THE AUTOMORPHISM GROUP OF AN APÉRY-FERMI $K3$ SURFACE: COMPUTATIONAL DATA

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This note is an explanation of the computational data that appeared in the paper

[P] Ichiro Shimada: The automorphism group of an Apéry-Fermi $K3$ surface, preprint.

The data are written in two files

`AperyFermiCompData.txt`,

`AperyFermiFacesData.txt`,

in plain text format. For the computation, we used

[G] `GAP`–Groups, Algorithms, and Programming, Version 4.13.0 of 2024-03-15 (http://www.gap-system.org).

Some parts of the data are presented in the `Record` format of `GAP`.

1. The file `AperyFermiCompData.txt`

The file `AperyFermiCompData.txt` contains the following data.

1.1. Data in Sections 2, 3.2, 3.3, 3.4 of the paper [P].

A vector in S_X is expressed by a row vector of length 19 with respect to the basis fixed in the paper [P]. An element of $\mathrm{O}(S_X)$ is expressed by a 19×19 matrix with respect to the fixed basis.

- `GramSX` is the Gram matrix of S_X with respect to the fixed basis.
- `L32` is the list of records `rat` of the 32 rational curves C in $\mathcal{L}_{32}$. Each record `rat` has the following items:
  - `rat.vect` is the class of the smooth rational curve C.
  - `rat.LMtype` is the type $L_{\gamma_1\gamma_2\gamma_3}$ or $M_{k\alpha\beta}$ of $C \subset X_s$; for example,

    `rec(index:=[-1,-1,-1], type:="L")` for L_{---},

    `rec(index:=[ 3, 1, 1], type:="M")` for M_{3++}.
  - `rat.MOtype` is the type $P_i, Q_{ij}, T''_\nu, C''_{\mu\nu,\rho}$ of $C \subset Y''_t$; for example,

    `rec(index:=3, root:="NA",type:="P")` for P_3,

    `rec(index:=[3,4], root:="NA", type:="Q")` for Q_{34},

    `rec(index:=3, root:="NA", type:="T")` for T''_3,

    `rec(index:=[1,2], root:="a", type:="C")` for $C''_{12,\tau}$,

    `rec(index:=[1,2], root:="b", type:="C")` for $C''_{12,1/\tau}$

    where `"NA"` means "not available", `root := "a"` means $\rho = \tau$, whereas `root := "b"` means $\rho = 1/\tau$.
- `h8` is the class of a hyperplane section of the $(2, 2, 2)$-complete intersection model $\overline{X}_s$ in the hyperplane of $\mathbb{P}^6$.

- `h4` is the class of a hyperplane section of the quartic surface model $Y_{t(s)} \subset \mathbb{P}^3$.
- `a32` is the ample class a_{32}.
- `OSXL32` is the group $\mathrm{O}(S_X, \mathcal{L}_{32})$ of order 96, which is a list of 96 isometries of S_X.
- `nongeommu` is the involution $\mu \in \mathrm{O}(S_X, \mathcal{L}_{32})$ given by $L^{\mu}_{\gamma_1\gamma_2\gamma_3} = L_{\gamma_1\gamma_2\gamma_3}$ and $M^{\mu}_{k\alpha\beta} = M_{k(-\alpha)\beta}$. This involution is *not* contained in $\mathrm{Aut}(X, \mathcal{L}_{32})$.
- `AutXL32` is the group $\mathrm{Aut}(X, \mathcal{L}_{32})$ of order 48, which is a sublist of `OSXL32`.
- `enrinvol` is the Enriques involution $\varepsilon \in \mathrm{Aut}(X, \mathcal{L}_{32})$ given by $L^{\varepsilon}_{\gamma_1\gamma_2\gamma_3} = L_{(-\gamma_1)(-\gamma_2)(-\gamma_3)}$ and $M^{\varepsilon}_{k\alpha\beta} = M_{k(-\alpha)\beta}$.

1.2. **Data in Section 3.5 of the paper [P].** We fix a basis of L_{26}. (This basis is not described explicitly in the paper [P].)

- `GramL` is the Gram matrix of L_{26} with respect to the fixed basis.
- `GramR` is the Gram matrix of the negative definite root lattice R of type $D_5 + A_2$ with respect to the basis $f_1, \dots, f_7$ given in the paper [P].
- `EmbSXL` is the matrix representation of the primitive embedding $S_X \hookrightarrow L_{26}$, which is a 19×26 matrix.
- `EmbRL` is the matrix representation of the primitive embedding $R \hookrightarrow L_{26}$, which is a 7×26 matrix.
- `weyl0` is the Weyl vector $\mathbf{w}_0$.
- `weyl0prime` is the vector $\mathbf{w}'_0$ such that $\langle \mathbf{w}'_0, \mathbf{w}'_0 \rangle = 0$ and $\langle \mathbf{w}_0, \mathbf{w}'_0 \rangle = 1$.
- `OSXD0` is the finite group $\mathrm{O}(S_X, D_0)$, which is a list of 32 isometries of S_X.
- `AutXD0` is the finite group $\mathrm{Aut}(X, D_0)$, which is a sublist of `OSXD0`.
- `WallsD0` is the list of 80 records `wrec`, each of which describes a wall w of D_0 by the following items:
  - `wrec.vect` is the primitive defining vector of w.
  - `wrec.orbnumb` is the number ν of the orbit $o_\nu \subset \mathrm{Aut}(X, D_0)$ to which w belongs.
  - `wrec.innout` is either `"inner"` or `"outer"`.
  - `wrec.adjg` is an element g of $\mathrm{O}(S_X, \mathcal{P}_X)$ such that D_0^g is the L_{26}/S_X-chamber adjacent to D_0 across w. This isometry g satisfies $\eta(g) \in \{\pm 1\}$. See Section 5.1 of the paper [P].

1.3. **Data in Section 4 of the paper [P].** The dataset

`AutXgenerators`

is a list of nine records `grec`. Each `grec` describes the automorphism $g^{[\nu]} \in \mathrm{Aut}(X)$, where $\nu = 0, 3, 4, \dots, 10$. Each `grec` has the following items.

- `grec.nu` is the index ν.
- `grec.g` is the isometry of S_X induced by the automorphism $g^{[\nu]} \in \mathrm{Aut}(X)$.
- `grec.w` is the primitive defining vector of the inner wall $w \in o_\nu$ across which $(D_0)^{g^{[\nu]}}$ is adjacent to D_0. When $\nu = 0$, this item is `"NA"`. (Recall that $g^{[0]}$ is an element of $\mathrm{Aut}(X, D_0)$.)

Except for the case $\nu = 3$, the record `grec` has the following items that describe the Jacobian fibration $\phi^{[\nu]} \colon X \to \mathbb{P}^1$. When $\nu = 3$, these items are `"NA"`. (Recall that $g^{[3]}$ is an element of $\mathrm{Aut}(X, \mathcal{L}_{32})$.) Note that we use the same Jacobian fibration for $\nu = 5$ and $\nu = 6$. (Recall that $g^{[6]} = (g^{[5]})^{-1}$.) Hence the items below for $\nu = 5$ and $\nu = 6$ are identical.

- `grec.whitenodes` is the list of classes of smooth rational curves corresponding to the white nodes in the configuration for $\phi^{[\nu]}$.
- `grec.graynode` is the class of the smooth rational curve corresponding to the gray node in the configuration for $\phi^{[\nu]}$.
- `grec.blacknodes` is the list of classes of smooth rational curves corresponding to the black nodes in the configuration for $\phi^{[\nu]}$.
- `grec.f` is the class of a fiber f of $\phi^{[\nu]}$.
- `grec.z` is the class of the zero section z of $\phi^{[\nu]}$.
- `grec.inversion` is the isometry of S_X induced by the inversion $\iota(\phi^{[\nu]})$ of the generic fiber of $\phi^{[\nu]}$.
- `grec.singfibs` is the list of the records `fibrec`, each of which describes a reducible fiber $F := \phi^{[\nu]-1}(p)$ of $\phi^{[\nu]}$. Each `fibrec` has the following items.
  - `fibrec.ADEtype` is the ADE-type of the reducible fiber F (for example, `"E8"`).
  - `fibrec.C0` is the class of the reduced irreducible component of F that intersects the zero section z.
  - `fibrec.Cs` is the list of classes of reduced irreducible components of F that are disjoint from z.
- `grec.MW` describes the Mordell–Weil group

$$\mathrm{MW}(\phi^{[\nu]}) \cong \mathbb{Z}^r \oplus T$$

  of $\phi^{[\nu]}$, where T is the torsion part of $\mathrm{MW}(\phi^{[\nu]})$. This is a pair

`[r, [ ]]`	when $T = 0$, or
`[r, [a]]`	when $T \cong \mathbb{Z}/a\mathbb{Z}$, or
`[r, [a, b]]`	when $T \cong \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$.

- `grec.MWgens` describes a generating set of the Mordell-Weil group $\mathrm{MW}(\phi^{[\nu]})$ of $\phi^{[\nu]}$. This list consists of $r+\mathrm{leng}\,T$ records, where r is the rank of the free part of $\mathrm{MW}(\phi^{[\nu]}) \cong \mathbb{Z}^r \oplus T$, and $\mathrm{leng}\,T$ is the minimal number of generators of the torsion part T. Each record `mwrec` describes a non-zero element s of $\mathrm{MW}(\phi^{[\nu]})$. Among the records in `grec.MWgens`, r records give a generating set of the free part $\cong \mathbb{Z}^r$ of $\mathrm{MW}(\phi^{[\nu]})$, and $\mathrm{leng}\,T$ records give a generating set of T. The record `mwrec` corresponding to a section $s \in \mathrm{MW}(\phi^{[\nu]})$ has the following items:
  - `mwrec.s` is the class of the image of the section $s\colon \mathbb{P}^1 \to X$.
  - `mwrec.g` is the isometry of S_X induced by the automorphism given by the translation by s of the generic fiber of $\phi^{[\nu]}$.
  - `mwrec.order` is the order of s, which is either $\infty =$ `infinity` or an integer > 1.

2. The file `AperyFermiFacesData.txt`

The file `AperyFermiFacesData.txt` contains the following data about the faces of D_0 of codimension up to 5.

- `wlfs`,
- `AutXD0wlfs`,
- `FacesList`,
- `Codim2FacesRecs`,
- `ADEFacesRecs`.

2.1. **The datasets `wlfs`, `AutXD0wlfs`, and `FacesList`.**

- `wlfs` is the list of primitive defining vectors of walls of D_0 written in terms of the basis of $S_X^\vee$ dual to the fixed basis of S_X. Here the term `"wlfs"` means `"wall linear forms"`. (Caution: to obtain the vector representation of a primitive defining vector with respect to the fixed basis of S_X, we have to multiply the linear form given in `wlfs` by the inverse matrix of the Gram matrix of S_X.) Each wall of D_0 is expressed by the position at which its primitive defining vector appears in this list `wlfs`. We denote by w_i the wall of D_0 defined by the ith linear form in `wlfs`.
- `AutXD0wlfs` is the list of permutations on the list `wlfs` induced by the group $\mathrm{Aut}(X, D_0)$. Each permutation is presented by the data type `Permutation` of `GAP`. For example,

  $$(2, 21)(3, 18)(4, 19)(5, 17)(6, 20)(7, 14)\cdots(73, 77)(74, 76)$$

  is an element of `AutXD0wlfs`.

A face f of D_0 is expressed by a list of indexes $[i_1, \dots, i_\mu]$ of the walls passing through f; that is, if

$$f = w_{i_1} \cap \cdots \cap w_{i_\mu},$$

then f is expressed by the list $[i_1, \dots, i_\mu]$.

- `FacesList` is the list

  $$[\, \texttt{Faces1},\ \texttt{Faces2},\ \dots,\ \texttt{Faces5}\,]$$

  of five lists. The list `Faces`μ, where $\mu = 1, \dots, 5$, describes the set of $\mathrm{Aut}(X, D_0)$-orbits in $\mathcal{F}^\mu(D_0)$. Each orbit is expressed by a list of lists of indexes $[i_1, \dots, i_\mu]$ expressing faces $f = w_{i_1} \cap \cdots \cap w_{i_\mu}$.

For example, the chamber D_0 has exactly 1746 faces of codimension 2, and they are divided into 128 orbits by the action of $\mathrm{Aut}(X, D_0)$. Hence the list `Faces2` consists of 128 lists, each list consists of pairs $[i_1, i_2]$ of indexes of walls, and these pairs form an orbit under the action of the permutation group `AutXD0wlfs`. The sum of the sizes of orbits is equal to 1746.

2.2. **Diagrams for the `Record` format of `GAP`.**

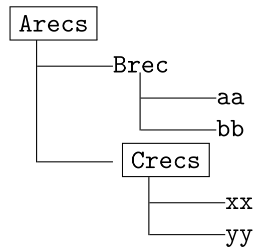
To explain the datasets `Codim2FacesRecs` and `ADEFacesRecs`, we use the following convention. A diagram

```
Arec
  |
  |------aa
  |
  |------bb
```

means that `Arec` is a record with items `Arec.aa` and `Arec.bb`, whereas a diagram

```
[Brecs]
  |
  |------xx
  |
  |------yy
```

means that `Brecs` is a list of records `Brec` with items `Brec.xx` and `Brec.yy`. Hence, for example, a diagram

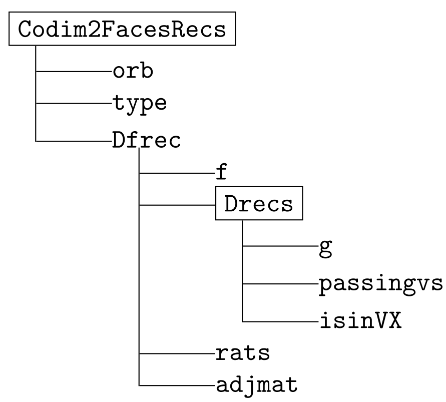


means that `Arecs` is a list of records `Arec`, each of which has two items `Arec.Brec` and `Arec.Crecs`. The item `Arec.Brec` is a record with items

`Arec.Brec.aa` and `Arec.Brec.bb`,

whereas `Arec.Crecs` is a list of records `Crec` with items `Crec.xx` and `Crec.yy`.

2.3. **The dataset `Codim2FacesRecs`.** This dataset is used in Section 5.2 of the paper [P], and has the following structure.



The dataset `Codim2FacesRecs` is a list of 128 records `Codim2FacesRec`, each of which describes an $\mathrm{Aut}(X, D_0)$-orbit in $\mathcal{F}^2(D_0)$. Each record `Codim2FacesRec` has the following items:

- `Codim2FacesRec.orb` is the list of faces in the orbit. Each face is expressed by the pair of indexes $[i_1, i_2]$ of the walls of D_0 passing through the face.
- `Codim2FacesRec.type` is the type n_{lr} = `[n, l, r]`, or 8_{80a} = `[8,8,0,"a"]`, or 8_{80b} = `[8,8,0,"b"]`, of the faces in the orbit.
- `Codim2FacesRec.Dfrec` is the record `Dfrec` describing $\mathcal{D}(f)$, where f is a representative of the orbit and is given by `Dfrec.f`. In particular, `Dfrec.f` is a member of `Codim2FacesRec.orb`.

We explain the items of the record `Dfrec` describing $\mathcal{D}(f)$, where f is a representative of the orbit `Codim2FacesRec.orb`.

- `Dfrec.f` is the face f given by the pair $[i_1, i_2]$ of indexes of the walls passing through f.
- `Dfrec.Drecs` is the list of records `Drec` describing the elements of $\mathcal{D}(f)$. Each `Drec` has the following items for describing $D \in \mathcal{D}(f)$.
  - `Drec.g` is an isometry $g \in \mathrm{O}(S_X, \mathcal{P}_X)$ such that $D = D_0^g$ and that $\eta(g) \in \{\pm 1\}$.

  – `Drec.passingvs` is the list of primitivive defining vectors v of the two walls $(v)^\perp \cap D$ of D passing through f. The vector v is written with respect to the fixed basis of S_X (not of $S_X^\vee$).
  – `Drec.isinVX` is `true` or `false`, and indicates whether $D \in V_X$ or not.

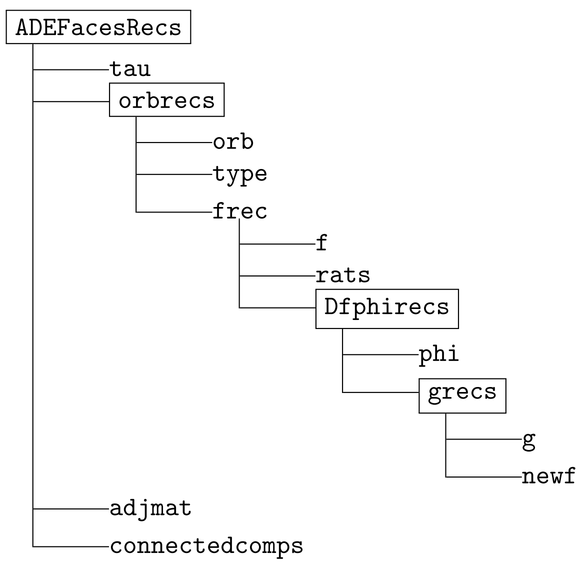
The initial member `Drecs[1]` of the list `Drecs` is the record for $D_0 \in \mathcal{D}(f)$; that is, the matrix `Drecs[1].g` is an element of $\mathrm{Aut}(X, D_0)$.

- `Dfrec.rats` is the list of $r \in \mathrm{Rats}(X)$ such that $f \subset (r)^\perp$.
- `Dfrec.adjmat` is the square matrix of size $|\mathcal{D}(f)|$ whose (i, j)-component is

$$\begin{cases} 1 & \text{if } i \neq j, \text{ and } D_i \text{ and } D_j \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

where D_i is the element of $\mathcal{D}(f)$ described by the record at the ith position in the list `Dfrec.Drecs`.

2.4. **The dataset `ADEFacesRecs`.** This dataset is used in Section 5.3 of the paper [P], and has the following structure.



The dataset `ADEFacesRecs` is a list of records `ADEFacesRec`, each of which describes the set $\mathcal{F}^\mu(D_0, \tau)$ for an ADE-types τ with Milnor number μ satisfying $2 \le \mu \le 4$. (The record for $\mu = 1$ is missing.) Each `ADEFacesRec` has the following items.

- `ADEFacesRec.tau` is the ADE-type τ, which is written, for example, by

```
[ "A1", "A1", "A2" ]
```

  for $2A_1 + A_2$.
- `ADEFacesRec.orbrecs` is the list of records `orbrec`, each of which describes an $\mathrm{Aut}(X, D_0)$-orbit in $\mathcal{F}^\mu(D_0, \tau)$. The items of the record `orbrec` are explained below.
- `ADEFacesRec.adjmat` is the adjacency matrix of the graph

$$[\mathcal{F}] := \mathcal{F}^\mu(D_0, \tau) / \mathrm{Aut}(X, D_0).$$

The ith node $[f_i]$ of this graph is the $\mathrm{Aut}(X, D_0)$-orbit in $\mathcal{F}^\mu(D_0, \tau)$ described by the record at the ith position of `ADEFacesRec.orbrecs`. The (i, j)-component of `ADEFacesRec.adjmat` is

$$\begin{cases} 1 & \text{if } i \neq j, \text{ and the nodes } [f_i] \text{ and } [f_j] \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

- `ADEFacesRec.connectedcomps` is the list of connected components of the graph $[\mathcal{F}]$. Each member of this list is a list $[i_1, \dots, i_n]$ of indexes of nodes that corresponds to a connected component $\{[f_{i_1}], \dots, [f_{i_n}]\}$ of $[\mathcal{F}]$.

Each member `orbrec` of `ADEFacesRec.orbrecs` has the following items. The record `orbrec` corresponds to an $\mathrm{Aut}(X, D_0)$-orbit in $\mathcal{F}^\mu(D_0, \tau)$, which is a node of $[\mathcal{F}]$.

- `orbrec.orb` is the list of faces in the orbit. Each face is given by the list of indexes of the walls passing through the face.
- `orbrec.type` is the type n_{lr} for the faces in the orbit when $\mu = 2$. When $\mu > 2$, the item `orbrec.type` is `"NA"`.
- `orbrec.frec` is a record describing a representative face of the orbit. Its items are explained below.

Let f be a representative face of the $\mathrm{Aut}(X, D_0)$-orbit in $\mathcal{F}^\mu(D_0, \tau)$ given by `orbrec`. The record `frec=orbrec.frec` has the following items:

- `frec.f` is the face f, which is given by the list of indexes of the walls passing through f. Hence `frec.f` is a member of `orbrec.orb`.
- `frec.rats` is the list of $r \in \mathrm{Rats}(X)$ such that $f \subset (r)^\perp$. They form the dual graph of ADE-type τ.
- `frec.Dfphirecs` is the list of records `Dfphirec`, each of which describes the set $\mathcal{D}(f, \varphi) := \widetilde{V}_{\mathcal{C}(f)} \cap \mathcal{D}(\varphi)$ for each $\varphi \in \mathcal{F}^{\mu+1}\langle f \rangle$. The items of `Dfphirec` are explained below.

Let φ be a face of D_0 with codimension $\mu + 1$ contained in f. The record `Dfphirec` describes the set $\mathcal{D}(f, \varphi) := \widetilde{V}_{\mathcal{C}(f)} \cap \mathcal{D}(\varphi)$ by the following items:

- `Dfphirec.phi` is the face φ, which is given by the list of indexes of the walls passing through φ.
- `Dfphirec.grecs` is the list of records `grec`, each of which describes an element $D \in \mathcal{D}(f, \varphi)$ by the following items:
  - `grec.g` is an automorphism $g \in \mathrm{Aut}(X)$ such that $D = D_0^g$.
  - `grec.newf` is the face $f' := \mathcal{P}_{\mathcal{C}(f)}{}^{(g^{-1})} \cap D_0$ of D_0, which is given by the list of indexes of the walls passing through f'.

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