < 0$ holds, and in this case, the hyperplane $(r_S)^\perp \in \mathcal{H}_S$ of $\mathcal{P}_S$ is equal to the intersection $\mathcal{P}_S \cap (r)^\perp$. We put

$$\mathcal{R}_{\mathbf{L}|S} := \{\, r_S \mid r \in \mathcal{R}_{\mathbf{L}},\ r_S^2 < 0 \,\},$$

and show that the subset $\mathcal{R}_{\mathbf{L}|S}$ of $\mathcal{N}_S \cap S^\vee$ has the properties [V1]-[V4] given in Section 3 with L replaced by S.

Proposition 5.1. *If $v \in \mathcal{R}_{\mathbf{L}|S}$, then $-v^2 \le 2$. In particular, $\mathcal{R}_{\mathbf{L}|S}$ satisfies* [V1].

Proof. Since R is negative-definite and $r_S^2 + r_R^2 = -2$ holds for any $r \in \mathcal{R}_{\mathbf{L}}$, we have $-2 \le v^2$ for any $v = r_S \in \mathcal{R}_{\mathbf{L}|S}$. □

By Lemma 3.4, the family of hyperplanes

$$\begin{aligned} \mathcal{R}^*_{\mathbf{L}|S} &= \{\, (r_S)^\perp \in \mathcal{H}_S \mid r \in \mathcal{R}_{\mathbf{L}},\ r_S^2 < 0 \,\} \\ &= \{\, \mathcal{P}_S \cap (r)^\perp \mid r \in \mathcal{R}_{\mathbf{L}},\ \mathcal{P}_S \cap (r)^\perp \neq \emptyset \,\} \end{aligned}$$

is locally finite in $\mathcal{P}_S$.

Proposition 5.2. *Let g be an element of G. Then there exists an element $\tilde{g} \in \mathrm{O}^+(\mathbf{L})$ that leaves S invariant and is equal to g on S.*

Proof. By [SG3], there exists an element $h \in \mathrm{O}(R)$ such that $\delta_{\mathbf{L}*}(\eta_S(g)) = \eta_R(h)$. Then the action of $(\eta_S(g), \eta_R(h))$ on $A_S \oplus A_R$ preserves the graph $\mathbf{L}/(S \oplus R)$ of $\delta_{\mathbf{L}}$, and hence the action of (g, h) on $S^\vee \oplus R^\vee$ preserves $\mathbf{L} \subset S^\vee \oplus R^\vee$. The restriction $\tilde{g} \in \mathrm{O}(\mathbf{L})$ of (g, h) to $\mathbf{L}$ belongs to $\mathrm{O}^+(\mathbf{L})$, because $\tilde{g}$ leaves $\mathcal{P}_S \subset \mathcal{P}_{\mathbf{L}}$ invariant. Thus we obtain a desired lift $\tilde{g} \in \mathrm{O}^+(\mathbf{L})$. □

Proposition 5.3. *The action of G on $\mathcal{N}_S \cap S^\vee$ leaves $\mathcal{R}_{\mathbf{L}|S}$ invariant; that is, $\mathcal{R}_{\mathbf{L}|S}$ satisfies* [V2].

Proof. For any $v \in \mathcal{R}_{\mathbf{L}|S}$ and $g \in G$, we have $r \in \mathcal{R}_{\mathbf{L}}$ such that $v = r_S$ and a lift $\tilde{g} \in \mathrm{O}^+(\mathbf{L})$ of g. Then $v^g = (r^{\tilde{g}})_S \in \mathcal{R}_{\mathbf{L}|S}$ holds. □

By the definition of $\mathcal{R}_{\mathbf{L}|S}$, every $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D is written as

$$D = \mathcal{D} \cap \mathcal{P}_S$$

by some $\mathcal{R}^*_{\mathbf{L}}$-chamber $\mathcal{D}$.

Definition 5.4. An $\mathcal{R}^*_{\mathbf{L}}$-chamber $\mathcal{D}$ is said to be *S-nondegenerate* if $D = \mathcal{D} \cap \mathcal{P}_S$ is an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber. In other words, $\mathcal{D}$ is S-nondegenerate if and only if $\mathcal{D} \cap \mathcal{P}_S$ contains a non-empty open subset of $\mathcal{P}_S$.

Definition 5.5. Let D be an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber, and let $\mathcal{D}$ be an $\mathcal{R}^*_{\mathbf{L}}$-chamber such that $D = \mathcal{D} \cap \mathcal{P}_S$. By certain abuse of terminology, we say that the Weyl vector $w \in \mathbf{L}$ of $\mathcal{D}$ is a *Weyl vector of D*.

Remark 5.6. The set $\{r \in \mathcal{R}_{\mathbf{L}} \mid \mathcal{P}_S \subset (r)^\perp\}$ is equal to $\mathcal{R}_R$. In particular, the number of hyperplanes $(r)^\perp \in \mathcal{R}^*_{\mathbf{L}}$ passing through a point $v \in \mathcal{P}_S$ is at least $|\mathcal{R}_R|/2$, and is equal to $|\mathcal{R}_R|/2$ if v is general. We remark two consequences of this fact.

Let D be an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber. Then there exists a canonical one-to-one correspondence between the set of $\mathcal{R}^*_{\mathbf{L}}$-chambers $\mathcal{D}$ satisfying $D = \mathcal{D} \cap \mathcal{P}_S$ and the set of connected components of

$$(R \otimes \mathbb{R}) \setminus \bigcup_{\rho \in \mathcal{R}_R} [\rho]^\perp, \quad \text{where} \quad [\rho]^\perp := \{x \in R \otimes \mathbb{R} \mid \langle x, \rho \rangle_R = 0\}.$$

If $\mathcal{R}^*_{\mathbf{L}}$-chambers $\mathcal{D}$ and $\mathcal{D}'$ satisfy $D = \mathcal{D} \cap \mathcal{P}_S = \mathcal{D}' \cap \mathcal{P}_S$, then there exists a sequence of reflections s_i $(i = 1, \ldots, N)$ of $\mathbf{L}$ with respect to $\rho_i \in \mathcal{R}_R \subset \mathcal{R}_{\mathbf{L}}$ such that their product $s_1 \cdots s_N$ maps $\mathcal{D}$ to $\mathcal{D}'$. Then the Weyl vector w of $\mathcal{D}$ is mapped to the Weyl vector w' of $\mathcal{D}'$ by $s_1 \cdots s_N$. Since each (-2)-vector ρ_i is contained in R, we have $w_S = w'_S$. Therefore, for an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D, $w_S \in S^\vee$ is independent of the choice of a Weyl vector w of D.

Let $\mathcal{D}$ be an $\mathcal{R}^*_{\mathbf{L}}$-chamber, and let v be a vector in $\mathcal{D} \cap \mathcal{P}_S$. If the number of walls of $\mathcal{D}$ passing though v is equal to $|\mathcal{R}_R|/2$, then a small neighborhood of v in $\mathcal{P}_S$ is contained in $\mathcal{D} \cap \mathcal{P}_S$, and hence $\mathcal{D}$ is S-nondegenerate, and $D = \mathcal{D} \cap \mathcal{P}_S$ contains v in its interior.

Next we consider the property [V3] for $\mathcal{R}_{\mathbf{L}|S}$. Let w be a Weyl vector of $\mathbf{L}$, and let $\mathcal{D}$ be the corresponding $\mathcal{R}^*_{\mathbf{L}}$-chamber (not necessarily S-nondegenerate). Recall that the $\mathcal{R}_{\mathbf{L}}$-minimal defining set $\Delta_{R_{\mathbf{L}}}(\mathcal{D})$ of $\mathcal{D}$ is equal to $\{r \in \mathcal{R}_{\mathbf{L}} \mid \langle w, r \rangle_{\mathbf{L}} = 1\}$. We put

$$\Delta_w := \{\, r \in \Delta_{R_{\mathbf{L}}}(\mathcal{D}) \;\mid\; r_S^2 < 0 \,\} = \{\, r \in \Delta_{R_{\mathbf{L}}}(\mathcal{D}) \;\mid\; (r)^\perp \cap \mathcal{P}_S \neq \emptyset \,\} \tag{5.1}$$

and

$$\mathrm{pr}_S(\Delta_w) := \{r_S \mid r \in \Delta_w\}. \tag{5.2}$$

Then we have $\mathcal{D} \cap \mathcal{P}_S = \Sigma_S(\mathrm{pr}_S(\Delta_w)) \cap \mathcal{P}_S$. Therefore, if $\mathcal{D}$ is S-nondegenerate, then $\mathrm{pr}_S(\Delta_w)$ is a defining set of the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber $D = \mathcal{D} \cap \mathcal{P}_S$.

Proposition 5.7. *For any Weyl vector w in $\mathbf{L}$, the set Δ_w is finite. In particular, any $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D has a finite defining set, and hence $\mathcal{R}_{\mathbf{L}|S}$ satisfies* [V3].

Proof. First we show $w_S^2 > 0$. Note that $w^2 = w_S^2 + w_R^2$ and $w_R^2 \le 0$. Hence $w_S^2 > 0$ holds when $n = 10$ or $n = 18$ by Theorem 4.2. Suppose that $n = 26$ and $w_S^2 = 0$. Then $w_R = 0$ and $w = w_S$ hold. Therefore $\langle w \rangle^\perp / \langle w \rangle$ would contain R, which contradicts Theorem 4.6 and the assumption [SG2].

We denote by d_R the order of the discriminant group A_R. Then $d_R v^2 \in \mathbb{Z}$ and $d_R^2 v^2 \in 2\mathbb{Z}$ hold for any $v \in R^\vee$. We put

$$n_R := \{\, c \in \mathbb{Q} \;\mid\; d_R c \in \mathbb{Z},\;\; d_R^2 c \in 2\mathbb{Z},\;\; -2 < c \le 0 \,\}, \tag{5.3}$$

which is obviously finite. Since R is negative-definite, the set

$$R^\vee[c] := \{\, v \in R^\vee \;\mid\; v^2 = c \,\} \tag{5.4}$$

is finite for each $c \in n_R$. In particular, the set

$$a_R[c] := \{\, \langle w_R, v \rangle_R \;\mid\; v \in R^\vee[c] \,\} \tag{5.5}$$

is finite for each $c \in n_R$. Suppose that $r \in \Delta_w$, so that $r_S^2 < 0$. Note that

$$r^2 = r_S^2 + r_R^2 = -2 \quad \text{and} \quad \langle w, r \rangle_{\mathbf{L}} = \langle w_S, r_S \rangle_S + \langle w_R, r_R \rangle_R = 1.$$

Since $r_R^2 \le 0$, we have $-2 \le r_S^2 < 0$ and $-2 < r_R^2 \le 0$. Hence $n' := r_R^2$ belongs to n_R, and r_R is an element of $R^\vee[n']$. Since $w_S^2 > 0$, the quadratic part of the inhomogeneous quadratic form $x \mapsto x^2$ on the affine hyperplane

$$\{\, x \in S \otimes \mathbb{R} \;\mid\; \langle w_S, x \rangle_S = b \,\}$$

of $S \otimes \mathbb{R}$ is negative-definite for any $b \in \mathbb{R}$. If we put $a' := \langle w_R, r_R \rangle_R \in a_R[n']$, then r_S belongs to the finite set

$$S^\vee[n', a'] := \{\, v \in S^\vee \;\mid\; \langle w_S, v \rangle_S = 1 - a',\;\; v^2 = -2 - n' \,\}. \tag{5.6}$$

Since n_R, $R^\vee[n']$, $a_R[n']$ and $S^\vee[n', a']$ are finite, Δ_w is finite. □

We state the above proof in the form of an algorithm.

Algorithm 5.8. Suppose that a Weyl vector $w \in \mathbf{L}$ is given. This algorithm calculates the set Δ_w defined by (5.1).

Step 0. We calculate $w_S \in S^\vee$, $w_R \in R^\vee$, and the set n_R defined by (5.3). We set $\Delta' := \{\}$.

Step 1. For each $c \in n_R$, we calculate $R^\vee[c]$ defined by (5.4) by Algorithm 2.1, and calculate $a_R[c]$ defined by (5.5).

Step 2. Using Algorithm 2.2, we calculate the finite set $S^\vee[n', a']$ defined by (5.6) for each pair of $n' \in n_R$ and $a' \in a_R[n']$.

Step 3. For each triple $n' \in n_R$, $v_R \in R^\vee[n']$ and $v_S \in S^\vee[n', a']$, where $a' = \langle w_R, v_R \rangle_R$, we determine whether $v_S + v_R \in S^\vee \oplus R^\vee$ belongs to $\mathbf{L}$ or not, and if $v_S + v_R \in \mathbf{L}$, then we append $r := v_S + v_R$ to Δ'.

Step 4. Output Δ' as Δ_w.

We give a criterion of the S-nondegeneracy of a given $\mathcal{R}^*_{\mathbf{L}}$-chamber.

Criterion 5.9. Let $\mathcal{D}$ be an $\mathcal{R}^*_{\mathbf{L}}$-chamber with the Weyl vector w. Then $\mathcal{D}$ is S-nondegenerate if and only if there exists a vector $v \in \mathcal{P}_S$ that satisfies the finite number of strict inequalities

$$\langle v, r_S \rangle_S > 0 \quad \text{for any } r \in \Delta_w. \tag{5.7}$$

If $v \in \mathcal{P}_S$ satisfies these inequalities, then the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber $\mathcal{D} \cap \mathcal{P}_S$ contains v in its interior.

Next we consider the property [V4] for $\mathcal{R}_{\mathbf{L}|S}$.

Proposition 5.10. *For any $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D, we have $\pi_S(\overline{D}) \cap \partial\,\overline{\mathbb{H}}_S \subset \partial\,\overline{\mathbb{H}}_S^{\mathbb{Q}}$. In particular, $\mathcal{R}_{\mathbf{L}|S}$ satisfies* [V4].

Proof. Let $\mathcal{D}$ be an $\mathcal{R}^*_{\mathbf{L}}$-chamber such that $D = \mathcal{D} \cap \mathcal{P}_S$. Then we have $\overline{D} = \overline{\mathcal{D}} \cap \overline{\mathcal{P}}_S$. Since $\partial\,\overline{\mathbb{H}}_S^{\mathbb{Q}} = \partial\,\overline{\mathbb{H}}_S \cap \partial\,\overline{\mathbb{H}}_{\mathbf{L}}^{\mathbb{Q}}$ under the canonical inclusion $\overline{\mathbb{H}}_S \hookrightarrow \overline{\mathbb{H}}_{\mathbf{L}}$, the assertion follows from Proposition 4.4. □

Thus the subset $\mathcal{R}_{\mathbf{L}|S}$ of $\mathcal{N}_S \cap S^\vee$ has the properties [V1]-[V4]. Hence we can apply Algorithms 3.17, 3.18 and 3.19 described in Section 3 to $\mathcal{V} = \mathcal{R}_{\mathbf{L}|S}$.

Algorithm 5.11. Suppose that a Weyl vector $w \in \mathbf{L}$ of an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D is given. This algorithm calculates the primitively minimal defining set $\Delta_{S^\vee}(D)$ of D.

Step 0. We calculate Δ_w by Algorithm 5.8.

Step 1. We calculate the defining set $\mathrm{pr}_S(\Delta_w)$ of D.

Step 2. We then calculate $\Delta_{S^\vee}(D)$ from $\mathrm{pr}_S(\Delta_w)$ by Algorithm 3.17.

Remark 5.12. Let D and D' be $\mathcal{R}^*_{\mathbf{L}|S}$-chambers. By Algorithms 3.18, 3.19 and 5.11, we can calculate $Aut_G(D)$ and determine whether D and D' are G-congruent or not from Weyl vectors w of D and w' of D'. Note that, by Remark 5.6, the defining set $\mathrm{pr}_S(\Delta_w)$ of D is independent of the choice of the Weyl vector w. Moreover, by Proposition 5.2, if $g \in Aut_G(D)$, then g preserves $\mathrm{pr}_S(\Delta_w)$, and if $D' = D^g$, then g maps $\mathrm{pr}_S(\Delta_w)$ to $\mathrm{pr}_S(\Delta_{w'})$ bijectively. Hence, when we apply Algorithms 3.18 and 3.19 to $\mathcal{R}^*_{\mathbf{L}|S}$-chambers, we can use $\mathrm{pr}_S(\Delta_w)$ and $\mathrm{pr}_S(\Delta_{w'})$ in the place of the primitively defining sets $\Delta_{S^\vee}(D)$ and $\Delta_{S^\vee}(D')$.

Let w be a Weyl vector of an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D, and let v be an element of $\Delta_{S^\vee}(D)$. Then $(v)^\perp$ is a wall of D. We calculate the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber that is adjacent to D across $(v)^\perp$. First we prepare an auxiliary algorithm.

Algorithm 5.13. Suppose that $v \in \mathcal{N}_S \cap S^\vee$ is given. We regard the hyperplane $(v)^\perp$ of $\mathcal{P}_S$ as a linear subspace of the larger space $\mathcal{P}_{\mathbf{L}}$; that is, $(v)^\perp$ is of codimension rank $R+1$ in $\mathcal{P}_{\mathbf{L}}$, whereas $(r)^\perp$ for $r \in \mathcal{R}_{\mathbf{L}}$ is of codimension 1 in $\mathcal{P}_{\mathbf{L}}$. This algorithm calculates the set

$$P_v := \{\, r \in \mathcal{R}_{\mathbf{L}} \mid (v)^\perp \subset (r)^\perp \,\} = \{\, r \in \mathcal{R}_{\mathbf{L}} \mid r_S \in \mathbb{R}v \,\}. \tag{5.8}$$

We set $P := \{\}$. There exist only a finite number of $\alpha \in \mathbb{Q}$ such that $\alpha v \in S^\vee$ and $\alpha^2 v^2 \geq -2$. For each such rational number α and each $u \in R^\vee[c]$, where $c = -2 - \alpha^2 v^2$ and $R^\vee[c]$ defined by (5.4), we determine whether $\alpha v + u \in S^\vee \oplus R^\vee$ belongs to $\mathbf{L}$ or not, and if $\alpha v + u \in \mathbf{L}$, then we append $r := \alpha v + u \in \mathbf{L}$ to P. Then we output P as P_v. (Remark that P_v includes the subset $\mathcal{R}_R$ of $\mathcal{R}_{\mathbf{L}}$.)

We consider the linear subspace

$$V := \mathbb{R}v \oplus (R \otimes \mathbb{R})$$

of $\mathbf{L} \otimes \mathbb{R}$. Let $\langle\ ,\ \rangle_V \colon V \times V \to \mathbb{R}$ denote the restriction of $\langle\ ,\ \rangle_{\mathbf{L}}$ to V. Note that $\langle\ ,\ \rangle_V$ is negative-definite. Let $x \mapsto x_V$ denote the orthogonal projection from $\mathbf{L} \otimes \mathbb{R}$ to V. Each element r of P_v defined by (5.8) belongs to V. Hence $r_V = r$ holds for any $r \in P_v$. We denote by $\{\mathcal{D}_0, \dots, \mathcal{D}_m\}$ the set of $\mathcal{R}^*_{\mathbf{L}}$-chambers containing the hyperplane $(v)^\perp$ of $\mathcal{P}_S$ (that is, the linear subspace of $\mathcal{P}_{\mathbf{L}}$ with codimension rank $R+1$), and put

$$\mathcal{D}^\circ_{j,V} := \{\, x \in V \mid \langle x, r\rangle_V > 0 \text{ for any } r \in P_v \cap \Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D}_j) \,\}.$$

Then $\mathcal{D}_j \mapsto \mathcal{D}^\circ_{j,V}$ gives a one-to-one correspondence from $\{\mathcal{D}_0, \dots, \mathcal{D}_m\}$ to the set of connected components of

$$V \setminus \bigcup_{r \in P_v} [r]^\perp_V, \quad \text{where} \quad [r]^\perp_V := \{\, x \in V \mid \langle x, r\rangle_V = 0 \,\}.$$

Let w_j be the Weyl vector of $\mathcal{D}_j$. By renumbering $\mathcal{D}_0, \dots, \mathcal{D}_m$, we can assume that $D = \mathcal{P}_S \cap \mathcal{D}_0$ and that w_0 is the given Weyl vector w of D. Since $\langle w_{j,V}, r\rangle_V = \langle w_j, r\rangle_{\mathbf{L}} = 1$ for any $r \in P_v \cap \Delta_{\mathcal{R}_{\mathbf{L}}}(\mathcal{D}_j)$, the vector $w_{j,V}$ belongs to $\mathcal{D}^\circ_{j,V}$. There exists an $\mathcal{R}^*_{\mathbf{L}}$-chamber $\mathcal{D}_{\mathrm{opp}}$ among $\{\mathcal{D}_0, \dots, \mathcal{D}_m\}$ such that

$$\mathcal{D}^\circ_{\mathrm{opp},V} = -\mathcal{D}^\circ_{0,V}.$$

Then $\mathcal{P}_S \cap \mathcal{D}_{\mathrm{opp}}$ is the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D' adjacent to D across $(v)^\perp$. We calculate the Weyl vector of $\mathcal{D}_{\mathrm{opp}}$. Let u be a sufficiently general vector of $\mathbf{L} \otimes \mathbb{Q}$, and let ε be a sufficiently small positive real number. Consider the oriented line segment

$$p(t) := (1-t)w_V + t(-w_V + \varepsilon u_V) \qquad (0 \leq t \leq 1)$$

in V from $w_V = w_{0,V} \in \mathcal{D}^\circ_{0,V}$ to $-w_V + \varepsilon u_V \in \mathcal{D}^\circ_{\mathrm{opp},V}$. Let $P'_v = \{r_1, \dots, r_N\} \subset P_v$ be a complete set of representatives of $P_v/\{\pm 1\}$. For $i = 1, \dots, N$, let t_i be the value of t such that $p(t_i) \in [r_i]_V^\perp$. Since $\langle x, r_i\rangle_{\mathbf{L}} = \langle x_V, r_i\rangle_V$ for any $x \in \mathbf{L}\otimes\mathbb{R}$, we have

$$t_i = \left(2 - \varepsilon\frac{\langle u, r_i\rangle_{\mathbf{L}}}{\langle w, r_i\rangle_{\mathbf{L}}}\right)^{-1}.$$

Since u is general, we can assume that $t_1, \dots, t_N$ are distinct. We put the numbering of the elements $r_1, \dots, r_N$ of P'_v so that $t_1 < \dots < t_N$ holds. Let $s_i \in \mathrm{O}^+(\mathbf{L})$ denote the reflection with respect to r_i. Then $\mathcal{D}_0$ and $\mathcal{D}_{\mathrm{opp}}$ are related by

$$\mathcal{D}_{\mathrm{opp}} = \mathcal{D}_0^{s_1 s_2 \dots s_N}.$$

Therefore $w^{s_1 s_2 \dots s_N}$ is the Weyl vector of $\mathcal{D}_{\mathrm{opp}}$. By this consideration, we obtain the following:

Algorithm 5.14. Suppose that a Weyl vector $w \in \mathbf{L}$ of an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D and an element v of $\Delta_{S^\vee}(D)$ are given. This algorithm calculates a Weyl vector w' of the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D' adjacent to D across the wall $(v)^\perp$.

We calculate the set P_v by Algorithm 5.13, and choose a complete set of representatives $P'_v = \{r_1, \dots, r_N\}$ of $P_v/\{\pm 1\}$. We also choose a vector u of $\mathbf{L}\otimes\mathbb{Q}$ such that $i \ne j$ implies $\langle u, r_i\rangle_{\mathbf{L}}/\langle w, r_i\rangle_{\mathbf{L}} \ne \langle u, r_j\rangle_{\mathbf{L}}/\langle w, r_j\rangle_{\mathbf{L}}$. We sort the elements r_i of P'_v so that

$$i < j \implies \frac{\langle u, r_i\rangle_{\mathbf{L}}}{\langle w, r_i\rangle_{\mathbf{L}}} < \frac{\langle u, r_j\rangle_{\mathbf{L}}}{\langle w, r_j\rangle_{\mathbf{L}}}$$

holds. Then $w^{s_1 s_2 \dots s_N}$ is a Weyl vector of D', where $s_i \in \mathrm{O}^+(\mathbf{L})$ is the reflection with respect to r_i.

6. The main algorithm

We present our main algorithm, and prove its termination and correctness. Let G, S and $\mathbf{L}$ be as in the previous section. Let N be an $\mathcal{R}^*_S$-chamber. Since $\mathcal{R}_S$ is contained in $\mathcal{R}_{\mathbf{L}|S}$, the $\mathcal{R}^*_S$-chamber N is a union of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers. We fix an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_0 contained in N. An *N-chain* is a finite sequence

$$D^{(0)}, D^{(1)}, \dots, D^{(l)}$$

of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers contained in N such that $D^{(a+1)}$ is adjacent to $D^{(a)}$ for each a. The *length* of an N-chain $D^{(0)}, \dots, D^{(l)}$ is defined to be l. Let D be an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber contained in N. Since N is connected, there exists an N-chain $D^{(0)}, \dots, D^{(l)}$ such that $D^{(0)} = D_0$ and $D^{(l)} = D$. The *level of* D is defined to be the minimum of the lengths of all N-chains from D_0 to D.

In the actual execution of the following algorithm, we use a Weyl vector to store an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber in the computer memory. Thus, for example, the set $\mathbb{D}$ is realized as a set of vectors of $\mathbf{L}$ in the computer.

Algorithm 6.1. Suppose that a Weyl vector w_0 of an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_0 is given. Let N be the $\mathcal{R}^*_S$-chamber containing D_0. This algorithm calculates a finite set

$\mathit{\Gamma}$ of generators of $Aut_G(N)$, a finite set $\mathbb{D}$ of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers contained in N such that the union

$$F_N := \bigcup_{D_i \in \mathbb{D}} D_i \tag{6.1}$$

satisfies the property (2) in Proposition 6.3 below, and a set $\mathcal{B}$ of (-2)-vectors of S that satisfies the property (3) in Proposition 6.3. In fact, the set $\mathbb{D}$ is a complete set of representatives of G-congruence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers, and it is a union of non-empty subsets $\mathbb{D}_\ell$, where elements of $\mathbb{D}_\ell$ are of level ℓ.

Step 0. Set $\mathit{\Gamma}$ to be $\{\}$, $\mathbb{D}_0$ to be $\{D_0\}$, and $\mathcal{B}$ to be $\{\}$.

Step 1. Calculate the set $\Delta_{S^\vee}(D_0)$ from w_0 by Algorithm 5.11.

Step 2. Execute $\mathtt{adj}(0)$, where $\mathtt{adj}(\ell)$ is the following procedure, which calls $\mathtt{adj}(\ell+1)$ at the last step if the condition for termination is not fulfilled.

The procedure $\mathtt{adj}(\ell)$*.* Suppose that, for each $\lambda = 0, \dots, \ell$, a non-empty finite set $\mathbb{D}_\lambda$ of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers with the following properties has been calculated.

[D1] Each $D \in \mathbb{D}_\lambda$ is contained in N, and is of level λ, and
[D2] if $D, D' \in \cup_{\lambda=0}^{\ell} \mathbb{D}_\lambda$ are distinct, then D and D' are not G-congruent.

Suppose also that, for each $D \in \cup_{\lambda=0}^{\ell} \mathbb{D}_\lambda$, we have calculated $\Delta_{S^\vee}(D)$. This procedure calculates $\mathbb{D}_{\ell+1}$ and $\Delta_{S^\vee}(D)$ for each $D \in \mathbb{D}_{\ell+1}$.

Let $D_{k+1}, \dots, D_{k+m}$ be the elements of $\mathbb{D}_\ell$, where k is the sum of the cardinalities of $\mathbb{D}_\lambda$ with $\lambda < \ell$ (we put $k = -1$ if $\ell = 0$), and m is the cardinality of $\mathbb{D}_\ell$.

1. Put $\mathbb{D}' := \{\}$ and set $l := k + m + 1$.
2. For each $D_i \in \mathbb{D}_\ell$, we make the following calculation.
   - 2-1. Calculate the finite group $Aut_G(D_i)$ from $\Delta_{S^\vee}(D_i)$ by Algorithm 3.18, and append a finite set of generators of $Aut_G(D_i)$ to $\mathit{\Gamma}$.
   - 2-2. Note that $Aut_G(D_i)$ acts on $\Delta_{S^\vee}(D_i)$. Decompose $\Delta_{S^\vee}(D_i)$ into the $Aut_G(D_i)$-orbits $o_1, \dots, o_t$. Since $\mathcal{R}^*_S$ is G-invariant, the set $o^*_\nu := \{(v)^\perp \mid v \in o_\nu\}$ is either disjoint from $\mathcal{R}^*_S$ or entirely contained in $\mathcal{R}^*_S$. Let v be an element of o_ν. Since v is primitive in $S^\vee$, we have $o^*_\nu \subset \mathcal{R}^*_S$ if and only if there exists a positive integer α such that $\alpha^2 v^2 = -2$ and $\alpha v \in S$.
   - 2-3. For each orbit o_ν such that o^*_ν is disjoint from $\mathcal{R}^*_S$, we make the following calculation.
     - 2-3-1. Choose a vector $v \in o_\nu$, and calculate a Weyl vector $w' \in \mathbf{L}$ of the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D' adjacent to D_i across the wall $(v)^\perp$ by Algorithm 5.14. We then calculate $\Delta_{S^\vee}(D')$ by Algorithm 5.11. Since $(v)^\perp \notin \mathcal{R}^*_S$ and $D_i \subset N$, we have $D' \subset N$. Since D_i is of level ℓ, we see that D' is of level $\le \ell + 1$.

2-3-2. By Algorithm 3.19, determine whether D' is G-congruent to an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D'' in

$$\tilde{\mathbb{D}} := \mathbb{D}_0 \cup \cdots \cup \mathbb{D}_\ell \cup \mathbb{D}' = \{D_0, D_1, \dots, D_{l-1}\}$$

or not. If $D' = D''^h$ for some $D'' \in \tilde{\mathbb{D}}$ and $h \in G$, then append h to Γ. If there exist no such $D'' \in \tilde{\mathbb{D}}$ and $h \in G$, then D' represents a new G-congruence class and its level is $\ell + 1$, and hence we put $D_l := D'$, append D_l to $\mathbb{D}'$ and increment l by 1.

2-4. For each orbit o_ν such that $o^*_\nu \subset \mathcal{R}^*_S$, choose a vector $v \in o_\nu$, find a positive integer α such that $r := \alpha v \in \mathcal{R}_S$, and append r to $\mathcal{B}$.

3. If $\mathbb{D}' \neq \emptyset$, then put $\mathbb{D}_{\ell+1} := \mathbb{D}'$, which has the properties [D1] and [D2] above, and execute `adj`$(\ell + 1)$. If $\mathbb{D}' = \emptyset$, then put

$$\mathbb{D} := \mathbb{D}_0 \cup \mathbb{D}_1 \cup \cdots \cup \mathbb{D}_\ell,$$

and terminate.

Proposition 6.2. *Algorithm 6.1 terminates.*

Proof. By construction, any two distinct $\mathcal{R}^*_{\mathbf{L}|S}$-chambers in $\mathbb{D}_0 \cup \cdots \cup \mathbb{D}_\ell \cup \mathbb{D}'$ are not G-congruent during the calculation. Thus Proposition 6.2 follows from Theorem 3.7. □

Proposition 6.3. (1) *The group $Aut_G(N)$ is generated by Γ.*

(2) *For any $v \in N$, there exists an element $g \in Aut_G(N)$ such that $v^g \in F_N$, where F_N is defined by* (6.1)*.*

(3) *Let r be an element of the $\mathcal{R}_S$-minimal defining set $\Delta_{\mathcal{R}_S}(N)$ of N. Then there exists an element $g \in Aut_G(N)$ such that $r^g \in \mathcal{B}$.*

Proof. Since each $D_i \in \mathbb{D}$ is contained in N, we have $Aut_G(D_i) \subset Aut_G(N)$ by (3.2), and hence all elements of Γ appended in Step 2-1 is an element of $Aut_G(N)$. If $h \in G$ is appended to Γ in Step 2-3-2, then we have $D' = D''^h$ for some $\mathcal{R}^*_{\mathbf{L}|S}$-chambers D' and D'' contained in N, and hence $h \in Aut_G(N)$ by (3.2). Therefore the subgroup $\langle \Gamma \rangle$ of G generated by Γ is contained in $Aut_G(N)$.

To prove the rest of Proposition 6.3, it is enough to show the following. This claim also proves that $\mathbb{D}$ is a complete set of representatives of G-congruence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers contained in N.

Claim 6.4. For an arbitrary $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D contained in N, there exists an element $\gamma \in \langle \Gamma \rangle$ such that $D^\gamma \in \mathbb{D}$.

Indeed, suppose that Claim 6.4 is proved. Let g be an arbitrary element of $Aut_G(N)$. Since $D_0^g \subset N$, there exists an element $\gamma \in \langle \Gamma \rangle$ such that $(D_0^g)^\gamma$ is equal to some $D_i \in \mathbb{D}$. Since D_i and D_0 are G-congruent and $D_i, D_0 \in \mathbb{D}$, we have $D_0 = D_i$. Therefore $g\gamma \in Aut_G(D_0) \subset \langle \Gamma \rangle$ follows, and hence we have $g \in \langle \Gamma \rangle$.

Let $v \in N$ be an arbitrary vector, and let D be an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber containing v and contained in N. By Claim 6.4, we have $\gamma \in \langle \varGamma \rangle$ such that $D^\gamma = D_i \in \mathbb{D}$. Then we have $v^\gamma \in F_N$. Suppose that $r \in \Delta_{\mathcal{R}_S}(N)$. Then there exist an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D contained in N and a vector $v \in \Delta_{S^\vee}(D)$ such that $r = \alpha v$ for some positive integer α. By Claim 6.4, we have $\gamma \in \langle \varGamma \rangle$ such that $D^\gamma = D_i \in \mathbb{D}$. Then $v^\gamma \in \Delta_{S^\vee}(D_i)$ is contained in an $Aut_G(D_i)$-orbit o_ν such that $o_\nu^* \subset \mathcal{R}^*_S$. Hence there exists an element $\gamma' \in Aut_G(D_i)$ such that $r^{\gamma\gamma'}$ is appended to $\mathcal{B}$ in Step 2-3-3.

Now we prove Claim 6.4. We fix an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D contained in N, and prove the existence of $\gamma \in \langle \varGamma \rangle$ such that $D^\gamma \in \mathbb{D}$. An N-chain $D^{(0)}, D^{(1)}, \dots, D^{(l)}$ of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers is said to be *D-admissible* if $D^{(0)}$ belongs to $\mathbb{D}$ and there exists an element $\gamma \in \langle \varGamma \rangle$ such that $D^{(l)} = D^\gamma$. Since N is connected, there exists at least one D-admissible N-chain. It is enough to show that there exists a D-admissible N-chain of length 0. We suppose that the D-admissible N-chain with *minimal* length

$$D^{(0)} = D_i,\ D^{(1)},\ \dots,\ D^{(l)} = D^\gamma \qquad (D_i \in \mathbb{D},\ \gamma \in \langle \varGamma \rangle)$$

is of length $l > 0$, and derive a contradiction. Let $v' \in \Delta_{S^\vee}(D_i)$ be the vector such that $(v')^\perp$ is the wall between $D^{(0)} = D_i$ and $D^{(1)}$. Since D_i and $D^{(1)}$ are contained in N, the $Aut_G(D_i)$-orbit $o_\nu \subset \Delta_{S^\vee}(D_i)$ containing v' satisfies $o_\nu^* \cap \mathcal{R}^*_S = \emptyset$. Let v be the vector of o_ν chosen in Step 2-3-1, and let $g \in Aut_G(D_i)$ be an element that maps v' to v. Then $D^{(1)g}$ is the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber adjacent to $D^{(0)} = D_i$ across $(v)^\perp$. Since $g \in \langle \varGamma \rangle$, the N-chain

$$D^{(0)} = D_i,\ D^{(1)g},\ \dots,\ D^{(l)g} = D^{\gamma g}$$

is D-admissible. By the minimality of l, $D^{(1)g}$ does not belong to $\mathbb{D}$. Hence, in Step 2-3-2, the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber $D^{(1)g}$ is not appended to $\mathbb{D}'$, which means that there exist a chamber $D'' \in \mathbb{D}$ and and an element $h \in G$ such that $D^{(1)gh} = D''$. The element h^{-1} is appended to $\varGamma$ in Step 2-3-2, and hence $\gamma gh \in \langle \varGamma \rangle$. Then

$$D^{(1)gh} = D'',\ D^{(2)gh},\ \dots, D^{(l)gh} = D^{\gamma gh}$$

is a D-admissible N-chain of length $l-1$, which is a contradiction. □

Remark 6.5. For $D_i \in \mathbb{D}$, let $E(D_i) \subset D_i$ be a fundamental domain of the action of the finite group $Aut_G(D_i)$ on D_i. Then their union $\bigcup_{D_i \in \mathbb{D}} E(D_i)$ is a fundamental domain of the action of $Aut_G(N)$ on N. In particular, if we have $|Aut_G(D)| = 1$ for any $D \in \mathbb{D}$, then F_N is a fundamental domain of the action of $Aut_G(N)$ on N.

Remark 6.6. When Algorithm 6.1 is applied to the case $G = \mathrm{O}^+(S)$, we see that $\mathrm{O}^+(S)$ is generated by $\varGamma$ and the reflections $\{s_r \mid r \in \mathcal{B}\}$.

Remark 6.7. Suppose that $n = 26$ and that R contains a root lattice as a sublattice of finite index. By Borcherds [4, Lemma 5.1], there exist only a small number of $\mathrm{O}^+(S)$-congruence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers.

Remark 6.8. By Remark 5.12, we can make Step 2-3 faster; namely, we can determine whether D' represents a new G-congruence class or not by using $\mathrm{pr}_S(\Delta_{w'})$ instead of $\Delta_{S^\vee}(D')$.

7. The automorphism group of a $K3$ surface

In this section, we review the classical theory of the automorphism groups of $K3$ surfaces. Let X be an algebraic $K3$ surface with the Néron-Severi lattice S_X of rank > 1. Then S_X is an even hyperbolic lattice. Let $\mathcal{P}(X)$ denote the positive cone of S_X containing an ample class. The nef cone $\mathrm{Nef}(X)$ of X is defined by

$$\mathrm{Nef}(X) := \{\ v \in S_X \otimes \mathbb{R}\ \mid\ \langle v, [C] \rangle \geq 0\ \text{ for any curve } C \text{ on } X\ \},$$

where $[C] \in S_X$ is the class of a curve C. Note that, by Kleiman's criterion, $\mathrm{Nef}(X)$ is contained in the closure $\overline{\mathcal{P}}(X)$ of $\mathcal{P}(X)$ in $S_X \otimes \mathbb{R}$. We put

$$N(X) := \mathrm{Nef}(X) \cap \mathcal{P}(X) = \mathrm{Nef}(X) \setminus (\mathrm{Nef}(X) \cap \partial\, \overline{\mathcal{P}}(X)).$$

It is known that $N(X)$ is an $\mathcal{R}^*_{S_X}$-chamber, and that the $\mathcal{R}_{S_X}$-minimal defining set $\Delta_{\mathcal{R}_{S_X}}(N(X))$ consists of the classes of smooth rational curves on X (see, for example, Rudakov-Shafarevich [25]). For simplicity, we put

$$\mathit{Aut}(N(X)) := \{\ g \in \mathrm{O}^+(S_X)\ \mid\ N(X)^g = N(X)\ \} = \mathit{Aut}_{\mathrm{O}^+(S_X)}(N(X)).$$

By a (-2)-*wall*, we mean a wall bounding $N(X)$; that is, a wall $([C])^\perp$, where C is a smooth rational curve on X.

7.1. **Complex $K3$ surfaces.**

Suppose that X is defined over $\mathbb{C}$. With the cup-product, the second cohomology group $H := H^2(X, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$. Let T_X denote the orthogonal complement of S_X in H, which we call the *transcendental lattice* of X. We regard a non-zero holomorphic 2-form ω_X on X as a vector of $T_X \otimes \mathbb{C}$, and put

$$C_X := \{\ g \in \mathrm{O}(T_X)\ \mid\ \omega_X^g = \lambda\, \omega_X\ \text{ for some }\ \lambda \in \mathbb{C}^\times\ \}. \tag{7.1}$$

Since H is unimodular, the subgroup $H/(S_X \oplus T_X)$ of the discriminant group $A_{S_X} \oplus A_{T_X}$ of $S_X \oplus T_X$ is the graph of an isomorphism

$$\delta_{ST} \colon (A_{S_X}, q_{S_X})\ \xrightarrow{\sim}\ (A_{T_X}, -q_{T_X})$$

by Nikulin [17, Proposition 1.6.1], which induces an isomorphism of the automorphism groups $\delta_{ST*} \colon \mathrm{O}(q_{S_X}) \xrightarrow{\sim} \mathrm{O}(q_{T_X})$ of discriminant forms. Recall that, for an even lattice L, we have a natural homomorphism $\eta_L \colon \mathrm{O}(L) \to \mathrm{O}(q_L)$.

Theorem 7.1 (Piatetski-Shapiro and Shafarevich [23])**.** *Via the natural actions of* $\mathrm{Aut}(X)$ *on the lattices* S_X *and* T_X*, the automorphism group* $\mathrm{Aut}(X)$ *of* X *is identified with*

$$\{\ (g, h) \in \mathit{Aut}(N(X)) \times C_X\ \mid\ \delta_{ST*}(\eta_{S_X}(g)) = \eta_{T_X}(h)\ \}.$$

Since $\mathrm{O}(q_{T_X})$ is finite, the subgroup

$$G_X := \{\ g \in \mathrm{O}^+(S_X)\ \mid\ \delta_{ST*}(\eta_{S_X}(g)) \in \eta_{T_X}(C_X)\ \} \tag{7.2}$$

of $\mathrm{O}^+(S_X)$ has finite index.

Corollary 7.2. *The kernel of the natural homomorphism* $\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X)$ *is isomorphic to* $\mathrm{Ker}(\eta_{T_X}) \cap C_X$*, and its image is equal to*

$$Aut_{G_X}(N(X)) = \{g \in G_X \mid N(X)^g = N(X)\}.$$

7.2. **Supersingular $K3$ surfaces in odd characteristics.** Suppose that X is a supersingular $K3$ surface defined over an algebraically closed field k of *odd* characteristic p. By Artin [1], we know that the discriminant group A_{S_X} is a p-elementary abelian group of rank 2σ, where σ is a positive integer ≤ 10, which is called the *Artin invariant of* X. The $\mathbb{F}_p$-vector space $S_0 := pS_X^\vee/pS_X$ of dimension 2σ has a natural quadratic form

$$Q_0 \colon px \bmod pS_X \mapsto px^2 \bmod p \quad (x \in S_X^\vee)$$

that takes values in $\mathbb{F}_p$. We denote by $\mathrm{O}(Q_0)$ the finite group of automorphisms of (S_0, Q_0). We have a natural homomorphism $\mathrm{O}(S_X) \to \mathrm{O}(Q_0)$. We denote by $\varphi \colon S_0 \otimes k \to S_0 \otimes k$ the map $\mathrm{id}_{S_0} \otimes F_k$, where F_k is the Frobenius map of k. Let $c_{\mathrm{DR}} \colon S_X \to H^2_{\mathrm{DR}}(X/k)$ denote the Chern class map. Then the kernel $\mathrm{Ker}(\bar{c}_{\mathrm{DR}})$ of the induced homomorphism $\bar{c}_{\mathrm{DR}} \colon S_X \otimes k \to H^2_{\mathrm{DR}}(X/k)$ from $S_X \otimes k = S_X/pS_X$ to $H^2_{\mathrm{DR}}(X/k)$ is contained in $S_0 \otimes k$. (Note that we have $pS_X^\vee \subset S_X$.) The subspace

$$K := \varphi^{-1}(\mathrm{Ker}(\bar{c}_{\mathrm{DR}}))$$

of $S_0 \otimes k$ is called the *period of* X.

Theorem 7.3 (Ogus [21], [22])**.** *Via the natural action of* $\mathrm{Aut}(X)$ *on* S_X*, the automorphism group* $\mathrm{Aut}(X)$ *of* X *is identified with*

$$\{\ g \in Aut(N(X))\ \mid\ K^g = K\ \}.$$

Since $\mathrm{O}(Q_0)$ is finite, the subgroup

$$G_X := \{\ g \in \mathrm{O}^+(S_X)\ \mid\ K^g = K\ \} \tag{7.3}$$

of $\mathrm{O}^+(S_X)$ has finite index.

Corollary 7.4. *The natural homomorphism* $\mathrm{Aut}(X) \to \mathrm{O}(S_X)$ *is injective, and its image is equal to* $Aut_{G_X}(N(X))$.

8. Geometric application of Algorithm 6.1

Let X be a complex algebraic $K3$ surface or a supersingular $K3$ surface in odd characteristic. Let S_X be the Néron-Severi lattice of X, and let $N(X)$ be the $\mathcal{R}^*_{S_X}$-chamber $\mathrm{Nef}(X) \cap \mathcal{P}(X)$ in the positive cone $\mathcal{P}(X)$ containing an ample class. Let G_X be the subgroup of $\mathrm{O}^+(S_X)$ defined by (7.2) or (7.3). Applying Algorithm 6.1 to the case $S = S_X$, $G = G_X$ and $N = N(X)$, we can calculate a finite set of generators of

$$\mathrm{Im}(\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X)) = Aut_{G_X}(N(X))$$

and a closed domain $F_{N(X)}$ of $N(X)$ with the properties given in Proposition 6.3, provided that the following hold:

(1) The subgroup G_X of $\mathrm{O}^+(S_X)$ satisfies the condition [G] of the existence of a membership algorithm in Section 3.
(2) We can find a primitive embedding of S_X into an even unimodular hyperbolic lattice $\mathbf{L}$ of rank 10, 18 or 26 such that the orthogonal complement R of S_X in $\mathbf{L}$ satisfies [SG2] and [SG3] in Section 5.
(3) We can find a Weyl vector of an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_0 contained in the $\mathcal{R}^*_S$-chamber $N(X)$.

We discuss these requirements for a complex $K3$ surface X. Let ρ_X denote the Picard number of X, and let T_X denote the transcendental lattice of X.

8.1. **Requirement (1).** By the definition (7.2) of G_X, we have a membership algorithm for the subgroup G_X of $\mathrm{O}^+(S_X)$ if we have explicit descriptions of the isomorphism $\delta_{ST} \colon (A_{S_X}, q_{S_X}) \xrightarrow{\sim} (A_{T_X}, -q_{T_X})$ induced by the even unimodular overlattice $H^2(X, \mathbb{Z})$ of $S_X \oplus T_X$, the homomorphisms $\eta_{S_X} \colon \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$, $\eta_{T_X} \colon \mathrm{O}(T_X) \to \mathrm{O}(q_{T_X})$, and the subgroup C_X of $\mathrm{O}(T_X)$.

Suppose that $C_X = \{\pm 1\}$. Then the condition $g \in G_X$ is reduced to the condition $\eta_{S_X}(g) \in \{\pm 1\}$, and hence all we need to do is to calculate the homomorphism η_{S_X}. This assumption $C_X = \{\pm 1\}$ holds in many cases. For example, if $\rho_X < 20$ and the period ω_X is generic in $T_X \otimes \mathbb{C}$, then $C_X = \{\pm 1\}$. Indeed, since the eigenspaces in $T_X \otimes \mathbb{C}$ of any $g \in \mathrm{O}(T_X) \setminus \{\pm 1\}$ are proper subspaces, a period ω_X for which $C_X \neq \{\pm 1\}$ must lie in a countable union of proper subspaces of $T_X \otimes \mathbb{C}$.

8.2. **Requirement (2).** We have the following:

Proposition 8.1. *For a complex algebraic $K3$ surface X, the lattice S_X has a primitive embedding into an even unimodular hyperbolic lattice $\mathbf{L}_{26}$ of rank 26.*

Proof. Recall that T_X is an even lattice of signature $(2, 20-\rho_X)$ such that (A_{T_X}, q_{T_X}) is isomorphic to $(A_{S_X}, -q_{S_X})$. By Nikulin [17, Theorem 1.10.1], the existence of the lattice T_X implies the existence of an even lattice R of signature $(0, 26-\rho_X)$ with $(A_R, q_R) \cong (A_{T_X}, q_{T_X})$. Hence, by Nikulin [17, Proposition 1.6.1], S_X can be embedded primitively into an even unimodular hyperbolic lattice of rank 26 with

R being the orthogonal complement. Since an even unimodular hyperbolic lattice of rank 26 is unique up to isomorphism, Proposition 8.1 follows. □

Therefore S_X always satisfies the condition [SG1] in Section 5. It is, however, difficult in general to obtain a primitive embedding $S_X \hookrightarrow \mathbf{L}$ explicitly. In fact, by Nikulin [17, Proposition 1.6.1], this problem is equivalent to construct a negative-definite lattice R of rank equal to $\operatorname{rank}\mathbf{L} - \operatorname{rank} S_X$ such that $(A_R, q_R) \cong (A_{S_X}, -q_{S_X})$ holds. For the algorithm to construct an integral lattice in a given genus explicitly, see Conway and Sloane [7, Chapter 15].

Remark 8.2. In the case where X is supersingular, we can use the table [20] of positive-definite integral lattices of rank 4.

Once a primitive embedding $S_X \hookrightarrow \mathbf{L}$ is obtained, we can calculate a Gram matrix of R of the orthogonal complement, the isomorphism $\delta_{\mathbf{L}}\colon (A_{S_X}, q_{S_X}) \xrightarrow{\sim} (A_R, -q_R)$ of discriminant forms induced by $\mathbf{L} \subset S_X^\vee \oplus R^\vee$, and the induced isomorphism $\delta_{\mathbf{L}*}\colon \mathrm{O}(q_{S_X}) \xrightarrow{\sim} \mathrm{O}(q_R)$. Since both R and Λ are negative-definite, we can enumerate all embeddings of R into Λ, and hence the condition [SG2] can be checked. In practice, [SG2] is often verified simply by showing that $\mathcal{R}_R \neq \emptyset$. Note that $\mathrm{O}(R)$ is a finite group, and hence its image by $\eta_R\colon \mathrm{O}(R) \to \mathrm{O}(q_R)$ can be calculated. By the definition of G_X, the subgroup $\eta_{S_X}(G_X)$ of $\mathrm{O}(q_{S_X})$ is equal to

$$\operatorname{Im}\eta_{S_X} \cap \delta_{ST*}^{-1}(\eta_{T_X}(C_X)).$$

Hence, if $\delta_{\mathbf{L}*}^{-1}(\operatorname{Im}\eta_R)$ contains $\delta_{ST*}^{-1}(\eta_{T_X}(C_X))$, then the liftability condition [SG3] is satisfied. This sufficient condition for [SG3] is fulfilled in the following cases that occur frequently: the case where η_R is surjective, or the case where $C_X = \{\pm 1\}$.

8.3. **Requirement (3).** In order to find an initial $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_0 contained in $N(X)$, we can employ one of the following two methods.

Let $\mathcal{D}_0$ be the $\mathcal{R}^*_{\mathbf{L}}$-chamber corresponding to the standard Weyl vector w_0 given in Section 4. We choose an interior point u_0 of $\mathcal{D}_0$. When $\operatorname{rank}\mathbf{L}$ is 10 or 18, w_0 is an interior point of $\mathcal{D}_0$. When $\operatorname{rank}\mathbf{L}$ is 26 and $w_0 = f_U$, we can use

$$u_0 := 3f_U + z_U \in \mathbf{L} = U \oplus \Lambda,$$

because we have $u_0^2 = 4$, $\langle u_0, w_0\rangle_{\mathbf{L}} = 1$ and $\langle u_0, r_\lambda\rangle_{\mathbf{L}} = 1 - \lambda^2/2 > 0$ for all $\lambda \in \Lambda$.

Next we find an ample class $a_0 \in S_X$. The method of this step depends, of course, on what kind of geometric information of X is available. Once an ample class a_0 is obtained, we can determine whether a given vector $v \in S_X$ is ample or not by means of the following criterion and Algorithms 2.2 and 2.3. Note that a vector of S_X is ample if and only if it belongs to the interior of $N(X)$. Therefore a vector $v \in S_X$ is ample if and only if

(i) $v^2 > 0$ and $\langle v, a_0\rangle > 0$, so that $v \in \mathcal{P}(X)$,
(ii) the set $\{r \in \mathcal{R}_{S_X} \mid \langle v, r\rangle = 0\}$ is empty, and

(ii) the set $\{r \in \mathcal{R}_{S_X} \mid \langle v, r\rangle < 0,\ \langle a_0, r\rangle > 0\}$ is empty, so that the line segment in $\mathcal{P}(X)$ connecting a_0 and v does not intersect any hyperplane $(r)^\perp$ perpendicular to some (-2)-vector r.

For example, we can produce many ample classes $Aa_0 + u$ by putting A to be sufficiently large integers and $u \in S_X$ to be vectors with relatively small coordinates.

We choose a general ample class $a \in S_X$. (The cases where a is not general enough so that we have to re-choose a are indicated below.) Let $i_0 \colon S_X \hookrightarrow \mathbf{L}$ be a primitive embedding. We consider the oriented line segment

$$\ell(t) := (1-t)\ i_0(a) + t\ u_0 \quad (0 \le t \le 1)$$

in $\mathcal{P}_{\mathbf{L}}$ from $i_0(a)$ to u_0. By Algorithm 2.3, we calculate the finite set

$$\{\ r \in \mathcal{R}_{\mathbf{L}}\ \mid\ \langle u_0, r\rangle_{\mathbf{L}} > 0,\ \ \langle i_0(a), r\rangle_{\mathbf{L}} < 0\ \} = \{r_1, \dots, r_N\}$$

and sort the elements of this set in such a way that

$$t_1 \le t_2 < \cdots \le t_N, \quad \text{where } t_i \text{ satisfies } \langle \ell(t_i), r_i\rangle_{\mathbf{L}} = 0.$$

If $t_1, \dots, t_N$ are not distinct, we re-choose a. We assume that $t_1, \dots, t_N$ are distinct. Then the oriented line segment ℓ intersects the hyperplanes $(r_1)^\perp, \dots, (r_N)^\perp$ in this order. We replace the embedding i_0 by

$$i := s_{r_N} \circ \cdots \circ s_{r_1} \circ i_0,$$

and consider S_X as a primitive sublattice of $\mathbf{L}$ by i. Then the ample class a is contained in $\mathcal{D}_0 \cap \mathcal{P}(X)$. We check whether the inequalities (5.7) in Criterion 5.9 are satisfied for $v = a$ and $w = w_0$. If not, then we re-choose a and repeat the process again. If (5.7) is satisfied, then $D_0 := \mathcal{D}_0 \cap \mathcal{P}(X)$ is an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber containing a in its interior. In particular, D_0 is contained in $N(X)$ and the standard Weyl vector w_0 is a Weyl vector of D_0.

The other method does not necessarily succeed, but is practically useful. In fact, we have used this method in [12] and [15]. Suppose that $\operatorname{rank} \mathbf{L} = 26$. We choose an ample class $a \in S_X$ that is primitive in S_X. By Algorithm 2.1, we can calculate the list of all vectors $v \in R$ such that $a^2 + v^2 = 0$. For each v in this list, we determine whether the vector $w := a + v \in S_X \oplus R$ of $\mathbf{L}$ is a Weyl vector or not by Theorem 4.6; we check whether $\langle w\rangle^\perp/\langle w\rangle$ is a negative-definite unimodular lattice with no (-2)-vectors by applying Algorithm 2.1 to a Gram matrix of $\langle w\rangle^\perp/\langle w\rangle$. Suppose that a Weyl vector $w_0 := a + v_0$ of this form is found, and let $\mathcal{D}_0$ be the corresponding $\mathcal{R}^*_{\mathbf{L}}$-chamber. We check whether the inequalities (5.7) in Criterion 5.9 are satisfied for $v = a$ and $w = w_0$. If (5.7) is satisfied, then $D_0 := \mathcal{D}_0 \cap \mathcal{P}(X)$ is an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber containing a in its interior. In particular, D_0 is contained in $N(X)$ and the Weyl vector w_0 is a Weyl vector of D_0.

9. Complex elliptic $K3$ surfaces with Picard number 3

We demonstrate Algorithm 6.1 on certain complex $K3$ surfaces with Picard number 3, because, in this case, we can draw pictures of the closed domain $F_{N(X)}$ defined by (6.1) in the hyperbolic plane.

Let X be a complex $K3$ surface with Picard number 3 and with a Jacobian fibration

$$\phi\colon X \to \mathbb{P}^1$$

whose Mordell-Weil group MW_ϕ is of rank 1. We assume that the period ω_X of X is generic in $T_X \otimes \mathbb{C}$, which implies that the group C_X defined by (7.1) is equal to $\{\pm 1\}$. We denote by $f_\phi \in S_X$ the class of a fiber of ϕ and by $z_\phi \in S_X$ the class of the zero section of ϕ. Then there exists a vector $v_3 \in S_X$, unique up to sign, such that f_ϕ, z_ϕ, v_3 form a basis of S_X, and that the Gram matrix of S_X with respect to f_ϕ, z_ϕ, v_3 is

$$M := \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2k \end{bmatrix},$$

where $k := -v_3^2/2$. Since MW_ϕ is of rank 1, there exist no reducible fibers of ϕ and hence we have $k > 1$ (see, for example, [35]). A vector $\xi f_\phi + \eta z_\phi + \zeta v_3$ of $S_X \otimes \mathbb{R}$ is written as $[\xi, \eta, \zeta]$. The discriminant group A_{S_X} of S_X is a cyclic group of order $2k$ generated by $u_3 \bmod S_X$, where $u_3 := [0, 0, -1/2k]$. Since $k > 1$, we have $-1 \notin \mathrm{Ker}(\eta_{T_X})$. Therefore, by Corollary 7.2 and the assumption $C_X = \{\pm 1\}$, the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}^+(S_X)$ is injective, and $\mathrm{Aut}(X) \cong \mathit{Aut}_{G_X}(N(X))$ holds, where G_X is defined by (7.2). We describe the group G_X in terms of matrices. Note that the class $a := [2, 1, 0] \in S_X$ is nef, and hence $g \in \mathrm{O}(S_X)$ belongs to $\mathrm{O}^+(S_X)$ if and only if $\langle a^g, a\rangle_{S_X} > 0$. Therefore G_X is canonically identified with

$$\{\ g \in \mathrm{GL}_3(\mathbb{Z}) \ \mid\ g\, M\, {}^t g = M,\ \ a\, g\, M\, {}^t a > 0,\ \ u_3 g \equiv \pm u_3 \bmod \mathbb{Z}^3\ \}.$$

This identification provides us with the membership algorithm for G_X in the condition [G] in Section 3.

We embed S_X into the even unimodular hyperbolic lattice $\mathbf{L} = U \oplus E_8$ of rank 10 primitively. Note that [SG2] is irrelevant in this case, and [SG3] holds because $C_X = \{\pm 1\}$. Let $\mathcal{D}_0$ be the $\mathcal{R}^*_{\mathbf{L}}$-chamber with the Weyl vector $w_0 \in \mathbf{L} = U \oplus E_8$ given in Section 4.1. We will find a primitive embedding $i\colon S_X \hookrightarrow \mathbf{L}$ such that $D_0 := i^{-1}(\mathcal{D}_0)$ is an $\mathcal{R}^*_{\mathbf{L}|S_X}$-chamber contained in $N(X)$ by the method described in Section 8.3. To find an initial primitive embedding i_0, it is enough to choose a primitive vector $v'_3 \in E_8$ such that $v'^2_3 = -2k$, and define $i_0\colon S_X \hookrightarrow \mathbf{L}$ by

$$i_0(f_\phi) = f_U, \quad i_0(z_\phi) = z_U, \quad i_0(v_3) = v'_3,$$

where f_U and z_U are the basis of the hyperbolic plane U with respect to which the Gram matrix is (4.1). As the general ample class a, we use $[3A, A, -1]$ with A large enough.

There exist two obvious elements in $\mathrm{Aut}(X)$. We have a canonical bijection

$$\mathrm{MW}_\phi \;\cong\; \{\, [ks^2, 1, s] \;\mid\; s \in \mathbb{Z} \,\}$$

by sending sections of ϕ to the classes of their images. The Mordell-Weil group $\mathrm{MW}_\phi \cong \mathbb{Z}$ acts on X as translations. We also have the inversion automorphism $\iota_X \colon X \to X$ induced by the multiplication by -1 on the generic fiber of ϕ. Therefore $\mathrm{Aut}(X)$ contains $\mathrm{MW}_\phi \rtimes \langle \iota_X \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which is generated by the two involutions

$$h_1 := \iota_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad h_2 := \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & -1 \\ 2k & 0 & -1 \end{bmatrix}.$$

To describe the closed domain $F_{N(X)}$ satisfying the properties given in Proposition 6.3, we use the following method. The norm of $[1, x, y] \in S_X \otimes \mathbb{R}$ is $2x - 2x^2 - 2ky^2$. Hence, by the map $[1, x, y] \mapsto (x, y)$, the hyperbolic plane $\mathbb{H}_{S_X}$ associated with S_X is identified with

$$H_X := \{\, (x, y) \in \mathbb{R}^2 \;\mid\; (x - 1/2)^2 + (\sqrt{k}y)^2 < 1/4 \,\}.$$

The vector f_ϕ corresponds to the point $(0, 0)$ of $\overline{H}_X$, and the hyperplane $(z_\phi)^\perp$ is given by $x = 1/2$. The Poincaré disk model of the hyperbolic plane $\mathbb{H}_{S_X}$ is given by the mapping

$$(x, y) \mapsto \frac{1 - 2\,x}{1 + \sqrt{2}\,r} + \sqrt{-1}\frac{2\sqrt{k}\,y}{1 + \sqrt{2}\,r}, \quad \text{where } r := \sqrt{2x - 2x^2 - 2ky^2} \tag{9.1}$$

from H_X to $\Delta := \{z \in \mathbb{C} \,|\, |z| < 1\}$. This map sends the boundary point corresponding to f_ϕ to $1 \in \bar{\Delta}$, and the hyperplane $(z_\phi)^\perp$ to the imaginary axis in Δ.

Example 9.1. We present the result for the case $-2k = -22$. Elements of $\mathbf{L}$ is written in terms of the basis $f_U, z_U, e_1, \dots, e_8$. The primitive embedding $i \colon S_X \hookrightarrow \mathbf{L}$ is given by

$$i(f_\phi) = f_U, \quad i(z_\phi) = z_U, \quad i(v_3) = [0, 0, 12, 8, 16, 24, 20, 16, 11, 6].$$

As the elements of $\mathbb{D}$, we obtained twenty-seven $\mathcal{R}^*_{\mathbf{L}|S_X}$-chambers $D_0, \dots, D_{26}$. It turns out that $Aut_{G_X}(D_i) = \{1\}$ holds for $i = 0, \dots, 26$. Their walls and adjacency relation are given in Table 9.1, where (a, b, c) is the wall defined by $a + bx + cy = 0$ in H_X. For example, the chamber D_{23} has three walls $\mu_1 : -1 - x - 10y = 0$, $\mu_2 : 2 + x + 17y = 0$ and $\mu_3 : 2x + 5y = 0$. The chamber adjacent to D_{23} across μ_1

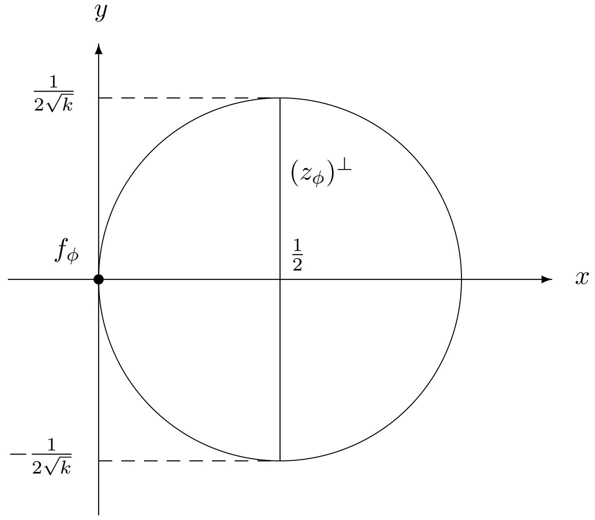


FIGURE 9.1. H_X

D_0	$(1,-2,0)$	(-2)-wall	D_1	$(0,1,5)$	D_3	D_2	$(1,-2,0)$	(-2)-wall
	$(0,1,6)$	D_1		$(0,-1,-6)$	D_0		$(-1,1,-6)$	D_1
	$(0,0,-1)$	$\cong D_0$ by h_1		$(1,-1,6)$	D_2		$(0,1,5)$	D_4
D_3	$(0,-1,-5)$	D_1	D_4	$(1,-1,5)$	D_6	D_5	$(0,-1,-4)$	D_3
	$(1,-1,6)$	D_4		$(0,1,4)$	D_7		$(1,-1,6)$	D_7
	$(0,1,4)$	D_5		$(-1,1,-6)$	D_3		$(0,1,3)$	D_8
				$(0,-1,-5)$	D_2			
D_6	$(1,-2,0)$	(-2)-wall	D_7	$(1,-1,5)$	D_9	D_8	$(0,-1,-3)$	D_5
	$(-1,1,-5)$	D_4		$(0,1,3)$	D_{10}		$(1,-1,6)$	D_{10}
	$(0,1,4)$	D_9		$(-1,1,-6)$	D_5		$(0,2,5)$	D_{11}
				$(0,-1,-4)$	D_4			
D_9	$(0,-1,-4)$	D_6	D_{10}	$(0,-1,-3)$	D_7	D_{11}	$(0,-2,-5)$	D_8
	$(-1,1,-5)$	D_7		$(1,1,11)$	D_{13}		$(1,1,11)$	D_{14}
	$(1,0,8)$	D_{12}		$(-1,1,-6)$	D_8		$(0,1,2)$	$\cong D_{11}$ by h_2
D_{12}	$(1,-1,4)$	D_{16}	D_{13}	$(1,0,8)$	D_{18}	D_{14}	$(0,1,2)$	$\cong D_{14}$ by h_2
	$(1,1,11)$	D_{15}		$(-1,-1,-11)$	D_{10}		$(-1,-1,-11)$	D_{11}
	$(-1,0,-8)$	D_9		$(0,2,5)$	D_{17}		$(1,-1,6)$	D_{17}
D_{15}	$(1,-1,4)$	D_{19}	D_{16}	$(1,-2,0)$	(-2)-wall	D_{17}	$(0,-2,-5)$	D_{13}
	$(-1,-1,-11)$	D_{12}		$(1,1,11)$	D_{19}		$(-1,1,-6)$	D_{14}
	$(0,1,3)$	D_{20}		$(-1,1,-4)$	D_{12}		$(1,0,8)$	D_{21}
D_{18}	$(1,-1,5)$	D_{20}	D_{19}	$(-1,1,-4)$	D_{15}	D_{20}	$(0,-1,-3)$	D_{15}
	$(0,2,5)$	D_{21}		$(-1,-1,-11)$	D_{16}		$(-1,1,-5)$	D_{18}
	$(-1,0,-8)$	D_{13}		$(2,-1,11)$	D_{22}		$(1,1,10)$	D_{23}
D_{21}	$(0,-2,-5)$	D_{18}	D_{22}	$(1,-2,0)$	(-2)-wall	D_{23}	$(-1,-1,-10)$	D_{20}
	$(-1,0,-8)$	D_{17}		$(-2,1,-11)$	D_{19}		$(2,1,17)$	$\cong D_{25}$ by h_3
	$(1,1,10)$	D_{24}		$(1,0,7)$	D_{25}		$(0,2,5)$	D_{26}
D_{24}	$(1,-1,5)$	D_{26}	D_{25}	$(1,-2,0)$	(-2)-wall	D_{26}	$(0,-2,-5)$	D_{23}
	$(2,3,22)$	(-2)-wall		$(2,1,17)$	$\cong D_{23}$ by h_3		$(2,3,22)$	(-2)-wall
	$(-1,-1,-10)$	D_{21}		$(-1,0,-7)$	D_{22}		$(-1,1,-5)$	D_{24}

TABLE 9.1. Chambers for the case $-2k = -22$

is D_{20}, the chamber adjacent to D_{23} across μ_3 is D_{26}, and the chamber D' adjacent

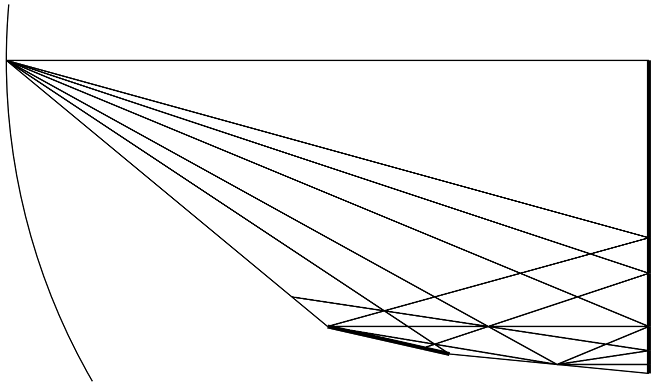


Figure 9.2. Fundamental domain for the case $-2k=-22$ in H_X

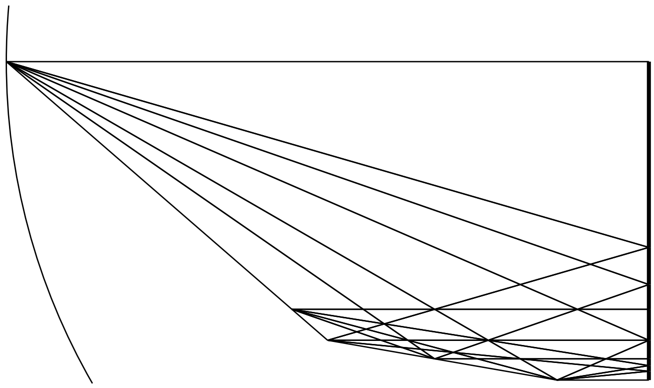


Figure 9.3. Fundamental domain for the case $-2k=-24$ in H_X

to D_{23} across μ_2 is isomorphic to D_{25} via

$$h_3 := \begin{bmatrix} 20 & 9 & -3 \\ 7 & 2 & -1 \\ 154 & 66 & -23 \end{bmatrix}.$$

Hence $\operatorname{Im}\varphi_X$ is generated by the three elements h_1, h_2, h_3. The shape of the union $F := F_{N(X)}$ of these D_i in H_X is given in Figure 9.2. Since the union F of these D_i has two (-2)-walls, which is depicted by thick lines in Figure 9.2, the set of smooth rational curves on X is decomposed into at most two orbits under the action of $\operatorname{Aut}(X)$. The left-hand side of Figure 9.4 shows the chambers

$$F,\ F^{h_1},\ F^{h_2},\ F^{h_3},\ F^{h_1h_2},\ F^{h_2h_1},\ F^{h_1h_3},\ F^{h_3h_1},\ F^{h_3h_2},\ F^{h_2h_3}$$

on the Poincaré disk model (9.1) of $\mathbb{H}_{S_X}$. The (-2)-walls are drawn by thick lines.

Remark 9.2. Example 1.1 in Introduction is the case $-2k=-24$. The set $\mathbb{D}$ contains 46 chambers, and their union $F := F_{N(X)}$ is given in Figure 9.3. Each chamber D_i

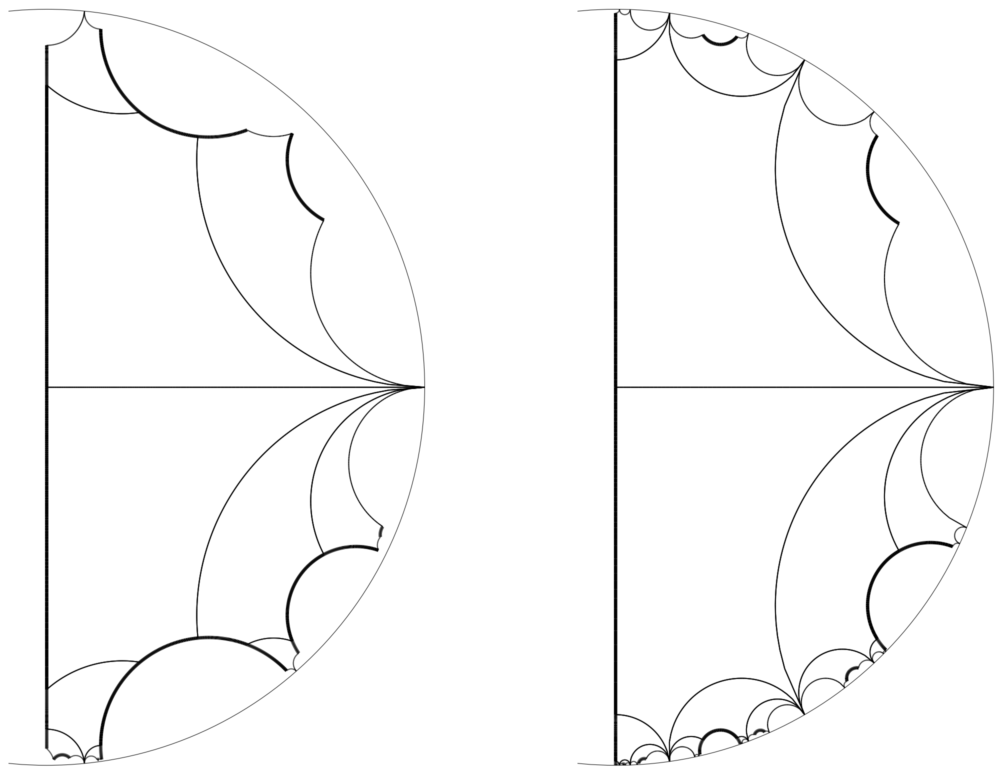


FIGURE 9.4. Tessellation of $N(X)$ for the cases $-2k = -22$ and $-2k = -24$

satisfies $Aut_{G_X}(D_i) = \{1\}$. Since $F_{N(X)}$ has only one (-2)-wall, $\mathrm{Aut}(X)$ acts on the set of smooth rational curves on X transitively. Observe that $F := F_{N(X)}$ has four non-(-2)-walls, which correspond to the four generators $h_1, \dots, h_4$ of $\mathrm{Aut}(X)$ given in Example 1.1. The right-hand side of Figure 9.4 shows the chambers

$$F,\ F^{h_1},\ F^{h_2},\ F^{h_3},\ F^{h_4},\ F^{h_i h_j},\ \ (1 \le i, j \le 4, i \ne j)$$

on the Poincaré disk model (9.1) of $\mathbb{H}_{S_X}$.

Remark 9.3. We have made the same calculation for $-2k = -4, -6, \dots, -30$. The results are presented in the author's web page [31].

10. Singular $K3$ surfaces

Recall that a $K3$ surface defined over $\mathbb{C}$ is singular if its Picard number attains the possible maximum 20. We demonstrate Algorithm 6.1 on singular $K3$ surfaces X such that their transcendental lattices T_X satisfy $\operatorname{disc} T_X \le 16$. There exist, up

No.	$\operatorname{disc} T_X$	$[a, b, c]$	root-R	$\lvert\mathbb{D}\rvert$	$\lvert C_X\rvert$	
1	3	$[2, 1, 2]$	E_6	1	6	Vinberg [39]
2	4	$[2, 0, 2]$	D_6	1	4	Vinberg [39]
3	7	$[2, 1, 4]$	A_6	1	2	Ujikawa [37]
4	8	$[2, 0, 4]$	$D_5 + A_1$	1	2	Section 10.2
5	11	$[2, 1, 6]$	none	1098	2	Section 10.3
6	12	$[2, 0, 6]$	$A_5 + A_1$	1	2	Section 10.2
7	12	$[4, 2, 4]$	$D_4 + A_2$	1	6	Keum and Kondo [13]
8	15	$[2, 1, 8]$	$A_4 + A_2$	1	2	Section 10.2
9	15	$[4, 1, 4]$	none	2051	2	Section 10.4
10	16	$[2, 0, 8]$	none	4539	2	Section 10.5
11	16	$[4, 0, 4]$	$2A_3$	1	4	Keum and Kondo [13]

Table 10.1. Transcendental lattices of small discriminants

to isomorphisms, exactly eleven singular $K3$ surfaces with $\operatorname{disc} T_X \le 16$. Table 10.1 shows Gram matrices

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

of the transcendental lattices of these $K3$ surfaces. We embed S_X into an even unimodular hyperbolic lattice $\mathbf{L}$ of rank 26. By the *root-R condition*, we mean that there should exist a negative-definite root lattice R of rank 6 that satisfies $(A_R, q_R) \cong (A_{T_X}, q_{T_X})$. If T_X satisfies the root-R condition, then there exists a primitive embedding $S_X \hookrightarrow \mathbf{L}$ such that the orthogonal complement is isomorphic to a root lattice R, and hence Borcherds' method terminates quickly (see Remark 6.7). The column root-R of Table 10.1 indicates the existence of such a root lattice R, and its ADE-type if it exists.

The singular $K3$ surfaces of Nos. 1, 2, 3, 7 and 11 have been studied previously. Therefore we treat the other six singular $K3$ surfaces Nos. 4, 5, 6, 8, 9 and 10.

10.1. **The period and ample classes of X.** The period ω_X is given in $T_X \otimes \mathbb{C}$ as a vector $(1, \alpha)$, where α is a root of

$$a + 2bx + cx^2 = 0.$$

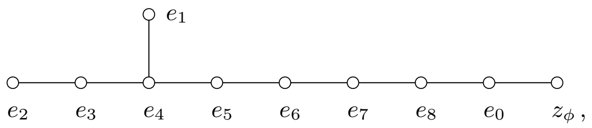
The choice of a root of this quadratic equation does not matter, because, in the eleven cases in Table 10.1, the orientation reversing of T_X yields an isomorphic oriented lattice; that is, the $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of T_X contains only one $\mathrm{SL}_2(\mathbb{Z})$-equivalence class (see [33]). Since $\mathrm{O}(T_X)$ is finite, we can calculate the subgroup C_X defined by (7.1). It turns out that, in the six cases Nos. 4, 5, 6, 8, 9 and 10, we have $C_X = \{\pm 1\}$. Therefore the conditions [G] and [SG3] for G_X and R are satisfied (see Sections 8.1 and 8.2).

In the following, we use the general theory of elliptic surfaces, for which we refer to [35] or [28]. Shioda and Inose [34] showed that every singular $K3$ surface X has a Jacobian fibration $\phi\colon X \to \mathbb{P}^1$ with two singular fibers $\phi^{-1}(p)$ and $\phi^{-1}(p')$ of type II^*. Hence S_X contains an even unimodular hyperbolic lattice $\mathbf{L}_{18}(\phi) := U_\phi \oplus E_8 \oplus E_8$ of rank 18 as a sublattice, where U_ϕ is spanned by the class f_ϕ of a fiber of ϕ and the class z_ϕ of the zero section of ϕ (and hence the Gram matrix of U_ϕ with respect the basis f_ϕ, z_ϕ is (4.1)), and each copy of E_8 is spanned by the classes of irreducible components of a singular fiber of type II^*. Since $\mathbf{L}_{18}(\phi)$ is unimodular, there exists a negative-definite lattice T_X^- of rank 2 such that

$$S_X = \mathbf{L}_{18}(\phi) \oplus T_X^-.$$

We have $(A_{T_X^-}, q_{T_X^-}) \cong (A_{T_X}, -q_{T_X})$. Since the Jacobian fibration ϕ is not unique, the isomorphism class of T_X^- is not unique in general (see [26] and [29]). However, for the eleven cases in Table 10.1, T_X^- is uniquely determined by the condition $(A_{T_X^-}, q_{T_X^-}) \cong (A_{T_X}, -q_{T_X})$ and is isomorphic to $(-1)T_X$.

We search for rational ample classes $a \in S_X \otimes \mathbb{Q}$. Let $e_1, \dots, e_8$ (resp. $e'_1, \dots, e'_8$) denote the classes of the irreducible components of $\phi^{-1}(p)$ (resp. of $\phi^{-1}(p')$) that are disjoint from the zero section and whose dual graph is



where we denote by e_0 (resp. e'_0) the class of the irreducible component of $\phi^{-1}(p)$ (resp. of $\phi^{-1}(p')$) that intersects the zero section. We have

$$e_0 = f_\phi - (3e_1 + 2e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8),$$

and a similar equality for e'_0.

Let e''_1 and e''_2 be a basis of T_X^- such that

$$\langle e''_1, e''_1 \rangle = -a, \;\; \langle e''_1, e''_2 \rangle = -b, \;\; \langle e''_2, e''_2 \rangle = -c.$$

Then the set $\mathcal{R}_{T_X^-} = \{v \in T_X^- \mid v^2 = -2\}$ is equal to

$$\begin{cases} \emptyset & \text{in No. 9,} \\ \{e''_1, -e''_1\} & \text{in Nos. 4, 5, 6, 8, 10.} \end{cases}$$

Therefore the set $\{t \in \mathbb{P}^1 \mid \phi^{-1}(t) \text{ is reducible}\}$ is equal to

$$\begin{cases} \{p, p'\} & \text{in No. 9,} \\ \{p, p', q\} & \text{in Nos. 4, 5, 6, 8, 10, where } q \in \mathbb{P}^1 \setminus \{p, p'\}. \end{cases}$$

Suppose that we are in the case of in Nos. 4, 5, 6, 8 or 10. Then the reducible fiber $\phi^{-1}(q)$ is either of type I_2 or III. Let C_0'' and C_1'' be the irreducible components of $\phi^{-1}(q)$ such that C_0'' intersects the zero section. Changing e_1'' and e_2'' to $-e_1''$ and $-e_2''$ if necessary, we can assume that e_1'' is the class of C_1'', and hence the class e_0'' of C_0'' is equal to $f_\phi - e_1''$.

We put

$$\mathcal{B} := \{\ [C]\ \mid\ C \text{ is a smooth rational curve on } X \text{ such that } \langle C, f_\phi\rangle = 0\ \}.$$

Then we have

$$\mathcal{B} = \begin{cases} \{e_0, e_1, \dots, e_8, e_0', e_1', \dots, e_8'\} & \text{in No. 9,} \\ \{e_0, e_1, \dots, e_8, e_0', e_1', \dots, e_8', e_0'', e_1''\} & \text{in Nos. 4, 5, 6, 8, 10.} \end{cases}$$

Let $e_1^\vee, \dots, e_8^\vee$ (resp. $e_1'^\vee, \dots, e_8'^\vee$) be the basis of E_8 dual to the basis $e_1, \dots, e_8$ (resp, $e_1', \dots, e_8'$). We have

$$u_p := e_1^\vee + \dots + e_8^\vee = -68e_1 - 46e_2 - 91e_3 - 135e_4 - 110e_5 - 84e_6 - 57e_7 - 29e_8,$$

and the similar formula for $u_{p'} := e_1'^\vee + \dots + e_8'^\vee$. Let $e_1''^\vee$ and $e_2''^\vee$ be the basis of $T_X^{-\vee}$ dual to the basis e_1'', e_2''. Then the vector

$$u_0 := \begin{cases} 30z_\phi + u_p + u_{p'} & \text{in No. 9,} \\ 30z_\phi + u_p + u_{p'} + e_1''^\vee & \text{in Nos. 4, 5, 6, 8, 10,} \end{cases}$$

satisfies

$$\langle u_0, v\rangle > 0 \quad \text{for any} \quad v \in \mathcal{B}.$$

(The coefficient 30 of z_ϕ in u_0 is determined by $\langle e_0, u_p\rangle = \langle e_0', u_{p'}\rangle = -29$.) Consider the projection $\pi_{S_X} : \mathcal{P}(X) \to \mathbb{H}_{S_X}$ to the 19-dimensional hyperbolic space $\mathbb{H}_{S_X}$. The point $b := \pi_{S_X}(f_\phi)$ is a rational boundary point of $\pi_{S_X}(N(X))$. By Corollary 3.12, if we choose a sufficiently small closed horoball HB_b with the base b, then $HB_b \cap \pi_{S_X}(N(X))$ is bounded in HB_b by the inequalities

$$\langle x, v\rangle \ge 0 \quad \text{for any} \quad v \in \mathcal{B}.$$

Therefore, if A is sufficiently large, then

$$a := Af_\phi + u_0 \ \in\ S_X \otimes \mathbb{Q}$$

is contained in the interior of $N(X)$. We use this rational ample class a in search for the primitive embedding $S_X \hookrightarrow \mathbf{L}$ such that the initial $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_0 is contained in $N(X)$.

We use $U \oplus E_8 \oplus E_8 \oplus E_8$ as $\mathbf{L}$ and the Weyl vector w_E given in Remark 4.9. Let $\mathcal{D}_0$ be the $\mathcal{R}^*_{\mathbf{L}}$-chamber corresponding to w_E. We have a natural isomorphism from $\mathbf{L}_{18}(\phi)$ to the sublattice $U \oplus E_8 \oplus E_8$ of $\mathbf{L}$. Hence, in order to find a primitive embedding $i_0 \colon S_X \hookrightarrow \mathbf{L}$, it is enough to find a primitive embedding $T_X^- \hookrightarrow E_8$; that is, to find vectors u_1, u_2 of E_8 satisfying

$$u_1^2 = -a, \quad u_2^2 = -c, \quad \langle u_1, u_2 \rangle_{E_8} = -b,$$

and generating a primitive sublattice of rank 2 in E_8. From this embedding i_0 and using the method described in Section 8.3 with the ample class $a = Af_\phi + u_0$ above, we find a primitive embedding $S_X \hookrightarrow \mathbf{L}$ such that $D_0 := \mathcal{D}_0 \cap \mathcal{P}(X)$ is an $\mathcal{R}^*_{\mathbf{L}|S}$-chamber contained in $N(X)$. In the examples below, we confirmed that the vector

$$h_E := w_{E,S} \in S_X^\vee$$

is in fact in the interior of D_0 by Criterion 5.9 and hence $\mathcal{D}_0$ is S-nondegenerate. Moreover, we confirmed that the interior point $Af_\phi + u_0$ of $N(X)$ is contained in D_0 by showing that

$$\langle f_\phi, v \rangle > 0 \;\text{ or }\; (\langle f_\phi, v \rangle = 0 \text{ and } \langle u_0, v \rangle \geq 0) \quad \text{for any } \; v \in \Delta_{S_X^\vee}(D_0),$$

and hence we can conclude that D_0 is contained in $N(X)$. In particular, h_E is also an ample class of X, and hence we can consider the automorphism group

$$\operatorname{Aut}(X, h_E) := \{\, g \in \operatorname{Aut}(X) \;\mid\; g^*(h_E) = h_E \,\}$$

of the polarized $K3$ surface (X, h_E). Since φ_X is injective in our six cases, we have

$$\operatorname{Aut}(X, h_E) \cong Aut_{G_X}(D_0).$$

If T_X satisfies the root-R condition, the orthogonal complement R of S_X in $\mathbf{L}$ satisfies [SG2], because $\mathcal{R}_R \neq \emptyset$. If T_X does not satisfy the root-R condition, we present below a Gram matrix of R explicitly, from which we immediately see that $\mathcal{R}_R \neq \emptyset$ and hence R satisfies [SG2].

10.2. **The cases where the root-R condition is satisfied.** In these cases (Nos. 4,6 and 8), we have $\mathbb{D} = \{D_0\}$, and hence the description of $\operatorname{Aut}(X)$ is simple. We have

$$h_E^2 = \begin{cases} 61/2 & \text{in No. } 4, \\ 18 & \text{in No. } 6, \\ 12 & \text{in No. } 8, \end{cases}$$

and

$$|\operatorname{Aut}(X, h_E)| = |Aut_{G_X}(D_0)| = \begin{cases} 48 & \text{in No. } 4, \\ 144 & \text{in No. } 6, \\ 720 & \text{in No. } 8. \end{cases}$$

Table 10.2 presents the orbit decomposition of the set of walls of D_0 by the action of $\operatorname{Aut}(X, h_E) \cong Aut_{G_X}(D_0)$. The column $|o|$ indicates the cardinality of the orbit

No. 4

$\lvert o\rvert$	n_v	a_v	
6	-2	1	$*$
12	-2	1	$*$
8	-2	1	$*$
3	$-3/2$	$3/2$	
6	-1	5	
4	-1	5	
24	$-3/4$	6	
8	$-3/4$	6	
2	$-1/2$	$11/2$	$*$
8	$-1/4$	$13/2$	

No. 6

$\lvert o\rvert$	n_v	a_v	
12	-2	1	$*$
18	-2	1	$*$
4	$-3/2$	$3/2$	
24	$-7/6$	$7/2$	
6	$-2/3$	4	
24	$-2/3$	5	
36	$-2/3$	5	
12	$-1/2$	$11/2$	$*$
36	$-1/2$	$11/2$	$*$
24	$-1/6$	$11/2$	

No. 8

$\lvert o\rvert$	n_v	a_v	
36	-2	1	$*$
12	$-4/3$	2	
40	$-6/5$	3	
90	$-4/5$	4	
30	$-8/15$	4	
30	$-8/15$	4	
120	$-2/15$	5	
120	$-2/15$	5	

TABLE 10.2. Orbit decomposition of the walls of D_0

o. Each wall of D_0 is uniquely written as $(v)^{\perp}$, where v is a primitive vector of $S_X^{\vee}$ satisfying $\langle v, h_E\rangle > 0$. In Table 10.2, we also present the values

$$n_v := v^2, \quad a_v := \langle v, h_E\rangle$$

for each orbit. Note that a wall $(v)^{\perp}$ of D_0 with v primitive in $S_X^{\vee}$ and $\langle v, h_E\rangle > 0$ is a (-2)-wall if and only if there exists a positive integer α such that $\alpha^2\, n_v = -2$ and $\alpha v \in S_X$. The orbits that consist of walls satisfying this condition are marked by $*$. From Table 10.2, we see the following.

In No. 4, the automorphism group $\mathrm{Aut}(X)$ is generated by the finite group $\mathrm{Aut}(X, h_E)$ of order 48 and six extra automorphisms corresponding to the six walls of D_0 that are not (-2)-walls. Exactly four orbits consist of (-2)-walls, and hence the number of orbits of $\mathrm{Aut}(X)$ on the set of smooth rational curves on X is at most 4.

In No. 6, $\mathrm{Aut}(X)$ is generated by the finite group $\mathrm{Aut}(X, h_E)$ of order 144 and six extra automorphisms. The number of orbits of $\mathrm{Aut}(X)$ on the set of smooth rational curves is at most 4.

In No. 8, $\mathrm{Aut}(X)$ is generated by the finite group $\mathrm{Aut}(X, h_E)$ of order 720 and seven extra automorphisms, and $\mathrm{Aut}(X)$ acts on the set of smooth rational curves transitively.

Remark 10.1. In [32], geometric realizations of generators of $\mathrm{Aut}(X)$ for No. 8 will be given. We also show that the finite group $\mathrm{Aut}(X, h_E)$ of order 720 for No. 8 is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_9)$.

10.3. **The case** $[a, b, c] = [2, 1, 6]$ **(No. 5).** The lattice R has a Gram matrix

$$\begin{bmatrix} -2 & 1 & 0 & 0 & -1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ -1 & 0 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

The number of G_X-equivalence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers is 1098, and the maximum of the level (see Section 6 for the definition) is 13. The results are presented in Example 1.2 in Introduction.

10.4. **The case** $[a, b, c] = [4, 1, 4]$ **(No. 9).** The lattice R has a Gram matrix

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & -1 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

The number of G_X-equivalence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers is 2051, and the maximum of the level is 16. We find that $\operatorname{Im}\varphi_X$ is generated by 1098 elements of $\mathrm{O}(S_X)$. Moreover the number of orbits of the action of $\mathrm{Aut}(X)$ on the set of smooth rational curves is at most 154.

10.5. **The case** $[a, b, c] = [2, 0, 8]$ **(No. 10).** The lattice R has a Gram matrix

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & -4 \end{bmatrix}.$$

The number of G_X-equivalence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers is 4539, and the maximum of the level is 17. We find that $\operatorname{Im}\varphi_X$ is generated by 3308 elements of $\mathrm{O}(S_X)$. Moreover the number of orbits of the action of $\mathrm{Aut}(X)$ on the set of smooth rational curves is at most 705.

Remark 10.2. The detailed computational data of the above three examples are presented in the author's web page [31].

Remark 10.3. The complex quartic surface X defined by $w^4 + x^4 + y^4 + z^4 = 0$ in $\mathbb{P}^3$ is a singular $K3$ surface with

$$T_X = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

We embed S_X into $\mathbf{L}$ primitively in such a way that the Weyl vector w_{0,S_X} of the initial $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_0 is in the interior of D_0 and is equal to the class of a hyperplane section of X in $\mathbb{P}^3$. We found that the number of G_X-congruence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers is *at least* 10000.

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