THE AUTOMORPHISM GROUPS OF CERTAIN SINGULAR $K3$ SURFACES AND AN ENRIQUES SURFACE

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Abstract. We present finite sets of generators of the full automorphism groups of three singular $K3$ surfaces, on which the alternating group of degree 6 acts symplectically. We also present a finite set of generators of the full automorphism group of an associated Enriques surface, on which the Mathieu group M_{10} acts.

1. Introduction

For a complex $K3$ surface X, we denote by S_X the Néron-Severi lattice of X with the intersection form $\langle\ ,\ \rangle_S \colon S_X \times S_X \to \mathbb{Z}$, and by T_X the orthogonal complement of S_X in $H^2(X,\mathbb{Z})$ with respect to the cup-product. We call T_X the *transcendental lattice* of X. A complex $K3$ surface is said to be *singular* if the rank of S_X attains the possible maximum 20. By the result of Shioda and Inose [36], the isomorphism class of a singular $K3$ surface X is determined uniquely by its transcendental lattice T_X with the orientation given by the class $[\omega_X] \in T_X \otimes \mathbb{C}$ of a nowhere-vanishing holomorphic 2-form ω_X on X. Shioda and Inose [36] also showed that the automorphism group $\mathrm{Aut}(X)$ of a singular $K3$ surface X is infinite. It is an important problem to determine the structure of the automorphism groups of singular $K3$ surfaces.

In this paper, we study the automorphism groups of the following three singular $K3$ surfaces X_0, X_1, X_2; the Gram matrices of the transcendental lattice $T_k := T_{X_k}$ of X_k is

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \text{ for } k=0, \quad \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix} \text{ for } k=1, \quad \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} \text{ for } k=2. \tag{1.1}$$

(Note that the inversion of the orientation of T_k does not affect the isomorphism class of the singular $K3$ surface in these three cases. See, for example, [35].) These three $K3$ surfaces have a common feature in that they admit a symplectic action by the alternating group $\mathfrak{A}_6$ of degree 6. By the classification due to Mukai [20], we know that $\mathfrak{A}_6$ is one of the eleven maximal finite groups that act symplectically on complex $K3$ surfaces. (See also Kondo [17] and Xiao [41].) It was proved in [14] that every $K3$ surface with a symplectic action by $\mathfrak{A}_6$ is singular. A characterization of singular $K3$ surfaces with a symplectic action by $\mathfrak{A}_6$ is given in [10] (see also Remark 4.4).

The purpose of this paper is to present a finite set of generators of the full automorphism group $\mathrm{Aut}(X_k)$ of X_k for $k = 0, 1, 2$. Moreover, we describe the action of $\mathrm{Aut}(X_k)$ on the Néron-Severi lattice $S_k := S_{X_k}$. Furthermore, we calculate the automorphism group $\mathrm{Aut}(Z_0)$ of an Enriques surface Z_0 whose universal cover is X_0.

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Let X be a $K3$ surface. We let $\mathrm{Aut}(X)$ act on X from the left, and hence on S_X *from the right* by the pull-back. We denote by

$$\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X)$$

the natural representation of $\mathrm{Aut}(X)$ on S_X, where $\mathrm{O}(S_X)$ is the orthogonal group of the lattice S_X. Since the action of $\mathrm{Aut}(X)$ on $H^2(X, \mathbb{C})$ preserves the one-dimensional subspace $H^{2,0}(X)$, we also have a natural representation

$$\lambda_X \colon \mathrm{Aut}(X) \to \mathrm{GL}(H^{2,0}(X)) = \mathbb{C}^\times.$$

An automorphism $g \in \mathrm{Aut}(X)$ is said to be *symplectic* if $\lambda_X(g) = 1$, whereas we say that g is *purely non-symplectic* if the order of g is > 1 and equal to the order of $\lambda_X(g) \in \mathbb{C}^\times$. For a subgroup G of $\mathrm{Aut}(X)$, the subgroup $\mathrm{Ker}\, \lambda_X \cap G$ consisting of symplectic automorphisms belonging to G is called the *symplectic subgroup of* G. Let $\iota \in \mathrm{Aut}(X)$ be an involution. If ι is symplectic, then the quotient $X/\langle \iota \rangle$ is birational to a $K3$ surface. Otherwise, $X/\langle \iota \rangle$ is birational to either an Enriques surface or a rational surface. According to these cases, we say that ι is an *Enriques involution* or a *rational involution*.

Recall that the Néron-Severi lattice S_X is canonically isomorphic to the Picard group of X. A vector $h \in S_X$ with $n := \langle h, h \rangle_S > 0$ is called a *polarization of degree* n if the complete linear system $|\mathcal{L}_h|$ associated with a line bundle $\mathcal{L}_h \to X$ whose class is h is non-empty and has no fixed-components. For a polarization $h \in S_X$, we denote the automorphism group of the projective model of the polarized $K3$ surface (X, h) by

$$\mathrm{Aut}(X, h) := \{\, g \in \mathrm{Aut}(X) \mid h^g = h \,\}.$$

It is easy to see that $\mathrm{Aut}(X, h)$ is a finite group. Let $h \in S_X$ be a polarization of degree 2. Then the Galois transformation of the generically finite morphism $X \to \mathbb{P}^2$ of degree 2 induced by $|\mathcal{L}_h|$ gives rise to a rational involution

$$\tau(h) \colon X \to X$$

of X, which we call the *double-plane involution* associated with h.

Let X_k $(k = 0, 1, 2)$ be the three singular $K3$ surfaces defined above. Recall that S_k is the Néron-Severi lattice of X_k. We have the following:

Proposition 1.1. *The action* φ_{X_k} *of* $\mathrm{Aut}(X_k)$ *on* S_k *is faithful.*

Hence $\mathrm{Aut}(X_k)$ can be regarded as a subgroup of the orthogonal group $\mathrm{O}(S_k)$.

Our main results are as follows:

Theorem 1.2. (0) *The group* $\mathrm{Aut}(X_0)$ *is generated by a purely non-symplectic automorphism* $\rho_0^{(0)}$ *of order* 4 *and* $3 + 12$ *double-plane involutions*

$$\tau(h_0^{[1]}),\ \ldots,\ \tau(h_0^{[3]}),\ \tau(\tilde{h}_0^{(1)}),\ \ldots,\ \tau(\tilde{h}_0^{(12)}).$$

There exists an ample class $a_0 \in S_0$ *with* $\langle a_0, a_0 \rangle_S = 20$ *such that* $\mathrm{Aut}(X_0, a_0)$ *is a finite group of order* 1440. *This group* $\mathrm{Aut}(X_0, a_0)$ *is generated by* $\rho_0^{(0)}$ *and* $\tau(h_0^{[1]}), \ldots, \tau(h_0^{[3]})$. *The symplectic subgroup of* $\mathrm{Aut}(X_0, a_0)$ *is isomorphic to* $\mathfrak{A}_6$. *There exists a unique Enriques involution* $\varepsilon_0^{(0)}$ *in* $\mathrm{Aut}(X_0, a_0)$, *and the center of* $\mathrm{Aut}(X_0, a_0)$ *is generated by* $\varepsilon_0^{(0)}$.

(1) *The group* $\mathrm{Aut}(X_1)$ *is generated by a symplectic involution* $\sigma_1^{(4)}$ *and* $3+(12-1)$ *double-plane involutions*

$$\tau(h_1^{[1]}),\ \dots,\ \tau(h_1^{[3]}),\ \tau(\tilde{h}_1^{(1)}),\ \dots,\ \tau(\tilde{h}_1^{(3)}),\ \tau(\tilde{h}_1^{(5)}),\ \dots\ \tau(\tilde{h}_1^{(12)}).$$

There exists an ample class $a_1 \in S_1$ *with* $\langle a_1, a_1\rangle_S = 30$ *such that* $\mathrm{Aut}(X_1, a_1)$ *is isomorphic to the group* $\mathrm{PGL}_2(\mathbb{F}_9)$ *of order* 720. *This group* $\mathrm{Aut}(X_1, a_1)$ *is generated by* $\tau(h_1^{[1]}), \dots, \tau(h_1^{[3]})$, *and its symplectic subgroup is isomorphic to* $\mathrm{PSL}_2(\mathbb{F}_9) \cong \mathfrak{A}_6$.

(2) *The group* $\mathrm{Aut}(X_2)$ *is generated by* $3+7$ *double-plane involutions*

$$\tau(h_2^{[1]}),\ \dots,\ \tau(h_2^{[3]}),\ \tau(\tilde{h}_2^{(1)}),\ \dots,\ \tau(\tilde{h}_2^{(7)}).$$

There exists an ample class $a_2 \in S_2$ *with* $\langle a_2, a_2\rangle_S = 12$ *such that* $\mathrm{Aut}(X_2, a_2)$ *is isomorphic to the group* $\mathrm{PGL}_2(\mathbb{F}_9)$. *This group* $\mathrm{Aut}(X_2, a_2)$ *is generated by* $\tau(h_2^{[1]}), \dots, \tau(h_2^{[3]})$, *and its symplectic subgroup is isomorphic to* $\mathrm{PSL}_2(\mathbb{F}_9) \cong \mathfrak{A}_6$.

Remark 1.3. Part of the assertion on $\mathrm{Aut}(X_0, a_0)$ in Theorem 1.2 was proved in [14], and the group structure of $\mathrm{Aut}(X_0, a_0)$ was completely determined in [15]. The problem of determining the full automorphism group $\mathrm{Aut}(X_0)$ was suggested in [14].

In fact, in Corollary 3.5, we give an explicit basis of S_k by means of a Shioda-Inose elliptic fibration on X_k (see Definition 3.1). Using this basis, we obtain automorphisms generating $\mathrm{Aut}(X_k)$ in the form of 20×20 matrices belonging to $\mathrm{O}(S_k)$ by Borcherds method ([1], [2]). We then extract geometric properties of these automorphisms from their matrix representations computationally. Because of the size of the data, however, it is impossible to present all of these matrices in this paper. Instead, in Tables 8.3, 8.4 and 8.5, we give the polarizations $h_k^{[i]}$ and $\tilde{h}_k^{(i)}$ of degree 2 that appear in Theorem 1.2 in the form of row vectors, from which we can recover the matrices of $\tau(h_k^{[i]})$ and $\tau(\tilde{h}_k^{(i)})$ by the method described in Section 7. Moreover, we present the ADE-type of the singularities of the branch curve of the double covering $X_k \to \mathbb{P}^2$ induced by these polarizations. The matrices of the purely non-symplectic automorphism $\rho_0^{(0)} \in \mathrm{Aut}(X_0)$, the Enriques involution $\varepsilon_0^{(0)} \in \mathrm{Aut}(X_0)$, and the symplectic involution $\sigma_1^{(4)} \in \mathrm{Aut}(X_1)$ are given in Tables 8.2, 8.1 and 8.6, respectively. We also present the ample classes a_k in Table 5.2. For the readers' convenience, we put the matrices of the generators of $\mathrm{Aut}(X_k)$ and other computational data in the author's web paper [33].

Let X be a $K3$ surface, and let $\mathcal{P}(X)$ denote the connected component of $\{x \in S_X \otimes \mathbb{R} \,|\, \langle x, x\rangle_S > 0\}$ containing an ample class. We put

$$N(X) := \{\ x \in \mathcal{P}(X)\ \mid\ \langle x, C\rangle_S \geq 0\ \text{ for any curve } C \text{ on } X\ \ \}.$$

Then $\mathrm{Aut}(X)$ acts on $N(X)$. Next we investigate this action for $X = X_0, X_1, X_2$.

Let L be an even hyperbolic lattice with the symmetric bilinear form $\langle\ ,\ \rangle_L$, and let $\mathcal{P}(L)$ be one of the two connected components of $\{x \in L \otimes \mathbb{R} \,|\, \langle x, x\rangle_L > 0\}$, which we call a *positive cone* of L. We let the orthogonal group $\mathrm{O}(L)$ on L *from the right*, and put

$$\mathrm{O}^+(L) := \{\ g \in \mathrm{O}(L)\ \mid\ \mathcal{P}(L)^g = \mathcal{P}(L)\ \},$$

which is a subgroup of $\mathrm{O}(L)$ with index 2. For $v \in L \otimes \mathbb{R}$ with $\langle v, v\rangle_L < 0$, we denote by $(v)^\perp$ the real hyperplane

$$(v)^\perp := \{\ x \in \mathcal{P}(L)\ \mid\ \langle x, v\rangle_L = 0\ \}$$

of $\mathcal{P}(L)$. We put

$$\mathcal{R}(L) := \{\ r \in L\ \mid\ \langle r, r\rangle_L = -2\ \}.$$

Let $W(L)$ denote the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections

$$s_r \colon x \mapsto x + \langle x, r\rangle_L \cdot r$$

in the mirrors $(r)^\perp$ for $r \in \mathcal{R}(L)$. We call $W(L)$ the *Weyl group* of L. The closure in $\mathcal{P}(L)$ of each connected component of the complement

$$\mathcal{P}(L) \setminus \bigcup_{r \in \mathcal{R}(L)} (r)^\perp$$

of the union of the mirrors of $W(L)$ is a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}(L)$.

We denote by $L^\vee$ the *dual lattice* $\mathrm{Hom}(L, \mathbb{Z})$ of L, which contains L as a submodule of finite index and hence is canonically embedded into $L \otimes \mathbb{Q}$. A closed subset Σ of $\mathcal{P}(L)$ with non-empty interior is said to be a *chamber* if there exists a set Δ of $L^\vee$ such that $\langle v, v\rangle_L < 0$ for every $v \in \Delta$, such that the family of hyperplanes $\{(v)^\perp \mid v \in \Delta\}$ is locally finite in $\mathcal{P}(L)$, and such that

$$\Sigma = \{\ x \in \mathcal{P}(L)\ \mid\ \langle x, v\rangle_L \geq 0\ \text{ for any } v \in \Delta\ \}$$

holds. Let Σ be a chamber. A hyperplane $(v)^\perp$ of $\mathcal{P}(L)$ is said to be a *wall* of Σ if $(v)^\perp$ is disjoint from the interior of Σ and $(v)^\perp \cap \Sigma$ contains a non-empty open subset of $(v)^\perp$. Then there exists a unique subset $\mathit{\Delta}(\Sigma)$ of $L^\vee$ consisting of all *primitive* vectors v in $L^\vee$ such that the hyperplane $(v)^\perp$ is a wall of Σ, and such that $\langle x, v\rangle_L > 0$ holds for an interior point x of Σ; that is, $\mathit{\Delta}(\Sigma)$ is the *set of primitive outward defining vectors of walls of* Σ. We say that Σ is *finite* if $\mathit{\Delta}(\Sigma)$ is finite.

By Riemann-Roch theorem, we know that the cone $N(X)$ is a chamber in the positive cone $\mathcal{P}(X)$ containing an ample class of X, and that $N(X)$ is a standard fundamental domain of the action of the Weyl group $W(S_X)$ on $\mathcal{P}(X)$. Moreover $\mathit{\Delta}(N(X))$ is equal to the set of all primitive vectors $v \in S_X^\vee$ such that nv is the class of a smooth rational curve on X for some positive integer n. (See, for example, [25].)

The next result describes the chamber $N(X_k)$ of the three singular $K3$ surfaces X_k.

Theorem 1.4. *Let k be 0, 1 or 2, and let a_k be the ample class of X_k given in Theorem 1.2. Then there exists a finite chamber $D^{(0)}$ in $\mathcal{P}(X_k)$ with the following properties;*

(i) *the ample class a_k is in the interior of $D^{(0)}$, and the stabilizer subgroup*

$$\{\ g \in \mathrm{Aut}(X_k)\ \mid\ D^{(0)g} = D^{(0)}\ \}$$

of $D^{(0)}$ in $\mathrm{Aut}(X_k)$ coincides with $\mathrm{Aut}(X_k, a_k)$,

(ii) *$D^{(0)}$ is contained in $N(X_k)$, and $N(X_k)$ is the union of all $D^{(0)g}$, where g ranges through $\mathrm{Aut}(X_k)$,*

(iii) *if $g \in \mathrm{Aut}(X_k)$ is not contained in $\mathrm{Aut}(X_k, a_k)$, then $D^{(0)g}$ is disjoint from the interior of $D^{(0)}$, and*

(iv) *if $(v)^\perp$ is a wall of $D^{(0)}$ that is not a wall of $N(X_k)$, then there exists a unique chamber of the form $D^{(0)g}$ with $g \in \mathrm{Aut}(X_k)$ such that the intersection $(v)^\perp \cap D^{(0)} \cap D^{(0)g}$ contains a non-empty open subset of $(v)^\perp$.*

Therefore $N(X_k)$ is tessellated by the chambers $D^{(0)g}$, where g runs through a complete set of representatives of $\mathrm{Aut}(X_k, a_k)\backslash \mathrm{Aut}(X_k)$. In fact, this tessellation extends to a tessellation of $\mathcal{P}(X_k)$ by the chambers $D^{(0)g}$, where g runs through a complete set of representatives of $Aut(D^{(0)})\backslash \mathrm{O}^+(S_k)$, where

$$Aut(D^{(0)}) := \{\ g \in \mathrm{O}^+(S_k)\ \mid\ D^{(0)g} = D^{(0)}\ \}$$

is the stabilizer subgroup of $D^{(0)}$ in $\mathrm{O}^+(S_k)$. We call each chamber $D^{(0)g}$ in this tessellation an *induced chamber*. (See Definition 5.6 for a more general definition.) For a wall $(v)^\perp$ of $D^{(0)}$ that is not a wall of $N(X_k)$, the induced chamber $D^{(0)g}$ such that $(v)^\perp \cap D^{(0)} \cap D^{(0)g}$ contains a non-empty open subset of $(v)^\perp$ is called the *induced chamber adjacent to* $D^{(0)}$ *across* $(v)^\perp$.

In fact, we can write all elements of the set $\Delta(D^{(0)})$ explicitly in terms of the fixed basis of S_k. Note that $\mathrm{Aut}(X_k, a_k)$ acts on $\Delta(D^{(0)})$. We describe this action and clarify the meaning of the generators of $\mathrm{Aut}(X_k)$ given in Theorem 1.2.

Theorem 1.5. *Let $D^{(0)}$ be the finite chamber in $N(X_k)$ given in Theorem 1.4. The set $\Delta(D^{(0)})$ is decomposed into the orbits o_i in Table 1.1 under the action of $\mathrm{Aut}(X_k, a_k)$.*

If $v \in o_0$, then v is the class of a smooth rational curve on X_k, and hence $(v)^\perp$ is a wall of $N(X_k)$. If $k = 1$ and $v \in o_0'$, then $2v$ is the class of a smooth rational curve on X_1, and hence $(v)^\perp$ is a wall of $N(X_1)$.

Suppose that $i > 0$. Then there exists a vector $v_i \in o_i$ such that the involution $\tau(\tilde{h}_k^{(i)})$, or $\sigma_1^{(4)}$ in the case $k = 1$ and $i = 4$, in Theorem 1.2 maps $D^{(0)}$ to the induced chamber $D^{(i)}$ in $N(X_k)$ adjacent to $D^{(0)}$ across the wall $(v_i)^\perp$.

In Table 1.1, the cardinality $|o_i|$ of each orbit o_i is presented. The rational number ν indicates the square-norm $\langle v, v\rangle_S$ of the primitive vectors $v \in o_i$, and α indicates $\langle a_k, v\rangle_S$ for $v \in o_i$.

An involution of X_k that maps $D^{(0)}$ to the adjacent chamber $D^{(i)}$ is not unique. For $i \geq 0$, we put

$$\mathrm{Invols}_k^{(i)} := \{\ \iota \in \mathrm{Aut}(X_k)\ \mid\ \iota \text{ is of order 2 and maps } D^{(0)} \text{ to } D^{(i)}\ \}.$$

Proposition 1.6. *The set $\mathrm{Invols}_k^{(0)}$ of involutions in $\mathrm{Aut}(X_k, a_k)$ has the cardinality*

$$\begin{aligned} |\mathrm{Invols}_0^{(0)}| &= 91 = 45 + 1 + 45, \\ |\mathrm{Invols}_1^{(0)}| &= 81 = 45 + 0 + 36, \\ |\mathrm{Invols}_2^{(0)}| &= 81 = 45 + 0 + 36, \end{aligned}$$

where the right-hand summation means

$$\begin{aligned} & (\textit{the number of symplectic involutions}) \\ + \quad & (\textit{the number of Enriques involutions}) \\ + \quad & (\textit{the number of rational involutions}) \quad . \end{aligned}$$

In Table 1.1, the cardinality of the set $\mathrm{Invols}_k^{(i)}$ is also presented for $i > 0$ in the same manner. Remark that $\mathrm{Invols}_1^{(4)}$ contains no rational involutions, and hence we have to put the symplectic involution $\sigma_1^{(4)}$ in the set of generators of $\mathrm{Aut}(X_1)$ in Theorem 1.2. Note that, for $\iota \in \mathrm{Invols}_k^{(i)}$ with $i > 0$, the vector

$$a_k^{(i)} := a_k^\iota$$

	orbit	$\lvert o_i\rvert$	ν	α	$\lvert\mathrm{Invols}_k^{(i)}\rvert$	$\langle a_k^{(i)}, a_k\rangle_S$
$k=0$						
	o_0	60	-2	2		
	o_1	40	$-3/2$	3	$10=0+0+10$	32
	o_2	180	$-4/3$	4	$4=0+0+4$	44
	o_3	10	-1	4	$24=12+0+12$	52
	o_4	144	$-5/6$	5	$6=0+0+6$	80
	o_5	144	$-5/6$	5	$6=0+0+6$	80
	o_6	240	$-2/3$	6	$4=0+0+4$	128
	o_7	360	$-2/3$	6	$4=0+0+4$	128
	o_8	180	$-1/3$	6	$4=0+0+4$	236
	o_9	240	$-1/6$	7	$4=0+0+4$	608
	o_{10}	240	$-1/6$	7	$4=0+0+4$	608
	o_{11}	720	$-1/6$	7	$2=0+0+2$	608
	o_{12}	720	$-1/6$	7	$2=0+0+2$	608
$k=1$						
	o_0	45	-2	2		
	o_0'	45	$-1/2$	7		
	o_1	10	$-3/2$	3	$12=0+0+12$	42
	o_2	30	$-4/3$	4	$10=0+0+10$	54
	o_3	72	$-5/4$	5	$6=0+0+6$	70
	o_4	60	-1	6	$6=6+0+0$	102
	o_5	12	$-5/6$	5	$16=0+1+15$	90
	o_6	40	$-3/4$	6	$6=0+0+6$	126
	o_7	120	$-7/12$	7	$4=0+0+4$	198
	o_8	120	$-7/12$	7	$4=0+0+4$	198
	o_9	120	$-1/3$	8	$4=0+0+4$	414
	o_{10}	180	$-1/3$	8	$4=0+0+4$	414
	o_{11}	120	$-1/12$	8	$4=0+0+4$	1566
	o_{12}	120	$-1/12$	8	$4=0+0+4$	1566
$k=2$						
	o_0	36	-2	1		
	o_1	12	$-4/3$	2	$16=0+1+15$	18
	o_2	40	$-6/5$	3	$6=0+0+6$	27
	o_3	90	$-4/5$	4	$4=0+0+4$	52
	o_4	30	$-8/15$	4	$10=0+1+9$	72
	o_5	30	$-8/15$	4	$10=0+1+9$	72
	o_6	120	$-2/15$	5	$4=0+0+4$	387
	o_7	120	$-2/15$	5	$4=0+0+4$	387

TABLE 1.1. The orbit decomposition of $\Delta(D^{(0)})$ by $\mathrm{Aut}(X_k, a_k)$

is an interior point of the adjacent chamber $D^{(i)}$, and does not depend on the choice of $\iota \in \mathrm{Invols}_k^{(i)}$. The column $\langle a_k^{(i)}, a_k\rangle_S$ shows the degree of $a_k^{(i)}$ with respect to a_k.

As a corollary, we obtain the following:

Corollary 1.7. *The action of* $\mathrm{Aut}(X_k)$ *on the set of smooth rational curves on* X_k *is transitive for* $k = 0$ *and* $k = 2$*, whereas this action has exactly two orbits for* $k = 1$*.*

Borcherds method ([1], [2]) has been applied to the studies of the automorphism groups of $K3$ surfaces by several authors. We briefly review these works. In [16], Kondo applied it to the Kummer surface associated with the Jacobian variety of a generic genus 2 curve. In [7], Kondo and Dolgachev applied it to the supersingular $K3$ surface in characteristic 2 with the Artin invariant 1. In [8], Keum and Dolgachev applied it to the quartic Hessian surface. In [13], Kondo and Keum applied it to the Kummer surfaces associated with the product of elliptic curves. In [18], Kondo and the author applied it to the supersingular $K3$ surface in characteristic 3 with the Artin invariant 1. In [39], Ujikawa applied it to the singular $K3$ surface whose transcendental lattice is of discriminant 7. The singular $K3$ surfaces whose transcendental lattices are of discriminant 3 and 4 had been studied by Vinberg [40] by another method. On the other hand, in [11], we have shown that, in some cases, Borcherds method requires too much computation to be completed.

The complexity of our results suggests that the computer-aided calculation is indispensable in the study of automorphism groups of $K3$ surfaces. The procedure to execute Borcherds method on a computer has been already described in [32]. In fact, a part of the result on $\mathrm{Aut}(X_2)$ has been obtained in [32]. In [32], however, we did not discuss the problem of converting a matrix in $\mathrm{O}(S_X)$ to a geometric automorphism of X. In the present article, we give a method to derive geometric information of automorphisms from their action on S_X. It turns out that the notion of *splitting lines* ([31], [34]) is useful to describe the geometry of double plane models of X_k associated with the double-plane involutions of X_k. See Section 9 for examples.

The Enriques involution $\varepsilon_0^{(0)}$ in $\mathrm{Aut}(X_0, a_0)$ has been detected also by Mukai and Ohashi [21]. The Enriques surface

$$Z_0 := X_0/\langle \varepsilon_0^{(0)} \rangle$$

plays an important role in their classification of finite semi-symplectic automorphism groups of Enriques surfaces.

By the explicit description of $\mathrm{Aut}(X_0)$ and the chamber $D^{(0)}$ in $N(X_0)$ presented above, we can calculate the full automorphism group $\mathrm{Aut}(Z_0)$ of the Enriques surface Z_0. Let S_Z denote the Néron-Severi lattice of Z_0 with the intersection form $\langle\ ,\ \rangle_Z$. Then S_Z is an even unimodular hyperbolic lattice of rank 10. We have the following:

Proposition 1.8. *The natural homomorphism*

$$\varphi_Z \colon \mathrm{Aut}(Z_0) \to \mathrm{O}(S_Z)$$

is injective.

Therefore we can regard $\mathrm{Aut}(Z_0)$ as a subgroup of $\mathrm{O}(S_Z)$. Let $\mathrm{Cen}(\varepsilon_0^{(0)})$ be the centralizer subgroup

$$\{\ g \in \mathrm{Aut}(X_0)\ \mid\ g\,\varepsilon_0^{(0)} = \varepsilon_0^{(0)} g\ \}$$

of $\varepsilon_0^{(0)}$ in $\mathrm{Aut}(X_0)$. Since X_0 is the universal covering of Z_0, we have a natural surjective homomorphism

$$\zeta \colon \mathrm{Cen}(\varepsilon_0^{(0)}) \twoheadrightarrow \mathrm{Aut}(Z_0),$$

which induces an isomorphism $\mathrm{Cen}(\varepsilon_0^{(0)})/\langle\varepsilon_0^{(0)}\rangle \xrightarrow{\sim} \mathrm{Aut}(Z_0)$. By Theorem 1.2 (0), we have $\mathrm{Aut}(X_0, a_0) \subset \mathrm{Cen}(\varepsilon_0^{(0)})$. The subgroup $\zeta(\mathrm{Aut}(X_0, a_0))$ of $\mathrm{Aut}(Z_0)$ with order 720 is generated by

$$\zeta(\rho_0^{(0)}),\ \ \zeta(\tau(h_0^{[1]})),\ \ \zeta(\tau(h_0^{[2]})),\ \ \zeta(\tau(h_0^{[3]})). \tag{1.2}$$

We have the following:

Theorem 1.9. *The finite subgroup* $\zeta(\mathrm{Aut}(X_0, a_0))$ *of* $\mathrm{Aut}(Z_0)$ *is isomorphic to the Mathieu group* M_{10}. *The double-plane involution* $\tau(\tilde{h}_0^{(3)})$ *of* X_0 *belongs to* $\mathrm{Cen}(\varepsilon_0^{(0)})$. *The automorphism group* $\mathrm{Aut}(Z_0)$ *of* Z_0 *is generated by* $\zeta(\mathrm{Aut}(X_0, a_0))$ *and* $\zeta(\tau(\tilde{h}_0^{(3)}))$.

In fact, we present the generators (1.2) and $\zeta(\tau(\tilde{h}_0^{(3)}))$ of $\mathrm{Aut}(Z_0)$ in the form of 10×10 matrices with respect to a certain basis of S_Z (see Table 10.4). Moreover, we describe a chamber $D_Z^{(0)}$ of S_Z that plays the same role to $\mathrm{Aut}(Z_0)$ as the role $D^{(0)}$ plays to $\mathrm{Aut}(X_0)$.

To the best knowledge of the author, Theorem 1.9 is the first example of the application of Borcherds method to the study of automorphism groups of Enriques surfaces.

This paper is organized as follows. In Section 2, we fix notions and notation about lattices, and present three elementary algorithms that are used throughout this paper. In Section 3, we give a basis of S_k in Corollary 3.5, and a computational criterion for a vector in S_k to be nef in Corollary 3.6. In Section 4, we give a computational characterization of the image of the natural homomorphism φ_{X_k} from $\mathrm{Aut}(X_k)$ to $\mathrm{O}(S_k)$ and prove Proposition 1.1. In Section 5, we confirm that the requirements to use Borcherds method given in [32] are fulfilled in the cases of our singular $K3$ surfaces X_k, obtain a finite set of generators of $\mathrm{Aut}(X_k)$ in the form of matrices in $\mathrm{O}(S_k)$ by this method, and prove Theorems 1.4 and 1.5. The embedding of S_k into the even unimodular hyperbolic lattice L_{26} of rank 26 given in Table 5.1 is the key of this method. In Section 6, we give an algorithm to calculate the set of classes of smooth rational curves of a fixed degree on a polarized $K3$ surface. This algorithm plays an important role in the study of splitting lines of double plane models of $K3$ surfaces. In Section 7, we review a general theory of the involutions of $K3$ surfaces. In Section 8, we prove Theorem 1.2. In Section 9, we investigate some automorphisms on X_k in details by means of the notion of splitting lines. In Section 10, we prove Proposition 1.8 and Theorem 1.9 on the Enriques surface Z_0.

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Conventions. Throughout this paper, we work over $\mathbb{C}$. Every $K3$ surface is assumed to be algebraic. The symbol Aut denotes a geometric automorphism group, whereas *Aut* denotes a lattice-theoretic automorphism group.

2. Computational tools

2.1. **Lattices.** A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle_L \colon L \times L \to \mathbb{Z}$. Suppose that a basis $e_1, \ldots, e_n$ of a lattice L is given. The $n \times n$ matrix $(\langle e_i, e_j\rangle_L)$ is called the *Gram matrix* of L with respect to the basis $e_1, \ldots, e_n$. The *discriminant* $\mathrm{disc}\, L$ of L is the determinant of a Gram matrix of L. The group of isometries of a lattice L is denoted by $\mathrm{O}(L)$. We let $\mathrm{O}(L)$ act on L *from the right*, and, when a basis of L is given, each vector of $L \otimes \mathbb{R}$ is

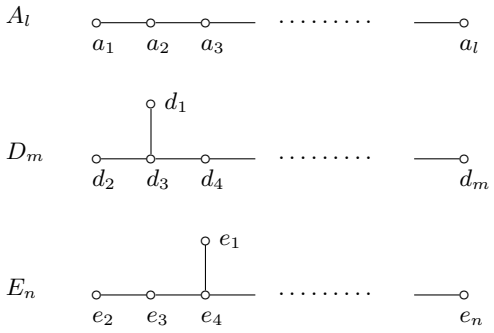

FIGURE 2.1. Indecomposable root systems

written as a *row* vector. A lattice L is *even* if $\langle v, v\rangle_L \in 2\mathbb{Z}$ holds for any $v \in L$. The *signature* of a lattice L is the signature of the real quadratic space $L \otimes \mathbb{R}$. A lattice L of rank n is *hyperbolic* if $n > 1$ and its signature is $(1, n-1)$, whereas L is *negative-definite* if its signature is $(0, n)$. A negative-definite lattice L is a *root lattice* if L is generated by the vectors in $\mathcal{R}(L) := \{r \in L \,|\, \langle r, r\rangle_L = -2\}$. The classification of root lattices is well-known (see, for example, Ebeling [9]). The roots in the indecomposable root systems of type A_l, D_m and E_n are labelled as in Figure 2.1. We denote by $L(m)$ the lattice obtained from L by multiplying $\langle\ ,\ \rangle_L$ by m, and we put $L^- := L(-1)$. For a subset A of a lattice L, we denote by $\langle A\rangle$ the $\mathbb{Z}$-submodule of L generated by the elements in A.

For an even lattice L, we denote by $L^\vee$ the dual lattice $\mathrm{Hom}(L, \mathbb{Z})$ of L, and by

$$q_L \colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z}$$

the *discriminant form* of L. See Nikulin [22] for the definition and basic properties of discriminant forms. The automorphism group of the finite quadratic form q_L is denoted by $\mathrm{O}(q_L)$. We have a natural homomorphism $\eta_L \colon \mathrm{O}(L) \to \mathrm{O}(q_L)$.

For square matrices $M_1, \dots, M_l$, let $\mathrm{diag}(M_1, \dots, M_l)$ denote the square matrix obtained by putting $M_1, \dots, M_l$ diagonally in this order and putting 0 on the other part.

2.2. **Three algorithms.** We use the following algorithms throughout this paper. See Section 3 of [34] for the details. Let L be a lattice. We assume that the Gram matrix of L with respect to a certain basis is given.

Algorithm 2.1. Suppose that L is negative-definite. Then, for a negative integer d, the finite set $\{v \in L \,|\, \langle v, v\rangle_L = d\}$ can be effectively calculated. ∎

Algorithm 2.2. Suppose that L is hyperbolic, and let a be a vector of L with $\langle a, a\rangle_L > 0$. Then, for integers b and d, the finite set

$$\{\ v \in L\ \mid\ \langle a, v\rangle_L = b,\ \ \langle v, v\rangle_L = d\ \}$$

can be effectively calculated. ∎

Algorithm 2.3. Suppose that L is hyperbolic. Let a_1 and a_2 be vectors of L satisfying $\langle a_1, a_1\rangle_L > 0$, $\langle a_2, a_2\rangle_L > 0$ and $\langle a_1, a_2\rangle_L > 0$. Then, for a negative integer d, the finite set

$$\{\ v \in L \ \mid\ \langle a_1, v\rangle_L > 0,\ \ \langle a_2, v\rangle_L < 0,\ \ \langle v, v\rangle_L = d\ \}$$

can be effectively calculated. ∎

3. Bases of the Néron-Severi lattices

In order to express elements of $\mathrm{Aut}(X_k)$ in the form of 20×20 matrices in $\mathrm{O}(S_k)$, we have to fix a basis of S_k. For this purpose, we review the theory of elliptic fibrations on $K3$ surfaces. See [37] or [29] for the details.

Let $\phi\colon X \to \mathbb{P}^1$ be an elliptic fibration on a $K3$ surface X with a zero-section $\sigma_0\colon \mathbb{P}^1 \to X$. We denote by $f_\phi \in S_X$ the class of a fiber of ϕ, by $z_\phi \in S_X$ the class of the image of σ_0, and by MW_ϕ the Mordell-Weil group of ϕ. We put

$$\mathcal{R}_\phi := \{\ v \in \mathbb{P}^1 \ \mid\ \phi^{-1}(v) \text{ is reducible }\},$$

and, for $v \in \mathcal{R}_\phi$, let $\Theta_{\phi,v} \subset S_X$ denote the sublattice spanned by the classes of irreducible components of $\phi^{-1}(v)$ that are disjoint from σ_0. Then each $\Theta_{\phi,v}$ is an indecomposable root lattice. We put

$$U_\phi := \langle f_\phi, z_\phi\rangle, \quad \Theta_\phi := \bigoplus_{v\in\mathcal{R}_\phi} \Theta_{\phi,v}.$$

Then U_ϕ is an even hyperbolic unimodular lattice of rank 2, and we have

$$\Theta_\phi = \langle\ r \in S_X \mid \langle r, f_\phi\rangle_S = \langle r, z_\phi\rangle_S = 0,\ \ \langle r, r\rangle_S = -2\ \rangle. \tag{3.1}$$

The sublattice

$$\mathrm{Triv}_\phi := U_\phi \oplus \Theta_\phi$$

of S_X is called the *trivial sublattice* of ϕ. For each element $\sigma\colon \mathbb{P}^1 \to X$ of MW_ϕ, let $[\sigma] \in S_X$ denote the class of the image of σ. Then the mapping $\sigma \mapsto [\sigma] \bmod \mathrm{Triv}_\phi$ induces an isomorphism

$$\mathrm{MW}_\phi \ \cong S_X/\mathrm{Triv}_\phi. \tag{3.2}$$

Recall that a reducible fiber $\phi^{-1}(v)$ is of type II^* if and only if $\Theta_{\phi,v}$ is the root lattice of type E_8.

Definition 3.1. An elliptic fibration on a $K3$ surface is called a *Shioda-Inose elliptic fibration* if it has a zero-section σ_0 and two singular fibers of type II^*.

Shioda and Inose [36] showed that every singular $K3$ surface has a Shioda-Inose elliptic fibration. Let X be a singular $K3$ surface with a Shioda-Inose elliptic fibration $\phi\colon X \to \mathbb{P}^1$. Let v and v' be the two points in $\mathcal{R}_\phi$ such that $\phi^{-1}(v)$ and $\phi^{-1}(v')$ are of type II^*, and let $e_1, \dots, e_8$ (resp. $e'_1, \dots, e'_8$) be the classes of the irreducible components of $\phi^{-1}(v)$ (resp. $\phi^{-1}(v')$) disjoint from σ_0 numbered in such a way that their dual graph is as in Figure 2.1. Then the 18 vectors

$$f_\phi, z_\phi, e_1, \dots, e_8, e'_1, \dots, e'_8 \tag{3.3}$$

span a hyperbolic *unimodular* sublattice $\mathrm{Triv}'_\phi := U_\phi \oplus \Theta_{\phi,v} \oplus \Theta_{\phi,v'}$ of Triv_ϕ. Let V_ϕ denote the orthogonal complement of Triv'_ϕ in S_X, so that we have an orthogonal direct-sum decomposition

$$S_X = \mathrm{Triv}'_\phi \oplus V_\phi. \tag{3.4}$$

Let V'_ϕ be the sublattice of V_ϕ generated by the vectors $r \in V_\phi$ with $\langle r, r\rangle_V = -2$, where $\langle\ ,\ \rangle_V$ is the symmetric bilinear form of the sublattice V_ϕ of S_X. By (3.1) and (3.2), we obtain

$$\Theta_\phi = \Theta_{\phi,v} \oplus \Theta_{\phi,v'} \oplus V'_\phi, \quad \mathrm{MW}_\phi \cong V_\phi/V'_\phi. \tag{3.5}$$

We apply these results to our three singular $K3$ surfaces X_k.

Proposition 3.2. *Let $\phi\colon X_k \to \mathbb{P}^1$ be a Shioda-Inose elliptic fibration on X_k. Then $V_\phi \cong T_k^-$.*

Proof. By (3.4) and the fact that Triv'_ϕ is unimodular, we have $q_{S_k} \cong q_{V_\phi}$. Since $H^2(X_k, \mathbb{Z})$ with the cup-product is an even unimodular overlattice of $S_k \oplus T_k$, we have $q_{S_k} \cong -q_{T_k}$ by Proposition 1.6.1 of [22]. Hence we have $q_{V_\phi} \cong q_{T_k^-}$. Note that V_ϕ is an even negative-definite lattice of rank 2 with discriminant 36 (resp. 24, resp. 15) if $k = 0$ (resp. $k = 1$, resp. $k = 2$). We can make a complete list of isomorphism classes of negative-definite lattices of rank 2 with a fixed discriminant d by the classical method of Gauss (see Chapter 15 of [6], for example). Looking at this list for $d = 36, 24$ and 15, we conclude that $V_\phi \cong T_k^-$ for $k = 0, 1, 2$. □

Remark 3.3. In general, the isomorphism class of the lattice V_ϕ depends on the choice of the Shioda-Inose elliptic fibration ϕ. See, for example, [27] or [30].

Proposition 3.4. *Let $\phi\colon X_k \to \mathbb{P}^1$ be a Shioda-Inose elliptic fibration on X_k, and let v and v' be as above.*

(0) *Suppose that $k = 0$. Then we have $\mathcal{R}_\phi = \{v, v'\}$, and MW_ϕ is a free $\mathbb{Z}$-module of rank 2 generated by elements σ_1, σ_2 such that the vectors*

$$s_1 := [\sigma_1] - 3f_\phi - z_\phi, \quad s_2 := [\sigma_2] - 3f_\phi - z_\phi \tag{3.6}$$

form a basis of V_ϕ with the Gram matrix

$$M_0 := \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}.$$

(1) *Suppose that $k = 1$. Then there exists a point $v'' \in \mathbb{P}^1$ such that $\mathcal{R}_\phi$ is equal to $\{v, v', v''\}$, and such that $\phi^{-1}(v'')$ is of type* I_2 *or* III. *Let C_1 be the irreducible component of $\phi^{-1}(v'')$ disjoint from the zero-section σ_0. Then MW_ϕ is a free $\mathbb{Z}$-module of rank 1 generated by an element σ_2 such that the vectors*

$$s_1 := [C_1], \quad s_2 := [\sigma_2] - 6f_\phi - z_\phi \tag{3.7}$$

form a basis of V_ϕ with the Gram matrix

$$M_1 := \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix}.$$

(2) *Suppose that $k = 2$. Then there exists a point $v'' \in \mathbb{P}^1$ such that $\mathcal{R}_\phi$ is equal to $\{v, v', v''\}$, and such that $\phi^{-1}(v'')$ is of type* I_2 *or* III. *Let C_1 be the irreducible component of $\phi^{-1}(v'')$ disjoint from σ_0. Then MW_ϕ is a free $\mathbb{Z}$-module of rank 1 generated by an element σ_2 such that the vectors*

$$s_1 := [C_1], \quad s_2 := -([\sigma_2] - 4f_\phi - z_\phi) \tag{3.8}$$

form a basis of V_ϕ with the Gram matrix

$$M_2 := \begin{bmatrix} -2 & -1 \\ -1 & -8 \end{bmatrix}.$$

Proof. By Proposition 3.2, V_ϕ has a basis s_1, s_2 with respect to which the Gram matrix of V_ϕ is M_k. Since

$$\{\ r \in V_\phi\ \mid\ \langle r, r\rangle_V = -2\ \} = \begin{cases} \emptyset & \text{if } k = 0, \\ \{s_1, -s_1\} & \text{if } k = 1 \text{ or } 2, \end{cases}$$

we have

$$\mathrm{MW}_\phi \cong V_\phi / V'_\phi = \begin{cases} \mathbb{Z}s_1 \oplus \mathbb{Z}s_2 & \text{if } k = 0, \\ \mathbb{Z}\bar{s}_2 & \text{if } k = 1 \text{ or } 2, \end{cases}$$

where $\bar{s}_2 := s_2 \bmod \langle s_1\rangle$. By (3.5), the assertions on $\mathcal{R}_\phi$, the type of $\phi^{-1}(v'')$ for $k = 1$ and 2, and the structure of MW_ϕ are proved. Note that, for an arbitrary element $\sigma \in \mathrm{MW}_\phi$, we have

$$\langle [\sigma], [\sigma]\rangle_S = -2, \quad \langle [\sigma], f_\phi\rangle_S = 1, \quad [\sigma] \perp \Theta_{\phi,v}, \quad [\sigma] \perp \Theta_{\phi,v'},$$

and, when $k = 1$ or 2, we have

$$\langle [\sigma], C_1\rangle_S = 0 \ \text{ or } \ 1.$$

The projection $\mathrm{pr}_V([\sigma])$ of $[\sigma]$ to V_ϕ with respect to the orthogonal direct-sum decomposition (3.4) is

$$[\sigma] - (2 + \langle [\sigma], z_\phi\rangle_S) f_\phi - z_\phi,$$

and its square-norm is $-4 - 2\langle [\sigma], z_\phi\rangle_S$.

Suppose that $k = 0$. Then we have generators σ_1, σ_2 of MW_ϕ such that $s_1 = \mathrm{pr}_V([\sigma_1])$ and $s_2 = \mathrm{pr}_V([\sigma_2])$. From $\langle s_1, s_1\rangle_S = \langle s_2, s_2\rangle_S = -6$, we obtain $\langle [\sigma_1], z_\phi\rangle_S = \langle [\sigma_2], z_\phi\rangle_S = 1$ and the equality (3.6) follows.

Suppose that $k = 1$ or 2. Changing s_1, s_2 to $-s_1, -s_2$ if necessary, we can assume that $s_1 = [C_1]$. Let σ_2 be a generator of $\mathrm{MW}_\phi \cong \mathbb{Z}$. Then $[C_1]$ and $\mathrm{pr}_V([\sigma_2])$ generate V_ϕ. In particular, we have $s_2 = x\,\mathrm{pr}_V([\sigma_2]) + y[C_1]$ for some $x, y \in \mathbb{Z}$. We put

$$t := \langle [\sigma_2], z_\phi\rangle_S, \quad u := \langle [\sigma_2], [C_1]\rangle_S = \langle [\sigma_2], s_1\rangle_S.$$

Note that $t \in \mathbb{Z}_{\geq 0}$ and $u \in \{0, 1\}$. Then we have

$$\langle s_1, s_2\rangle_S = xu - 2y, \tag{3.9}$$

$$\langle s_2, s_2\rangle_S = x^2(-4 - 2t) + 2xyu - 2y^2. \tag{3.10}$$

Suppose that $k = 1$. If $u = 1$, then we obtain $x = 2y$ from (3.9) and $\langle s_1, s_2\rangle_S = 0$, and hence $x^2(-7/2 - 2t) = -12$ holds from (3.10) and $\langle s_2, s_2\rangle_S = -12$. Since the equation $x^2(-7/2 - 2t) = -12$ has no integer solutions, we have $u = 0$. Then $y = 0$ and $x^2(-4 - 2t) = -12$ hold. The only integer solution of $x^2(-4 - 2t) = -12$ is $t = 4$ and $x = \pm 1$. Therefore, changing s_2 to $-s_2$ if necessary, we obtain (3.7). Suppose that $k = 2$. Since $\langle s_1, s_2\rangle_S = -1$, we obtain $u = 1$ and $x = 2y - 1$ from (3.9). Substituting $x = 2y - 1$ in $\langle s_2, s_2\rangle_S = -8$, we obtain a quadratic equation

$$(7 + 4t)y^2 - (7 + 4t)y + t - 2 = 0,$$

which has an integer solution only when $t = 2$. When $t = 2$, we have $(x, y) = (-1, 0)$ or $(1, 1)$. Changing s_2 to $-s_2 + s_1$ if necessary, we obtain (3.8). □

Corollary 3.5. *The Néron-Severi lattice S_k of X_k has a basis*

$$f_\phi, z_\phi, s_1, s_2, e_1, \ldots, e_8, e'_1, \ldots, e'_8, \tag{3.11}$$

where s_1, s_2 are obtained in Proposition 3.4. The Gram matrix of S_k with respect to this basis is

$$\mathrm{G}_k := \mathrm{diag}(\mathrm{U}_{\mathrm{ell}}, M_k, \mathrm{E}_8^-, \mathrm{E}_8^-),$$

where $\mathrm{U}_{\mathrm{ell}} := \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, *$M_k$ is defined in Proposition 3.4, and E_8^- is the Cartan matrix of type E_8 multiplied by -1.*

Throughout this paper, we use the basis (3.11) of S_k, and the Gram matrix G_k of S_k. Recall that $\mathrm{O}(S_k)$ acts on S_k *from the right*, so that we have

$$\mathrm{O}(S_k) = \{\ A \in \mathrm{GL}_{20}(\mathbb{Z})\ \mid\ A\,\mathrm{G}_k\,{}^tA = \mathrm{G}_k\ \}.$$

Next we investigate the chamber

$$\begin{aligned} N(X_k) &:= \{\ v \in \mathcal{P}(X_k)\ \mid\ \langle v, C\rangle_S \geq 0 \ \text{ for any curve } C \text{ on } X_k\ \} \\ &= \{\ v \in \mathcal{P}(X_k)\ \mid\ \langle v, C\rangle_S \geq 0 \ \text{ for any smooth rational curve } C \text{ on } X_k\ \} \end{aligned}$$

in the positive cone $\mathcal{P}(X_k)$ of S_k. By the definition of f_ϕ and z_ϕ, we see that the vector

$$a'_k := 2f_\phi + z_\phi$$

of square-norm 2 is nef, and hence is contained in $N(X_k)$. Moreover the set

$$\mathcal{B}_k := \{\ [C]\ \mid\ C \text{ is a smooth rational curve on } X_k \text{ with } \langle a'_k, C\rangle_S = 0\ \} \tag{3.12}$$

is equal to

$$\begin{cases} \{z_\phi, e_1, \ldots, e_8, e'_1, \ldots, e'_8\} & \text{if } k = 0, \\ \{z_\phi, s_1, e_1, \ldots, e_8, e'_1, \ldots, e'_8\} & \text{if } k = 1 \text{ or } 2. \end{cases} \tag{3.13}$$

Therefore we have the following criterion:

Corollary 3.6. *A vector $v \in S_k$ with $\langle v, v\rangle_S > 0$ is nef if and only if the following conditions are satisfied:*

(i) *$\langle v, a'_k\rangle_S > 0$, so that $v \in \mathcal{P}(X_k)$,*
(ii) *the set $\{r \in S_k \mid \langle r, r\rangle_S = -2,\ \langle r, a'_k\rangle_S > 0,\ \langle r, v\rangle_S < 0\}$ is empty, and*
(iii) *$\langle v, r\rangle_S \geq 0$ for all $r \in \mathcal{B}_k$.*

A nef vector $v \in S_k$ with $\langle v, v\rangle_S > 0$ is ample if and only if

$$\{\ r \in S_k\ \mid\ \langle r, r\rangle_S = -2,\ \langle r, v\rangle_S = 0\ \}$$

is empty.

Using Corollary 3.6 and Algorithms 2.2 and 2.3, we can determine whether a given vector $v \in S_k$ is nef or not, and ample or not.

4. Application of Torelli theorem to X_k

Let X be a $K3$ surface. The second cohomology group $H^2(X,\mathbb{Z})$ considered as an even unimodular lattice by the cup-product is denoted by H_X. By Proposition 1.6.1 of [22], the even unimodular overlattice H_X of $S_X \oplus T_X$ induces an isomorphism

$$\delta_H \colon q_{S_X} \xrightarrow{\sim} -q_{T_X}.$$

We regard the nowhere-vanishing holomorphic 2-form ω_X on X as a vector of $T_X \otimes \mathbb{C}$. If a $\mathbb{Q}$-rational subspace $T_{\mathbb{Q}}$ of $H_X \otimes \mathbb{Q}$ satisfies $\omega_X \in T_{\mathbb{Q}} \otimes \mathbb{C}$, then $T_{\mathbb{Q}}$ contains T_X. From this minimality of T_X, we see that, if $\gamma \in \mathrm{O}(H_X)$ preserves the subspace $H^{2,0}(X) = \mathbb{C}\,\omega_X$ of $H_X \otimes \mathbb{C}$, then γ preserves T_X. Moreover $\gamma \in \mathrm{O}(H_X)$ satisfies $\omega_X^\gamma = \omega_X$ if and only if γ acts on T_X trivially. We define the subgroup $\mathcal{C}_X$ of $\mathrm{O}(T_X)$ by

$$\mathcal{C}_X := \{\, \gamma \in \mathrm{O}(T_X) \mid \omega_X^\gamma = \lambda\,\omega_X \ \text{ for some } \ \lambda \in \mathbb{C}^\times \,\}. \tag{4.1}$$

For positive integers n, we define the subgroups $\mathcal{C}_X(n)$ of $\mathcal{C}_X$ by

$$\mathcal{C}_X(n) := \{\, \gamma \in \mathrm{O}(T_X) \mid \omega_X^\gamma = \lambda\,\omega_X \ \text{ for some } \ \lambda \in \mathbb{C}^\times \ \text{ with } \ \lambda^n = 1 \,\}.$$

Then we have $\mathcal{C}_X(1) = \{\mathrm{id}\}$. We denote by

$$\eta_S \colon \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X}), \quad \eta_T \colon \mathrm{O}(T_X) \to \mathrm{O}(q_{T_X})$$

the natural homomorphisms, and by

$$\delta_H^* \colon \mathrm{O}(q_{T_X}) \xrightarrow{\sim} \mathrm{O}(q_{S_X})$$

the isomorphism induced by the isomorphism $\delta_H \colon q_{S_X} \xrightarrow{\sim} -q_{T_X}$. By the definition of δ_H, an isometry $\gamma \in \mathrm{O}(S_X)$ of S_X extends to an isometry $\tilde{\gamma}$ of H_X that preserves the subspace $H^{2,0}(X) = \mathbb{C}\omega_X$ of $H_X \otimes \mathbb{C}$ if and only if

$$\eta_S(\gamma) \in \delta_H^*(\eta_T(\mathcal{C}_X)).$$

More precisely, an isometry $\gamma \in \mathrm{O}(S_X)$ extends to an isometry $\tilde{\gamma}$ of H_X that satisfies $\omega_X^{\tilde{\gamma}} = \lambda\,\omega_X$ with $\lambda^n = 1$ if and only if

$$\eta_S(\gamma) \in \delta_H^*(\eta_T(\mathcal{C}_X(n))).$$

By Torelli theorem for complex algebraic $K3$ surfaces due to Piatetski-Shapiro and Shafarevich [24], we have the following. Recall that we have the natural representations $\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X)$ and $\lambda_X \colon \mathrm{Aut}(X) \to \mathrm{GL}(H^{2,0}(X)) = \mathbb{C}^\times$ of $\mathrm{Aut}(X)$.

Theorem 4.1. *The kernel of φ_X is isomorphic to*

$$\{\, \gamma \in \mathcal{C}_X \mid \eta_T(\gamma) = \mathrm{id} \,\}.$$

The image of φ_X is equal to

$$\{\, \gamma \in \mathrm{O}(S_X) \mid N(X)^\gamma = N(X) \ \textit{and} \ \eta_S(\gamma) \in \delta_H^*(\eta_T(\mathcal{C}_X)) \,\}.$$

More precisely, the image of the subgroup $\{g \in \mathrm{Aut}(X) \mid \lambda_X(g)^n = 1\}$ of $\mathrm{Aut}(X)$ by φ_X is equal to

$$\{\, \gamma \in \mathrm{O}(S_X) \mid N(X)^\gamma = N(X) \ \textit{and} \ \eta_S(\gamma) \in \delta_H^*(\eta_T(\mathcal{C}_X(n))) \,\}.$$

We apply Theorem 4.1 to our singular $K3$ surfaces X_k. Let t_1, t_2 be the basis of T_k with the Gram matrix (1.1). We denote by $\langle\ ,\ \rangle_T$ the symmetric bilinear form of T_k. We have

$$|\mathrm{O}(T_k)| = \begin{cases} 8 & \text{if } k = 0, \\ 4 & \text{if } k = 1, 2, \end{cases} \qquad |\mathrm{O}(q_{T_k})| = \begin{cases} 16 & \text{if } k = 0, \\ 4 & \text{if } k = 1, 2. \end{cases}$$

Since $\langle \omega_{X_k}, \omega_{X_k} \rangle_T = 0$, we see that ω_{X_k} is equal to

$$\begin{cases} t_1 + \sqrt{-1}\, t_2 \ \text{ or } \ t_1 - \sqrt{-1}\, t_2 & \text{if } k = 0, \\ t_1 + \sqrt{-6}\, t_2 \ \text{ or } \ t_1 - \sqrt{-6}\, t_2 & \text{if } k = 1, \\ 8t_1 + (-1 + \sqrt{-15})\, t_2 \ \text{ or } \ 8t_1 + (-1 - \sqrt{-15})\, t_2 & \text{if } k = 2, \end{cases} \tag{4.2}$$

up to multiplicative constants, and the subgroup

$$\mathcal{C}_k := \mathcal{C}_{X_k}$$

of $\mathrm{O}(T_k)$ defined by (4.1) is equal to

$$\begin{cases} \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} & \text{if } k = 0, \\ \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} & \text{if } k = 1 \text{ or } 2. \end{cases} \tag{4.3}$$

(Note that $\mathcal{C}_k$ does not depend on the choice of the two possibilities of ω_{X_k} in (4.2).)

Proof of Proposition 1.1. By direct calculations, we see that η_T maps $\mathcal{C}_k$ into $\mathrm{O}(q_{T_k})$ injectively. □

The embedding $V_\phi = \langle s_1, s_2 \rangle \hookrightarrow S_k$ induces an isomorphism $q_{S_k} \cong q_{V_\phi}$. Let $\delta\colon q_{S_k} \xrightarrow{\sim} -q_{T_k}$ be the isomorphism induced from the isomorphism $V_\phi \xrightarrow{\sim} T_k^-$ given by $s_1 \mapsto t_1$, $s_2 \mapsto t_2$, and let

$$\delta^*\colon \mathrm{O}(q_{T_k}) \xrightarrow{\sim} \mathrm{O}(q_{S_k})$$

be the isomorphism induced by δ.

Lemma 4.2. *We have* $\delta_H^*(\eta_T(\mathcal{C}_k)) = \delta^*(\eta_T(\mathcal{C}_k))$.

Proof. By direct calculations, we see that $\eta_T(\mathcal{C}_k)$ is a normal subgroup of $\mathrm{O}(q_{T_k})$. Since δ_H^* and δ^* are conjugate, we obtain the proof. □

Therefore we can calculate the subgroups

$$\mathcal{C}'_k := \delta_H^*(\eta_T(\mathcal{C}_k)), \quad \mathcal{C}'_k(n) := \delta_H^*(\eta_T(\mathcal{C}_k(n))),$$

of $\mathrm{O}(q_{S_X})$, even though we do not know the isomorphism δ_H. Combining these with Proposition 1.1, we obtain the following computational criterion:

Corollary 4.3. *We put*

$$G_k := \{\ \gamma \in \mathrm{O}(S_k)\ \mid\ \eta_S(\gamma) \in \mathcal{C}'_k\ \}.$$

Let $a \in N(X_k)$ be an ample class. Then, by the natural representation φ_{X_k}, the group $\mathrm{Aut}(X_k)$ is identified with the subgroup $\{\gamma \in G_k \,|\, a^\gamma \text{ is ample}\}$ of $\mathrm{O}(S_k)$. Under this identification, for $g \in \mathrm{Aut}(X_k)$, we have $\lambda_{X_k}(g)^n = 1$ if and only if $\eta_S(g) \in \mathcal{C}'_k(n)$.

Remark 4.4. Let A_1 and A_2 be the positive-definite even lattices of rank 3 with Gram matrices

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 8 \end{bmatrix},$$

respectively. Suppose that X is a $K3$ surface on which $\mathfrak{A}_6$ acts symplectically. Then

$$H^2(X,\mathbb{Z})^{\mathfrak{A}_6} := \{v \in H^2(X,\mathbb{Z}) \,|\, v^g = v \text{ for any } g \in \mathfrak{A}_6\}$$

is isomorphic to A_1 or A_2 (see Table 10.3 of [10]). Hence X is singular, and its transcendental lattice is isomorphic to the orthogonal complement of an invariant polarization in $H^2(X,\mathbb{Z})^{\mathfrak{A}_6} \cong A_i$.

5. Borcherds method

Let L_{26} be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism (see, for example, Chapter V of [28]). We denote by $\langle\ ,\ \rangle_L$ the symmetric bilinear form of L_{26}. We choose a basis

$$f, z, \mathbf{e}_1, \dots, \mathbf{e}_8, \mathbf{e}'_1, \dots, \mathbf{e}'_8, \mathbf{e}''_1, \dots, \mathbf{e}''_8 \tag{5.1}$$

of L_{26} with respect to which the Gram matrix of L_{26} is equal to

$$\mathrm{diag}(\mathrm{U}_{\mathrm{ell}}, \mathrm{E}_8^-, \mathrm{E}_8^-, \mathrm{E}_8^-), \tag{5.2}$$

where $\mathrm{U}_{\mathrm{ell}}$ and E_8^- are given in Corollary 3.5. We consider the vector $w_0 \in L_{26}$ that is written as

$$\begin{aligned} w_0 \quad := \quad & (61, 30, -68, -46, -91, -135, -110, -84, -57, -29, \\ & -68, -46, -91, -135, -110, -84, -57, -29, \\ & -68, -46, -91, -135, -110, -84, -57, -29) \end{aligned} \tag{5.3}$$

in terms of the basis (5.1).

Remark 5.1. In terms of the basis of $L_{26}^\vee = L_{26}$ dual to (5.1), we have

$$w_0 = (30, 1, 1, \dots, 1)^\vee.$$

Note that we have $\langle w_0, w_0\rangle_L = 0$. Let $\mathcal{P}(L_{26})$ be the positive cone of L_{26} that contains w_0 in its closure. The real hyperplanes

$$(r)^\perp = \{\ x \in \mathcal{P}(L_{26})\ \mid\ \langle x, r\rangle_L = 0\ \}$$

of $\mathcal{P}(L_{26})$, where r ranges through $\mathcal{R}(L_{26}) = \{r \in L_{26} \,|\, \langle r, r\rangle_L = -2\}$, decompose $\mathcal{P}(L_{26})$ into the union of chambers, each of which is a standard fundamental domain of the action of the Weyl group $W(L_{26})$ on $\mathcal{P}(L_{26})$. We call these chambers *Conway chambers*. The action of $\mathrm{O}^+(L_{26})$ on $\mathcal{P}(L_{26})$ preserves this tessellation of $\mathcal{P}(L_{26})$ by Conway chambers.

Theorem 5.2. *We put*

$$\mathcal{W}_0 := \{\ r \in L_{26}\ \mid\ \langle r, r\rangle_L = -2,\ \ \langle r, w_0\rangle_L = 1\ \}.$$

Then the chamber

$$\mathcal{D}^{(0)} := \{\ x \in \mathcal{P}(L_{26})\ \mid\ \langle x, r\rangle_L \geq 0 \text{ for all } r \in \mathcal{W}_0\ \}$$

of $\mathcal{P}(L_{26})$ is a Conway chamber, and $(r)^\perp$ is a wall of $\mathcal{D}^{(0)}$ for any $r \in \mathcal{W}_0$.

Proof. By [5] and [3], it is enough to prove that $\langle w_0\rangle^\perp/\langle w_0\rangle$ is isomorphic the negative-definite Leech lattice; that is, $\langle w_0\rangle^\perp/\langle w_0\rangle$ is an even negative-definite unimodular lattice with no vectors of square-norm -2. The vector

$$\begin{aligned} w_0' \;:=\; (&62, 30, -71, -48, -95, -141, -115, -88, -60, -31,\\ &-68, -46, -91, -135, -110, -84, -57, -29,\\ &-68, -46, -91, -135, -110, -84, -57, -29) \end{aligned}$$

satisfies $\langle w_0, w_0'\rangle_L = 1$ and $\langle w_0', w_0'\rangle_L = 0$. Then the sublattice $\langle w_0, w_0'\rangle$ of L_{26} is an even unimodular hyperbolic lattice of rank 2, and $\langle w_0\rangle^\perp/\langle w_0\rangle$ is isomorphic to the orthogonal complement of $\langle w_0, w_0'\rangle$ in L_{26}. Hence $\langle w_0\rangle^\perp/\langle w_0\rangle$ is even, negative-definite and unimodular. Moreover we can calculate a Gram matrix of $\langle w_0\rangle^\perp/\langle w_0\rangle$. Using Algorithm 2.1, we can confirm that $\langle w_0\rangle^\perp/\langle w_0\rangle$ contains no vectors of square-norm -2. □

Corollary 5.3. *Any Conway chamber is equal to $\mathcal{D}^{(0)g}$ for some $g \in \mathrm{O}^+(L_{26})$.*

Since the vectors in $\mathcal{W}_0$ span L_{26}, the vector w_0 is uniquely determined by the condition $\langle w_0, r\rangle_L = 1$ for any $r \in \mathcal{W}_0$. Therefore $\mathcal{D}^{(0)g} = \mathcal{D}^{(0)g'}$ implies $w_0^g = w_0^{g'}$.

Definition 5.4. We call the vector w_0^g the *Weyl vector* of the Conway chamber $\mathcal{D}^{(0)g}$.

Let $\varepsilon_k\colon S_k \hookrightarrow L_{26}$ be the linear mapping given by

$$\varepsilon_k(f_\phi) = f, \quad \varepsilon_k(z_\phi) = z, \quad \varepsilon_k(e_i) = \mathbf{e}_i', \quad \varepsilon_k(e_i') = \mathbf{e}_i'',$$

and $\varepsilon_k(s_1), \varepsilon_k(s_2)$ are given in Table 5.1, in which $[c_1, \dots, c_8]$ denotes the vector

$$c_1\mathbf{e}_1 + \cdots + c_8\mathbf{e}_8$$

of L_{26}. We can easily confirm that ε_k is a primitive embedding of the lattice S_k into L_{26} by using the Gram matrices (3.5) and (5.2). From now on, we consider S_k as a primitive sublattice of L_{26} by ε_k. Let R_k denote the orthogonal complement of S_k in L_{26}. It turns out that R_k is a root lattice of type

$$\begin{cases} 2A_2 + 2A_1 & \text{if } k = 0,\\ A_3 + A_2 + A_1 & \text{if } k = 1,\\ A_4 + A_2 & \text{if } k = 2. \end{cases}$$

By Proposition 1.6.1 of [22], the even unimodular overlattice L_{26} of $S_k \oplus R_k$ induces an isomorphism

$$\delta_L\colon q_{R_k} \xrightarrow{\sim} -q_{S_k}.$$

Then δ_L induces an isomorphism

$$\delta_L^*\colon \mathrm{O}(q_{S_k}) \xrightarrow{\sim} \mathrm{O}(q_{R_k}).$$

Since R_k is negative-definite, we can calculate all elements of $\mathrm{O}(R_k)$ and their images by the natural homomorphism $\eta_R\colon \mathrm{O}(R_k) \to \mathrm{O}(q_{R_k})$. We have

$$|\mathrm{O}(R_k)| = \begin{cases} 2304 & \text{if } k = 0,\\ 1152 & \text{if } k = 1,\\ 2880 & \text{if } k = 2, \end{cases}$$

and see that η_R is surjective. In particular, by Proposition 1.4.2 of [22], the subgroup G_k of $\mathrm{O}(S_k)$

$$\begin{bmatrix} \varepsilon_0(s_1) \\ \varepsilon_0(s_2) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 & 6 & 6 & 6 & 4 & 2 \\ 6 & 4 & 8 & 12 & 9 & 6 & 4 & 2 \end{bmatrix}$$
$$\begin{bmatrix} \varepsilon_1(s_1) \\ \varepsilon_1(s_2) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 & 6 & 5 & 4 & 3 & 2 \\ 6 & 4 & 8 & 12 & 9 & 6 & 3 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \varepsilon_2(s_1) \\ \varepsilon_2(s_2) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 & 6 & 5 & 4 & 3 & 2 \\ 6 & 4 & 8 & 12 & 10 & 7 & 4 & 1 \end{bmatrix}$$

TABLE 5.1. The embeddings

defined in Corollary 4.3 satisfies the following:

Proposition 5.5. *Every element $\gamma \in G_k$ extends to an isometry $\tilde{\gamma} \in \mathrm{O}(L_{26})$.* □

It is easy to see that ε_k maps $\mathcal{P}(X_k)$ into $\mathcal{P}(L_{26})$.

Definition 5.6. A chamber D of $\mathcal{P}(X_k)$ is called an *induced chamber* if there exists a Conway chamber $\mathcal{D}$ such that $D = \mathcal{D} \cap \mathcal{P}(X_k)$. In this case, we say that D is *induced by* $\mathcal{D}$.

As will be seen in the proof of Theorems 1.4 and 1.5 below, this definition coincides with the definition of induced chambers in Introduction.

By definition, $\mathcal{P}(X_k)$ is tessellated by induced chambers, and for a wall $(v)^\perp$ of an induced chamber D, we can define the induced chamber adjacent to D across the wall $(v)^\perp$. By Proposition 5.5, we have the following:

Corollary 5.7. *The action of G_k on $\mathcal{P}(X_k)$ preserves the tessellation of $\mathcal{P}(X_k)$ by induced chambers.*

If $r \in S_k$ satisfies $\langle r, r\rangle_S = -2$, then we obviously have $\langle r, r\rangle_L = -2$. Therefore a wall of $N(X_k)$ is the intersection of a wall of a Conway chamber and $\mathcal{P}(X_k)$. Hence, if D is an induced chamber, then either D is contained in $N(X_k)$ or the interior of D is disjoint from $N(X_k)$. Therefore $N(X_k)$ is also tessellated by induced chambers.

We denote by

$$\mathrm{pr}_S \colon L_{26} \otimes \mathbb{Q} \to S_k \otimes \mathbb{Q}$$

the orthogonal projection. Note that $\mathrm{pr}_S(L_{26})$ is contained in $S_k^\vee$. For $r \in \mathcal{R}(L_{26})$, we put

$$r_S := \mathrm{pr}_S(r).$$

Using the fact that R_k contains a vector of square-norm -2 and hence cannot be embedded into the negative-definite Leech lattice, we have the following:

Proposition 5.8 (Algorithm 5.8 in [32])**.** *Suppose that the Weyl vector w of a Conway chamber $\mathcal{D}$ is given. Then the set*

$$\Delta_w := \{\ r \in \mathcal{R}(L_{26}) \ \mid\ \langle r, w\rangle_L = 1,\ \langle r_S, r_S\rangle_S < 0\ \}$$

is finite and can be effectively calculated.

$$\begin{aligned}
a_0 &= (122,\ 60,\ -11,\ -17,\ -136,\ -92,\ -182,\ -270,\ -220,\ -168,\ -114,\ -58,\\
&\qquad -136,\ -92,\ -182,\ -270,\ -220,\ -168,\ -114,\ -58)\\
a_1 &= (122,\ 60,\ -29,\ -8,\ -136,\ -92,\ -182,\ -270,\ -220,\ -168,\ -114,\ -58,\\
&\qquad -136,\ -92,\ -182,\ -270,\ -220,\ -168,\ -114,\ -58)\\
a_2 &= (61,\ 30,\ -12,\ -5,\ -68,\ -46,\ -91,\ -135,\ -110,\ -84,\ -57,\ -29,\\
&\qquad -68,\ -46,\ -91,\ -135,\ -110,\ -84,\ -57,\ -29)
\end{aligned}$$

TABLE 5.2. The ample vectors a_k

We put

$$a_k := \begin{cases} 2\,\mathrm{pr}_S(w_0) & \text{if } k = 0 \text{ or } 1,\\ \mathrm{pr}_S(w_0) & \text{if } k = 2. \end{cases}$$

Then a_k is a primitive vector of S_k contained in $\mathcal{P}(X_k)$. Its coordinates with respect to the basis (3.11) are given in Table 5.2. The square-norm $\langle a_k, a_k\rangle_S$ is given in Theorem 1.2.

Proposition 5.9. *The closed subset*

$$D^{(0)} := \mathcal{D}^{(0)} \cap \mathcal{P}(X_k)$$

of $\mathcal{P}(X_k)$ is an induced chamber that contains a_k in its interior and is contained in $N(X_k)$. In particular, $a_k \in S_k$ is ample.

Proof. For a vector $r \in L_{26}$ with $\langle r, r\rangle_L = -2$, the subset $(r)^\perp \cap \mathcal{P}(X_k) = \{x \in \mathcal{P}(X_k) \mid \langle r_S, x\rangle_S = 0\}$ of $\mathcal{P}(X_k)$ is equal to

$$\begin{cases} \text{the real hyperplane } (r_S)^\perp \text{ of } \mathcal{P}(X_k) & \text{if } \langle r_S, r_S\rangle_S < 0,\\ \mathcal{P}(X_k) & \text{if } r_S = 0,\\ \emptyset & \text{if } r_S \neq 0 \text{ and } \langle r_S, r_S\rangle_S \geq 0. \end{cases}$$

Moreover, because the embedding ε_k maps $\mathcal{P}(X_k)$ into $\mathcal{P}(L_{26})$, if $r \in \mathcal{W}_0$ satisfies $r_S \neq 0$ and $\langle r_S, r_S\rangle_S \geq 0$, then every point x of $\mathcal{P}(X_k)$ satisfies $\langle r_S, x\rangle_S > 0$. Note that $r \in \mathcal{W}_0$ satisfies $r_S = 0$ if and only if $r \in R_k$.

We first show that a_k is an interior point of the closed subset $D^{(0)}$ of $\mathcal{P}(X_k)$. We calculate the finite set $\Delta_{w_0} = \{r \in \mathcal{W}_0 \mid \langle r_S, r_S\rangle_S < 0\}$ by Proposition 5.8, and confirm that

$$\langle a_k, r\rangle_L > 0 \ \text{ for all } \ r \in \Delta_{w_0}.$$

Therefore, by the above consideration, we see that $\langle a_k, r_S\rangle_S = \langle a_k, r\rangle_L > 0$ for any $r \in \mathcal{W}_0$ with $r_S \neq 0$. Hence a_k is an interior point of $D^{(0)}$. Therefore $D^{(0)}$ is an induced chamber.

Next we show that a_k is ample. It is easy to see that $\langle a'_k, a_k\rangle_S > 0$, where a'_k is the nef vector $2f_\phi + z_\phi$. By Algorithms 2.2 and 2.3, we see that

$$\begin{aligned}
&\{\ r \in S_k \ \mid\ \langle r, a'_k\rangle_S > 0,\ \langle r, a_k\rangle_S < 0,\ \langle r, r\rangle_S = -2\ \} = \emptyset,\\
&\langle a_k, r\rangle_S > 0 \quad \text{for any } r \in \mathcal{B}_k,\\
&\{\ r \in S_k \ \mid\ \langle r, a_k\rangle_S = 0,\ \langle r, r\rangle_S = -2\ \} = \emptyset,
\end{aligned}$$

where $\mathcal{B}_k$ is defined by (3.12) and given in (3.13). By Corollary 3.6, we see that a_k is ample. Since $N(X_k)$ and the interior of $D^{(0)}$ have a common point a_k, we see that $D^{(0)}$ is contained in $N(X_k)$. □

Proof of Theorems 1.4 and 1.5. By the results proved so far, the assumptions required to use the main algorithm (Algorithm 6.1) of [32] are satisfied.

We calculate the set $\Delta(D^{(0)})$ of primitive outward defining vectors of walls of $D^{(0)}$ from the set Δ_{w_0} above by Algorithm 3.17 of [32]. Since $\Delta(D^{(0)})$ generate $S_k \otimes \mathbb{R}$, we can calculate the finite group

$$Aut(D^{(0)}) := \{\ \gamma \in \mathrm{O}(S_k)\ \mid\ D^{(0)\gamma} = D^{(0)}\ \} \tag{5.4}$$

by Algorithm 3.18 of [32]. Since a_k is an interior point of $D^{(0)}$ and the action of G_k preserves the decomposition of $\mathcal{P}(X_k)$ into the union of induced chambers by Proposition 5.5, we have

$$\mathrm{Aut}(X_k, a_k) = Aut(D^{(0)}) \cap G_k.$$

Indeed, a_k is proportional to the sum of the vectors in the orbit o_0 calculated bellow. Thus we can calculate all elements of the finite group $\mathrm{Aut}(X_k, a_k)$ in the form of matrices. Thus we obtain the set $\mathrm{Invols}_k^{(0)}$ of involutions in $\mathrm{Aut}(X_k, a_k)$.

We then calculate the orbits of the action of $\mathrm{Aut}(X_k, a_k)$ on $\Delta(D^{(0)})$. Let o_i be an orbit. We choose a vector $v_i \in o_i$. Suppose that there exists a positive integer n such that $nv_i \in S_k$ and $n^2\langle v_i, v_i\rangle_S = -2$. Then $(v_i)^\perp = (nv_i)^\perp$ is a wall of $N(X_k)$. This occurs only when $o_i = o_0$ or ($k = 1$ and $o_i = o_0'$). Suppose that there exists no such positive integer n. Then the induced chamber $D^{(i)}$ adjacent to $D^{(0)}$ across the wall $(v_i)^\perp$ is contained in $N(X_k)$. By Algorithm 5.14 of [32], we calculate the Weyl vector $w_i \in L_{26}$ such that the corresponding Conway chamber $\mathcal{D}^{(i)}$ induces $D^{(i)}$. From w_i, we calculate $\Delta(D^{(i)})$ by Algorithms 3.17 and 5.8 of [32]. We then use Algorithm 3.19 of [32] to search for an element $\tilde{\gamma}_k^{(i)} \in G_k$ such that $D^{(0)\tilde{\gamma}_k^{(i)}} = D^{(i)}$. It turns out that there *does* exist such an isometry $\tilde{\gamma}_k^{(i)}$. Hence all induced chambers in $N(X_k)$ are congruent under the action of G_k, and $\mathrm{Aut}(X_k)$ is generated by $\mathrm{Aut}(X_k, a_k)$ and the isometries $\tilde{\gamma}_k^{(i)}$. Finally, we calculate the set

$$\mathrm{Invols}_k^{(i)} = \{\ g \cdot \tilde{\gamma}_k^{(i)}\ \mid\ g \in \mathrm{Aut}(X_k, a_k),\ \text{and}\ \ g \cdot \tilde{\gamma}_k^{(i)}\ \text{is of order 2}\ \}$$

of involutions in G_k that map $D^{(0)}$ to $D^{(i)}$. □

Remark 5.10. The index of $\mathrm{Aut}(X_k, a_k)$ in the stabilizer subgroup $Aut(D^{(0)})$ of $D^{(0)}$ in $\mathrm{O}(S_k)$ is 4 for $k = 0$, and is 2 for $k = 1$ and $k = 2$. Under the action of this larger group $Aut(D^{(0)})$, the orbits o_i fuse as follows:

$$\begin{aligned}
&\text{for } k = 0: && o_0,\ o_1,\ o_2,\ o_3,\ o_4 \cup o_5,\ o_6,\ o_7,\ o_8,\ o_9 \cup o_{10},\ o_{11} \cup o_{12},\\
&\text{for } k = 1: && o_0,\ o_0',\ o_1,\ o_2,\ o_3,\ o_4,\ o_5,\ o_6,\ o_7 \cup o_8,\ o_9,\ o_{10},\ o_{11} \cup o_{12},\\
&\text{for } k = 2: && o_0,\ o_1,\ o_2,\ o_3,\ o_4 \cup o_5,\ o_6 \cup o_7.
\end{aligned}$$

By the work in this section, we have obtained a finite set of generators of $\mathrm{Aut}(X_k)$ in the form of matrices in $\mathrm{O}(S_k)$. Our next task is to realize them geometrically.

6. Smooth rational curves on a $K3$ surface

From now on to Section 9, we omit the subscript S in $\langle\ ,\ \rangle_S$.

6.1. **An algorithm to calculate the classes of smooth rational curves.** In order to obtain geometric information of an automorphism g of a $K3$ surface X from its action $v \mapsto v^g$ on the Néron-Severi lattice S_X of X, we introduce the following computational tool.

Proposition 6.1. *Let $h \in S_X$ be a polarization of degree $n := \langle h, h\rangle > 0$. Suppose that an ample class $a \in S_X$ is given. Then, for each non-negative integer d, we can calculate effectively the set*

$$\mathcal{C}_d(h) := \{\ [\Gamma] \in S_X \ \mid\ \Gamma \text{ is a smooth rational curve on } X \text{ such that } \langle h, \Gamma\rangle = d\ \}.$$

First we prove two lemmas. In the following, we fix a nef class $h \in S_X$ and an ample class $a \in S_X$.

Lemma 6.2. *Let D be an effective divisor on X with $\langle D, D\rangle < 0$, and let*

$$D = \Gamma_0 + \cdots + \Gamma_m + M$$

be a decomposition of D such that $\Gamma_0, \ldots, \Gamma_m$ are smooth rational curves and either $M = 0$ or M is effective with no fixed components in $|M|$. Then there exists a smooth rational curve Γ_i among $\Gamma_0, \ldots, \Gamma_m$ such that $\langle D, \Gamma_i\rangle < 0$.

Proof. If $\langle D, \Gamma_i\rangle \geq 0$ for $i = 0, \ldots, m$, then $\langle D, D\rangle = \sum \langle D, \Gamma_i\rangle + \langle D, M\rangle \geq 0$. □

Lemma 6.3. *Suppose that $v \in S_X$ satisfies $\langle v, v\rangle = -2$ and $\langle a, v\rangle > 0$. Then the following conditions are equivalent:*

(i) *The vector v is not the class of a smooth rational curve.*

(ii) *There exists a smooth rational curve Γ satisfying the following:*

$$\langle a, \Gamma\rangle < \langle a, v\rangle, \quad \langle h, \Gamma\rangle \leq \langle h, v\rangle, \quad \langle v, \Gamma\rangle < 0.$$

Suppose further that h is a polarization of degree $n := \langle h, h\rangle > 0$ and that $\langle h, v\rangle > 0$. Then the above two conditions are equivalent to the following:

(iii) *There exists a smooth rational curve Γ satisfying the following:*

$$\langle a, \Gamma\rangle < \langle a, v\rangle, \quad \langle h, \Gamma\rangle < \langle h, v\rangle, \quad \langle v, \Gamma\rangle < 0.$$

Proof. By $\langle v, v\rangle = -2$ and $\langle a, v\rangle > 0$, there exists an effective divisor D such that v is the class of D. Let $D = \Gamma_0 + \cdots + \Gamma_m + M$ be a decomposition of D such that $\Gamma_0, \ldots, \Gamma_m$ are smooth rational curves and either $M = 0$ or M is effective with no fixed components in $|M|$. By Lemma 6.2, we can assume that $\langle v, \Gamma_0\rangle = \langle D, \Gamma_0\rangle < 0$. Since h is nef, we have $\langle h, \Gamma_0\rangle \leq \langle h, D\rangle$.

Suppose that D is not irreducible. Then $m > 0$ or $M \neq 0$. In either case, we have $\langle a, \Gamma_0\rangle < \langle a, D\rangle$. Hence (ii) holds by taking Γ_0 as Γ. Suppose that (ii) holds. Since $\langle D, \Gamma\rangle < 0$, Γ is one of $\Gamma_0, \ldots, \Gamma_m$. Since $\langle a, \Gamma\rangle < \langle a, D\rangle$, we have $D \neq \Gamma$, and hence D is not irreducible. Thus the first part of Lemma 6.3 is proved.

Suppose that h is a polarization and that $d := \langle h, v\rangle > 0$. The implication (iii) $\Longrightarrow$ (ii) is obvious. We assume (i) and prove that (iii) holds. If $M \neq 0$, then $\langle h, M\rangle > 0$. Hence we have $\langle h, \Gamma_0\rangle < d$, and (iii) holds by taking Γ_0 as Γ. Therefore we can assume that $M = 0$ and $m > 0$. If $\langle h, \Gamma_0\rangle < d$, then (iii) holds by taking Γ_0 as Γ. Therefore we further assume that $\langle h, \Gamma_0\rangle = d$. Then we have

$$\langle h, \Gamma_i\rangle = 0 < d \ \text{ for } \ i = 1, \ldots, m. \tag{6.1}$$

If $\langle v, \Gamma_i\rangle < 0$ for some $i > 0$, then (iii) holds by taking Γ_i as Γ. Therefore we assume

$$\langle v, \Gamma_i\rangle \geq 0 \ \text{ for } i = 1, \ldots, m, \tag{6.2}$$

and derive a contradiction. For simplicity, we put

$$\Sigma_j := \sum_{i=0}^{j} \Gamma_i, \quad \Xi_j := \sum_{i=j+1}^{m} \Gamma_i.$$

Note that Γ_0 is distinct from any of $\Gamma_1, \ldots, \Gamma_m$. Since $\langle \Gamma_i, \Gamma_0 \rangle \geq 0$ for $i > 0$ and

$$\langle v, \Gamma_0 \rangle = \langle D, \Gamma_0 \rangle = -2 + \langle \Xi_0, \Gamma_0 \rangle < 0,$$

we have $\langle \Xi_0, \Gamma_0 \rangle = 0$ or 1. If $\langle \Xi_0, \Gamma_0 \rangle = 0$, then $\langle D, D \rangle = -2$ implies that $\langle \Xi_0, \Xi_0 \rangle = 0$. Since the class of Ξ_0 belongs to the orthogonal complement $[h]^\perp$ of h in S_X by (6.1), and $[h]^\perp$ is negative-definite because $\langle h, h \rangle > 0$, we obtain $\Xi_0 = 0$, which contradicts the assumption (i). Hence $\langle \Xi_0, \Gamma_0 \rangle = 1$, and therefore there exists a curve Γ_i among $\Gamma_1, \ldots, \Gamma_m$, say Γ_1, such that

$$\langle \Gamma_0, \Gamma_1 \rangle = 1, \qquad \langle \Gamma_0, \Gamma_i \rangle = 0 \quad (i = 2, \ldots, m).$$

We consider the following property P_k:

(a) $\{\Gamma_0, \ldots, \Gamma_k\} \cap \{\Gamma_{k+1}, \ldots, \Gamma_m\} = \emptyset$,
(b) $\Gamma_0, \ldots, \Gamma_k$ form an A_{k+1}-configuration of smooth rational curves.
(c) $\langle \Gamma_i, \Gamma_j \rangle = 0$ if $i < k$ and $j > k$.
(d) $\langle \Xi_k, \Gamma_k \rangle = 1$.

We have shown that the property P_0 holds. (The property (c) is vacuous for P_0.)

Claim 6.4. Suppose that the property P_k holds. Then, after renumbering of $\Gamma_{k+1}, \ldots, \Gamma_m$, the property P_{k+1} holds.

Proof of Claim 6.4. Since $\langle \Xi_k, \Gamma_k \rangle = 1$ and $\Gamma_k \notin \{\Gamma_{k+1}, \ldots, \Gamma_m\}$, there exists a unique element, say Γ_{k+1}, in the set $\{\Gamma_{k+1}, \ldots, \Gamma_m\}$ such that $\langle \Gamma_k, \Gamma_{k+1} \rangle = 1$ and $\langle \Gamma_k, \Gamma_j \rangle = 0$ for $j > k+1$. Then we have that

$$\Gamma_{k+1} \notin \{\Gamma_{k+2}, \ldots, \Gamma_m\}, \tag{6.3}$$

that $\Gamma_0, \ldots, \Gamma_{k+1}$ form an A_{k+2}-configuration of smooth rational curves, and that $\langle \Gamma_i, \Gamma_j \rangle = 0$ if $i < k+1$ and $j > k+1$. Therefore it is enough to show that $\langle \Xi_{k+1}, \Gamma_{k+1} \rangle = 1$. We have $\langle \Sigma_k, \Sigma_k \rangle = -2$ by (b) for the property P_k, and $\langle \Sigma_k, \Xi_k \rangle = 1$ by (c) and (d) for P_k. From $D^2 = (\Sigma_k + \Xi_k)^2 = -2$, we obtain $\Xi_k^2 = -2$. By Lemma 6.2, there exists an irreducible component Γ_l of Ξ_k such that $\langle \Xi_k, \Gamma_l \rangle < 0$. If $l > k+1$, then we have $\langle \Gamma_i, \Gamma_l \rangle = 0$ for $i \leq k$, and hence $\langle D, \Gamma_l \rangle = \langle \Xi_k, \Gamma_l \rangle < 0$, which contradicts the assumption (6.2). Hence we have $l = k+1$. From

$$\langle \Xi_k, \Gamma_{k+1} \rangle = -2 + \langle \Xi_{k+1}, \Gamma_{k+1} \rangle < 0$$

and $\langle \Xi_{k+1}, \Gamma_{k+1} \rangle \geq 0$ by (6.3), we see that $\langle \Xi_{k+1}, \Gamma_{k+1} \rangle = 0$ or 1. If $\langle \Xi_{k+1}, \Gamma_{k+1} \rangle = 0$, then $\langle \Xi_{k+1}, \Sigma_{k+1} \rangle = 0$ by (c) for P_{k+1} and, from $D^2 = (\Xi_{k+1} + \Sigma_{k+1})^2 = -2$ and $\Sigma_{k+1}^2 = -2$ by (b) for P_{k+1}, we have $\Xi_{k+1}^2 = 0$. Since the class of Ξ_{k+1} belongs to the negative-definite lattice $[h]^\perp$, we have $\Xi_{k+1} = 0$, and hence $D = \Sigma_{k+1}$. Then $\langle D, \Gamma_{k+1} \rangle < 0$, which contradicts the assumption (6.2). Therefore $\langle \Xi_{k+1}, \Gamma_{k+1} \rangle = 1$. □

Since the property P_0 holds, the property P_m holds by Claim 6.4, which says that $\Gamma_0, \ldots, \Gamma_m$ form an A_{m+1}-configuration. This contradicts (6.2) for $i = m$. □

Proof of Proposition 6.1. Since $\langle h, h\rangle > 0$, we can calculate the finite set

$$\mathcal{V}_d := \{\ v \in S_X\ \mid\ \langle h, v\rangle = d,\ \langle a, v\rangle > 0,\ \langle v, v\rangle = -2\ \}$$

by Algorithm 2.2. Suppose that $d = 0$. We decompose $\mathcal{V}_0$ into the disjoint union of subsets

$$\mathcal{V}_0[\alpha_i] := \{\ v \in \mathcal{V}_d\ \mid\ \langle a, v\rangle = \alpha_i\ \}$$

with $0 < \alpha_0 < \cdots < \alpha_N$. We calculate $\mathcal{C}_0[\alpha_i]$ inductively on i by setting $\mathcal{C}_0[\alpha_0] := \mathcal{V}_0[\alpha_0]$, and

$$\mathcal{C}_0[\alpha_i] := \{\ v \in \mathcal{V}_0[\alpha_i]\ \mid\ \text{there exist no vectors } \gamma \text{ in } \bigcup_{j<i} \mathcal{C}_0[\alpha_j] \text{ such that } \langle v, \gamma\rangle < 0\ \}.$$

Then the union of $\mathcal{C}_0[\alpha_0], \ldots, \mathcal{C}_0[\alpha_N]$ is the set $\mathcal{C}_0(h)$. Suppose that $d > 0$ and that the set $\mathcal{C}_{d'}(h)$ is calculated for every $d' < d$. Then

$$\{\ v \in \mathcal{V}_d\ \mid\ \text{there exist no vectors } \gamma \text{ in } \bigcup_{d'<d} \mathcal{C}_{d'}(h) \text{ such that } \langle v, \gamma\rangle < 0\ \}$$

is the set $\mathcal{C}_d(h)$. □

Suppose that $h \in S_X$ is a polarization of degree $n := \langle h, h\rangle > 0$. Let

$$\Phi_h \colon X \xrightarrow{\rho_h} X_h \longrightarrow \mathbb{P}^{1+n/2}$$

be the Stein factorization of the morphism Φ_h induced by the complete linear system $|\mathcal{L}_h|$ associated with a line bundle $\mathcal{L}_h \to X$ whose class is h. Then X_h has only rational double points as its singularities, and ρ_h is the minimal resolution of singularities. The set $\mathcal{C}_0(h)$ is equal to the set of classes of smooth rational curves contracted by ρ_h. In particular, the dual graph of $\mathcal{C}_0(h)$ is a disjoint union of indecomposable root systems of type A_l, D_m or E_n (see Figure 2.1). We can calculate the ADE-type of the singular points $\mathrm{Sing}(X_h)$ of X_h from $\mathcal{C}_0(h)$.

The set $\mathcal{C}_1(h)$ is the set of classes of smooth rational curves that are mapped to lines in $\mathbb{P}^{1+n/2}$ isomorphically by Φ_h; that is, $\mathcal{C}_1(h)$ is the set of classes of lines of the polarized $K3$ surface (X, h).

6.2. **Application to projective models.**

Definition 6.5. Let (X, h) and (X', h') be polarized $K3$ surfaces. We say that (X, h) and (X', h') have the *same line configuration* if there exists a bijection

$$\alpha \colon \mathcal{C}_0(h) \cup \mathcal{C}_1(h) \xrightarrow{\sim} \mathcal{C}_0(h') \cup \mathcal{C}_1(h')$$

such that we have

$$\langle \alpha(r), h'\rangle = \langle r, h\rangle \quad \text{for any } r \in \mathcal{C}_0(h) \cup \mathcal{C}_1(h), \tag{6.4}$$

(that is, $\alpha(\mathcal{C}_0(h)) = \mathcal{C}_0(h')$ and $\alpha(\mathcal{C}_1(h)) = \mathcal{C}_1(h')$ hold), and

$$\langle \alpha(r), \alpha(r')\rangle = \langle r, r'\rangle \quad \text{for any } r, r' \in \mathcal{C}_0(h) \cup \mathcal{C}_1(h). \tag{6.5}$$

We say that the line configuration on (X, h) is *full* if the union of $\mathcal{C}_0(h)$ and $\mathcal{C}_1(h)$ generates S_X.

Proposition 6.6. *Suppose that X is singular and that the line configuration on (X, h) is full. Then, up to isomorphism, there exist only a finite number of polarized $K3$ surfaces (X', h') that have the same line configuration as (X, h). Moreover all such polarized $K3$ surfaces (X', h') satisfy $\langle h', h'\rangle = \langle h, h\rangle$.*

Proof. Suppose that (X', h') has the same line configuration as (X, h), and let α be a bijection from $\mathcal{C}_0(h) \cup \mathcal{C}_1(h)$ to $\mathcal{C}_0(h') \cup \mathcal{C}_1(h')$ satisfying (6.4) and (6.5). Let S'' be the sublattice of $S_{X'}$ generated by the union of $\mathcal{C}_0(h')$ and $\mathcal{C}_1(h')$. Then α induces an isometry $\tilde{\alpha}$ from S_X to S''. Therefore X' is singular and

$$\operatorname{disc} T_{X'} = -\operatorname{disc} S_{X'} = -\operatorname{disc} S_X/m^2 = \operatorname{disc} T_X/m^2,$$

where m is the index of S'' in $S_{X'}$. Since the number of isomorphism classes of definite lattices of a fixed discriminant is finite, the number of isomorphism classes of singular $K3$ surfaces X' that admit a polarization h' with the same line configuration as (X, h) is finite. Note that the isometry $\tilde{\alpha}\colon S_X \xrightarrow{\sim} S'$ maps h to h', because h is uniquely determined by $\mathcal{C}_0(h)$ and $\mathcal{C}_1(h)$ as a unique vector satisfying $\langle r, h\rangle = 0$ for any $r \in \mathcal{C}_0(h)$ and $\langle r, h\rangle = 1$ for any $r \in \mathcal{C}_1(h)$. In particular, we have $\langle h, h\rangle = \langle h', h'\rangle$. For a fixed $K3$ surface X', the number of polarizations h' with a fixed degree is finite up to $\operatorname{Aut}(X')$ by Sterk [38]. □

We apply this consideration to our singular $K3$ surfaces X_k. Recall that the inversion of the orientation of T_k yields a singular $K3$ surface isomorphic to X_k.

Proposition 6.7. *Let h be a polarization on X_k of degree $n := \langle h, h\rangle > 0$ such that the line configuration on (X_k, h) is full. Suppose that (X', h') has the same line configuration as (X_k, h). Then either X' is isomorphic to X_k, or $k = 0$ and X' is the singular $K3$ surface with $T_{X'} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.*

Proof. Since X' is a singular $K3$ surface by Proposition 6.6, we have $\operatorname{disc} T_{X'} \equiv 0$ or $3 \bmod 4$. By the proof of Proposition 6.6, we see that $\operatorname{disc} T_{X'} = \operatorname{disc} T_k/m^2$, and if $m = 1$, then $T_{X'} \cong T_k$ by the proof of Proposition 3.2. □

Therefore, if the line configuration of (X_k, h) is full, then we can determine the projective model of the polarized $K3$ surface (X_k, h) up to finite possibilities.

7. Involutions of $K3$ surfaces

Let X be a $K3$ surface such that the representation $\varphi_X\colon \operatorname{Aut}(X) \to \mathrm{O}(S_X)$ is injective. Suppose that we are given the action of an involution $\iota \in \operatorname{Aut}(X)$ on S_X as a matrix. In this section, we discuss a method to obtain geometric properties of ι from this matrix.

7.1. **Types of the involution.** Note that we have $\lambda_X(\iota) = \pm 1$, where λ_X is the natural representation of $\operatorname{Aut}(X)$ on $H^{2,0}(X)$. Since we have assumed that φ_X is injective, we can determine, by Theorem 4.1, whether ι is symplectic or not by seeing whether $\eta_S(\varphi_X(\iota)) \in \mathrm{O}(q_{S_X})$ is the identity or not.

Suppose that ι is not symplectic. Then we can determine whether ι is Enriques or rational by the following:

Proposition 7.1 (Keum [12])**.** *Let $\iota\colon X \to X$ be an involution. We put*

$$S_X^+ := \{v \in S_X \mid v^\iota = v\}, \quad S_X^- := \{v \in S_X \mid v^\iota = -v\}.$$

Let $S_X^+(1/2)$ denote the $\mathbb{Q}$-lattice obtained from the lattice S_X^+ by multiplying the symmetric bilinear form with $1/2$. Then ι is an Enriques involution if and only if $S_X^+(1/2)$ is an even unimodular hyperbolic lattice of rank 10 and S_X^- contains no vectors r with $\langle r, r\rangle = -2$.

Remark 7.2. Since S_X^+ contains an ample class, its orthogonal complement S_X^- is negative-definite. Therefore we can calculate $\{r \in S_X^- \mid \langle r, r\rangle = -2\}$ by Algorithm 2.1.

7.2. **Polarizations of degree 2.** We have the following:

Proposition 7.3 (Theorem 5 of [19], Proposition 0.1 of [23])**.** *Let $h \in S_X$ be a nef class with $n := \langle h, h\rangle > 0$, and let $\mathcal{L}_h \to X$ be a line bundle whose class is h. Let*

$$|\mathcal{L}_h| = |M| + Z$$

be the decomposition of the complete linear system $|\mathcal{L}_h|$ into the movable part $|M|$ and the sum Z of the fixed components. Then either one of the following holds:

(i) *Z is empty, and $|\mathcal{L}_h|$ defines a morphism $\Phi_h \colon X \to \mathbb{P}^{1+n/2}$. In other words, h is a polarization of degree n.*
(ii) *Z is a smooth rational curve, and $|M|$ contains a member mE, where $m = 1 + n/2$ and E is a smooth curve of genus 1 satisfying $\langle E, Z\rangle = 1$. The complete linear system $|E|$ defines an elliptic fibration $\phi \colon X \to \mathbb{P}^1$ with a zero-section Z. In other words, we have $h = mf_\phi + z_\phi$, where f_ϕ and z_ϕ are defined in Section 3.*

Corollary 7.4. *Let $h \in S_X$ be a nef class with $n := \langle h, h\rangle > 0$. Then h is a polarization of degree n if and only if the set*

$$\mathcal{F}_h := \{\ f \in S_X \ \mid\ \langle f, h\rangle = 1,\ \langle f, f\rangle = 0\ \}$$

is empty.

Proof. If the case (ii) of Proposition 7.3 holds, then the class f_ϕ of E is an element of $\mathcal{F}_h$. Suppose that the case (i) of Proposition 7.3 holds and that $\mathcal{F}_h$ contains an element f. Then $\dim |\mathcal{L}_f| > 0$ and the movable part of $|\mathcal{L}_f|$ contains a curve that is mapped to a line in $\mathbb{P}^{1+n/2}$ by Φ_h isomorphically, which is absurd. □

Remark 7.5. Since $\langle h, h\rangle > 0$, we can calculate $\mathcal{F}_h$ by Algorithm 2.2.

Suppose that a polarization $h \in S_X$ of degree 2 is given, and let $\tau(h) \in \mathrm{Aut}(X)$ be the associated double-plane involution. We can calculate the matrix of the action of $\tau(h)$ on S_X by the following method, provided that we have an ample class $a \in S_X$. Let

$$\Phi_h \colon X \xrightarrow{\rho_h} X_h \xrightarrow{\pi_h} \mathbb{P}^2$$

be the Stein factorization of the morphism Φ_h induced by the complete linear system $|\mathcal{L}_h|$, and let B_h be the branch curve of $\pi_h \colon X_h \to \mathbb{P}^2$, which is a plane curve of degree 6 with only simple singularities. Recall that the dual graph of the set $\mathcal{C}_0(h)$ of classes of smooth rational curves contracted by the minimal resolution of singularities ρ_h is a disjoint union of indecomposable root systems of type A_l, D_m or E_n in Figure 2.1. The action of $\tau(h)$ on each indecomposable root system R is given as follows.

- If R is of type A_l, then $\tau(h)$ maps a_i to a_{l+1-i}.
- If R is of type D_{2k}, then $\tau(h)$ acts on R as the identity, whereas if R is of type D_{2k+1}, then $\tau(h)$ interchanges d_1 and d_2 and fixes $d_3, \ldots, d_{2k+1}$.
- If R is of type E_6, then $\tau(h)$ fixes e_1, e_4, and interchanges e_i and e_{8-i} for $i = 2, 3$. If R is of type E_7 or E_8, then $\tau(h)$ acts on R as the identity.

The eigenspace $(S_X \otimes \mathbb{Q})^+$ of the action of $\tau(h)$ on $S_X \otimes \mathbb{Q}$ with the eigenvalue 1 is generated over $\mathbb{Q}$ by the class h and the classes in the set

$$\{\ r + r^{\tau(h)}\ \mid\ r \in \mathcal{C}_0(h)\ \},$$

and the eigenspace $(S_X \otimes \mathbb{Q})^-$ with the eigenvalue -1 is orthogonal to $(S_X \otimes \mathbb{Q})^+$. Therefore we can determine the action of $\tau(h)$ on $S_X \otimes \mathbb{Q}$ and hence on S_X from the set $\mathcal{C}_0(h)$.

Conversely, suppose that the matrix $\varphi_X(\iota) \in \mathrm{O}(S_X)$ of an involution $\iota \in \mathrm{Aut}(X)$ is given. We search for a polarization h of degree 2 such that $\tau(h) = \iota$. Such a polarization does not necessarily exist. If it exists, however, we can detect it by the following method, with the help of an ample class $a \in S_X$. Let d be a positive integer. We calculate the finite set $\{v \in S_X \mid \langle v, v\rangle = 2, \langle v, a\rangle = d\}$ by Algorithm 2.2, and its subset

$$\mathcal{H}_d := \{\ v \in S_X\ \mid\ \langle v, v\rangle = 2,\ \ \langle v, a\rangle = d,\ \ v^\iota = v\ \}.$$

For each $h \in \mathcal{H}_d$, we see whether h is nef or not by Corollary 3.6. If h is nef, then we see whether h is a polarization of degree 2 or not by Corollary 7.4. If h is a polarization of degree 2, then we calculate the matrix $\varphi_X(\tau(h))$ by the method described above. If $\varphi_X(\tau(h))$ is equal to $\varphi_X(\iota)$, then we have $\tau(h) = \iota$. (Recall that we have assumed that φ_X is injective.) We start from $d = 1$ and repeat this process until we find the desired polarization h.

Remark 7.6. It often happens that two different polarizations of degree 2 yield the same double-plane involution. Let $h \in S_X$ be a polarization of degree 2. The morphism $\Phi_h \colon X \to \mathbb{P}^2$ factors as

$$X\ \xrightarrow{q}\ F\ \xrightarrow{\beta}\ \mathbb{P}^2,$$

where q is the quotient morphism by $\tau(h)$. Then F is a smooth rational surface and β is a succession of blowing-downs of (-1)-curves. There can exist a birational morphism $\beta' \colon F \to \mathbb{P}^2$ other than β. Let $h' \in S_X$ be the class of the pull-back of a line on $\mathbb{P}^2$ by $\beta' \circ q$. Then h' is a polarization of degree 2 with $\tau(h) = \tau(h')$. See Section 9.5 for a concrete example.

7.3. **Splitting lines.**

Definition 7.7. Let (X, h) be a polarized $K3$ surface of degree 2. A line ℓ on $\mathbb{P}^2$ is a *splitting line for* (X, h) if the strict transform of ℓ by Φ_h has two irreducible components.

Let B be a reduced projective plane curve of degree 6. A line ℓ is a *splitting line for* B if ℓ is not an irreducible component of B and the intersection multiplicity of ℓ and B at each intersection point is even.

By definition, a line ℓ is splitting for (X, h) if and only if ℓ is splitting for the branch curve B_h of $\pi_h \colon X_h \to \mathbb{P}^2$. Let Γ be a smooth rational curve on X such that $[\Gamma] \in \mathcal{C}_1(h)$. If $[\Gamma]^{\tau(h)} = [\Gamma]$, then Φ_h maps Γ to a line component of B_h isomorphically. If $[\Gamma]^{\tau(h)} \neq [\Gamma]$, then Φ_h maps Γ to a splitting line for B_h isomorphically.

8. Proof of Theorem 1.2, Proposition 1.6 and Table 1.1

In the proof of Theorems 1.4 and 1.5 in Section 5, we have already calculated, in the form of matrices, all the elements of the finite group $\mathrm{Aut}(X_k, a_k)$, the set $\mathrm{Invols}_k^{(0)}$ of involutions in $\mathrm{Aut}(X_k, a_k)$,

$$\begin{bmatrix}
405 & 200 & -35 & -57 & -451 & -303 & -606 & -902 & -736 & -560 & -377 & -194 & -451 & -303 & -606 & -902 & -736 & -560 & -377 & -194 \\
10 & 5 & -1 & -1 & -12 & -8 & -16 & -24 & -20 & -15 & -10 & -5 & -12 & -8 & -16 & -24 & -20 & -15 & -10 & -5 \\
426 & 210 & -37 & -60 & -474 & -318 & -636 & -948 & -774 & -588 & -396 & -204 & -474 & -318 & -636 & -948 & -774 & -588 & -396 & -204 \\
690 & 342 & -60 & -97 & -768 & -516 & -1032 & -1536 & -1254 & -954 & -642 & -330 & -768 & -516 & -1032 & -1536 & -1254 & -954 & -642 & -330 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 7 & -1 & -2 & -15 & -10 & -20 & -30 & -24 & -18 & -12 & -6 & -17 & -12 & -23 & -34 & -28 & -21 & -14 & -7 \\
22 & 11 & -2 & -3 & -25 & -17 & -34 & -50 & -41 & -31 & -21 & -11 & -24 & -16 & -32 & -48 & -39 & -30 & -21 & -11 \\
21 & 10 & -2 & -3 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10 & -22 & -15 & -30 & -44 & -36 & -27 & -18 & -9 \\
14 & 7 & -1 & -2 & -15 & -10 & -20 & -30 & -25 & -19 & -13 & -7 & -17 & -11 & -22 & -33 & -27 & -21 & -14 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 11 & -2 & -3 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10 & -26 & -17 & -34 & -51 & -41 & -31 & -21 & -11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 7 & -1 & -2 & -17 & -12 & -23 & -34 & -28 & -21 & -14 & -7 & -15 & -10 & -20 & -30 & -24 & -18 & -12 & -6 \\
22 & 11 & -2 & -3 & -24 & -16 & -32 & -48 & -39 & -30 & -21 & -11 & -25 & -17 & -34 & -50 & -41 & -31 & -21 & -11 \\
21 & 10 & -2 & -3 & -22 & -15 & -30 & -44 & -36 & -27 & -18 & -9 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10 \\
14 & 7 & -1 & -2 & -17 & -11 & -22 & -33 & -27 & -21 & -14 & -7 & -15 & -10 & -20 & -30 & -25 & -19 & -13 & -7 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 11 & -2 & -3 & -26 & -17 & -34 & -51 & -41 & -31 & -21 & -11 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10
\end{bmatrix}$$

TABLE 8.1. The Enriques involution $\varepsilon_0^{(0)}$

and the set $\mathrm{Invols}_k^{(i)}$ of involutions that map the induced chamber $D^{(0)}$ to the adjacent induced chamber $D^{(i)}$ for $i > 0$. By the method described in Section 7, we determine the types of the involutions in $\mathrm{Invols}_k^{(i)}$. Thus we prove Proposition 1.6 and complete Table 1.1.

We prove the assertions on X_0 in Theorem 1.2. The cardinalities of the conjugacy classes of $\mathrm{Aut}(X_0, a_0)$ are as follows:

order	1	2	2	2	3	4	4	4	4	5	6	8	8	8	8	10
card.	1	45	45	1	80	180	180	90	90	144	80	90	90	90	90	144.

The center of $\mathrm{Aut}(X_0, a_0)$ is therefore a cyclic group of order 2 generated by $\varepsilon_0^{(0)}$ given in Table 8.1. By Proposition 7.1, we see that $\varepsilon_0^{(0)}$ is an Enriques involution. One of the two conjugacy classes of order 2 with cardinality 45 consists of symplectic involutions, and the other consists of rational involutions. The matrix $\rho_0^{(0)}$ in Table 8.2 is an element of $\mathrm{Aut}(X_0, a_0)$ with order 4. Since $\eta_S(\rho_0^{(0)}) \in \mathrm{O}(q_{S_0})$ is of order 4, we see that $\rho_0^{(0)}$ is purely non-symplectic. There exist three double-plane involutions $\tau(h_0^{[1]})$, $\tau(h_0^{[2]})$, $\tau(h_0^{[3]})$ in $\mathrm{Aut}(X_0, a_0)$, where the polarizations $h_0^{[i]}$ of degree 2 are given in Table 8.3, such that $\tau(h_0^{[1]})$, $\tau(h_0^{[2]})$, $\tau(h_0^{[3]})$ and $\rho_0^{(0)}$ generate $\mathrm{Aut}(X_0, a_0)$. The subgroup

$$Aut(X_0, a_0)' := \langle \tau(h_0^{[1]}), \tau(h_0^{[2]}), \tau(h_0^{[3]}) \rangle$$

of $\mathrm{Aut}(X_0, a_0)$ is of index 2 and consists of elements $g \in \mathrm{Aut}(X_0, a_0)$ with $\lambda_{X_0}(g)^2 = 1$. The mapping

$$\tau(h_0^{[1]}) \mapsto ((12)(34), -1), \quad \tau(h_0^{[2]}) \mapsto ((35)(46), -1), \quad \tau(h_0^{[3]}) \mapsto ((23)(56), -1))$$

induces an isomorphism from $\mathrm{Aut}(X_0, a_0)'$ to $\mathfrak{A}_6 \times \{\pm 1\}$. By this isomorphism, the Enriques involution $\varepsilon_0^{(0)}$ is mapped to $(\mathrm{id}, -1)$, and the symplectic subgroup of $\mathrm{Aut}(X_0, a_0)'$ is mapped to $\mathfrak{A}_6 \times \{1\}$. For $i = 1, \ldots, 12$, the set $\mathrm{Invols}_0^{(i)}$ contains a double-plane involution $\tau(\tilde{h}_0^{(i)})$, where the polarization $\tilde{h}_0^{(i)}$ of degree 2 is given in Table 8.3.

Next we prove the assertions on X_1 and X_2 in Theorem 1.2. Suppose that $k = 1$ or 2. Then the cardinalities of the conjugacy classes of $\mathrm{Aut}(X_k, a_k)$ are as follows:

order	1	2	2	3	4	5	5	8	8	10	10
card.	1	45	36	80	90	72	72	90	90	72	72.

$$\begin{bmatrix}
318 & 159 & -29 & -46 & -342 & -236 & -465 & -684 & -560 & -430 & -289 & -148 & -350 & -237 & -474 & -700 & -569 & -438 & -296 & -148 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
324 & 162 & -30 & -47 & -348 & -240 & -474 & -696 & -570 & -438 & -294 & -150 & -354 & -240 & -480 & -708 & -576 & -444 & -300 & -150 \\
540 & 270 & -49 & -78 & -582 & -402 & -792 & -1164 & -954 & -732 & -492 & -252 & -594 & -402 & -804 & -1188 & -966 & -744 & -504 & -252 \\
21 & 10 & -2 & -3 & -22 & -15 & -30 & -44 & -36 & -27 & -18 & -9 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 10 & -2 & -3 & -22 & -15 & -30 & -44 & -36 & -28 & -19 & -10 & -24 & -16 & -32 & -47 & -38 & -29 & -20 & -10 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 6 & -1 & -2 & -14 & -10 & -19 & -28 & -23 & -18 & -12 & -6 & -14 & -10 & -19 & -28 & -23 & -18 & -12 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 11 & -2 & -3 & -25 & -17 & -34 & -50 & -41 & -31 & -21 & -11 & -24 & -17 & -33 & -48 & -39 & -30 & -20 & -10 \\
20 & 10 & -2 & -3 & -22 & -15 & -29 & -43 & -35 & -27 & -18 & -9 & -21 & -14 & -28 & -42 & -34 & -26 & -18 & -9 \\
8 & 4 & -1 & -1 & -8 & -6 & -11 & -16 & -13 & -10 & -7 & -4 & -9 & -6 & -12 & -18 & -15 & -12 & -8 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 11 & -2 & -3 & -25 & -17 & -34 & -50 & -41 & -32 & -22 & -11 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10 \\
12 & 6 & -1 & -2 & -12 & -8 & -16 & -24 & -20 & -15 & -10 & -5 & -13 & -9 & -18 & -26 & -21 & -16 & -11 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
22 & 11 & -2 & -3 & -24 & -17 & -33 & -48 & -39 & -30 & -20 & -10 & -25 & -17 & -34 & -50 & -41 & -31 & -21 & -11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
12 & 6 & -1 & -2 & -13 & -9 & -18 & -26 & -21 & -16 & -11 & -6 & -12 & -8 & -16 & -24 & -20 & -15 & -10 & -5
\end{bmatrix}$$

TABLE 8.2. The purely non-symplectic automorphism $\rho_0^{(0)}$ of order 4

The conjugacy class of order 2 with cardinality 45 consists of symplectic involutions, and the class of order 2 with cardinality 36 consists of rational involutions. There exist three double-plane involutions $\tau(h_k^{[1]})$, $\tau(h_k^{[2]})$, $\tau(h_k^{[3]})$ in $\mathrm{Aut}(X_k, a_k)$, where the polarizations $h_k^{[i]}$ of degree 2 are given in Tables 8.4 and 8.5. These three involutions generate $\mathrm{Aut}(X_k, a_k)$, and the mapping

$$\tau(h_k^{[1]}) \mapsto \begin{bmatrix} 0 & 1+\sqrt{2} \\ 1 & 0 \end{bmatrix}, \quad \tau(h_k^{[2]}) \mapsto \begin{bmatrix} 0 & 2+\sqrt{2} \\ 1 & 0 \end{bmatrix}, \quad \tau(h_k^{[3]}) \mapsto \begin{bmatrix} 2 & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

induces an isomorphism from $\mathrm{Aut}(X_k, a_k)$ to $\mathrm{PGL}_2(\mathbb{F}_9)$. Except for the case $k = 1$ and $i = 4$, the set $\mathrm{Invols}_k^{(i)}$ contains a double-plane involution $\tau(\tilde{h}_k^{(i)})$, where the polarization $\tilde{h}_k^{(i)}$ of degree 2 is given in Tables 8.4 and 8.5. The set $\mathrm{Invols}_1^{(4)}$ consists of 6 symplectic involutions, one of which is the matrix $\sigma_1^{(4)}$ given in Table 8.6.

Remark 8.1. According to [4], there exist exactly three non-splitting extensions of the cyclic group of order 2 by $\mathfrak{A}_6$; namely, the symmetric group $\mathfrak{S}_6$, the Mathieu group M_{10}, and the projective general linear group $\mathrm{PGL}_2(\mathbb{F}_9)$. In [6, Chapter 10, Section 1.5], these three groups are distinguished by the numbers of conjugacy classes of elements of order 3 and 5: $\mathfrak{S}_6$ has two classes of order 3 and one of order 5, M_{10} has one of each, and $\mathrm{PGL}_2(\mathbb{F}_9)$ has one of order 3 and two of order 5.

9. EXAMPLES

In this section, we investigate projective geometry of some of the automorphisms that appear in Theorem 1.2.

9.1. **The purely non-symplectic automorphism $\rho_0^{(0)}$.** We investigate the purely non-symplectic automorphism $\rho_0^{(0)}$ of order 4 in $\mathrm{Aut}(X_0, a_0)$. The vector

$$\begin{aligned} h_\rho \;:=\; (88,\ 43,\ -8,\ -12,\ -98,\ -66,\ -131,\ -195,\ -159,\ -121,\ -82,\ -42,\\ -99,\ -67,\ -133,\ -197,\ -161,\ -123,\ -84,\ -43) \end{aligned}$$

of S_0 with $\langle h_\rho, h_\rho \rangle = 4$ is invariant under the action of $\rho_0^{(0)}$. By Corollary 3.6, we see that h_ρ is nef, and by Corollary 7.4, we see that h_ρ is a polarization of degree 4. Moreover, by Algorithm 2.2, we

h	$\mathrm{Sing}(X_h)$	$\langle h, a_0\rangle$
$h_0^{[1]} = (43,\ 21,\ -4,\ -6,\ -47,\ -32,\ -63,\ -93,\ -76,\ -58,\ -39,\ -20,$ $-48,\ -33,\ -65,\ -96,\ -78,\ -60,\ -40,\ -20)$	$2A_2 + 7A_1$	10
$h_0^{[2]} = (64,\ 32,\ -6,\ -9,\ -71,\ -47,\ -94,\ -140,-114,\ -87,\ -59,\ -30,$ $-71,\ -48,\ -95,\ -141,-115,\ -88,\ -60,\ -30)$	$2A_2 + 7A_1$	10
$h_0^{[3]} = (49,\ 24,\ -4,\ -7,\ -56,\ -38,\ -75,\ -111,\ -90,\ -69,\ -47,\ -24,$ $-54,\ -36,\ -72,\ -107,\ -87,\ -66,\ -45,\ -23)$	$2A_2 + 7A_1$	10
$\tilde{h}_0^{(1)} = (64,\ 32,\ -6,\ -9,\ -71,\ -48,\ -95,\ -140,-114,\ -87,\ -59,\ -30,$ $-71,\ -48,\ -95,\ -141,-115,\ -87,\ -59,\ -30)$	$A_2 + 8A_1$	10
$\tilde{h}_0^{(2)} = (57,\ 28,\ -5,\ -8,\ -64,\ -43,\ -86,\ -127,-103,\ -78,\ -52,\ -26,$ $-64,\ -43,\ -85,\ -127,-103,\ -78,\ -53,\ -27)$	$4A_2 + 4A_1$	12
$\tilde{h}_0^{(3)} = (64,\ 32,\ -6,\ -9,\ -72,\ -48,\ -96,\ -142,-116,\ -89,\ -60,\ -31,$ $-69,\ -47,\ -93,\ -138,-113,\ -86,\ -59,\ -30)$	$3A_2 + 6A_1$	12
$\tilde{h}_0^{(4)} = (74,\ 37,\ -7,\ -10,\ -83,\ -56,\ -111,-164,-134,-103,\ -69,\ -35,$ $-82,\ -55,\ -110,-164,-134,-103,\ -70,\ -36)$	$5A_2 + 4A_1$	14
$\tilde{h}_0^{(5)} = (80,\ 40,\ -7,\ -11,\ -91,\ -61,\ -122,-181,-147,-112,\ -75,\ -38,$ $-89,\ -60,\ -119,-178,-145,-110,\ -75,\ -38)$	$5A_2 + 4A_1$	14
$\tilde{h}_0^{(6)} = (176,\ 88,\ -16,-25,-193,-130,-260,-383,-312,-238,-161,\ -81,$ $-197,-134,-264,-391,-318,-243,-165,\ -84)$	$3A_3 + 6A_1$	22
$\tilde{h}_0^{(7)} = (140,\ 70,\ -13,-20,-153,-102,-204,-303,-245,-187,-127,\ -64,$ $-155,-105,-209,-310,-254,-194,-131,\ -67)$	$4A_3 + 4A_1$	22
$\tilde{h}_0^{(8)} = (152,\ 76,\ -14,-21,-173,-115,-230,-342,-277,-212,-144,\ -72,$ $-167,-113,-222,-331,-270,-208,-142,\ -73)$	$3A_4 + A_2 + A_1$	24
$\tilde{h}_0^{(9)} = (252,126,-22,-35,-284,-191,-382,-563,-456,-349,-237,-121,$ $-280,-191,-378,-560,-457,-350,-238,-121)$	$3A_5 + 3A_1$	34
$\tilde{h}_0^{(10)} = (148,\ 74,\ -13,-21,-171,-114,-228,-338,-272,-206,-140,\ -70,$ $-160,-108,-212,-316,-260,-199,-134,\ -69)$	$3A_5 + 3A_1$	34
$\tilde{h}_0^{(11)} = (304,152,-27,-42,-341,-231,-456,-677,-551,-420,-284,-142,$ $-340,-230,-455,-680,-554,-424,-288,-147)$	$D_4 + 2A_5 + A_3$	38
$\tilde{h}_0^{(12)} = (206,103,-19,-29,-231,-156,-312,-457,-371,-285,-193,\ -97,$ $-224,-153,-300,-447,-365,-278,-191,\ -98)$	$D_4 + 2A_5 + A_3$	38

TABLE 8.3. The polarizations $h_0^{[i]}$ and $\tilde{h}_0^{(i)}$ of degree 2

have

$$\{\ v \in S_0\ \mid\ \langle v, v\rangle = 0,\ \ \langle h_\rho, v\rangle = 2\ \ \} = \emptyset.$$

Hence, by Theorem 5.2 of Saint-Donat [26], the polarization h_ρ is not hyperelliptic; that is, h_ρ is the class of the pull-back of a hyperplane section by a birational morphism from X_0 to a normal quartic surface $Y \subset \mathbb{P}^3$ given by $|\mathcal{L}_{h_\rho}|$. Since h_ρ is invariant under the action of $\rho_0^{(0)}$, we conclude that $\rho_0^{(0)}$ is induced by a projective linear automorphism of $\mathbb{P}^3$ that leaves Y invariant. By a direct calculation, we see that the line configuration of (X_0, h_ρ) is full, and hence, up to finite possibilities, the projective equivalence class of the quartic surface Y is determined by the line configuration of (X_0, h_ρ). We describe this line configuration in details, hoping that we can obtain a defining equation of Y in future. Let S be a set on which the group $\langle \rho_0^{(0)} \rangle$ of order 4 acts transitively. By $S = [s_0, s_1, s_2, s_3]$, we mean that $|S| = 4$ and that $\rho_0^{(0)}$ maps s_i to s_{i+1} for $i = 0, 1, 2$ and s_3 to s_0, and by $S = [s_0, s_1]$, we

h	$\mathrm{Sing}(X_h)$	$\langle h, a_1\rangle$
$h_1^{[1]} = (30,\ 15,\ -7,\ -2,\ -33,\ -22,\ -44,\ -66,\ -54,\ -41,\ -28,\ -14,$ $-34,\ -23,\ -45,\ -67,\ -55,\ -42,\ -28,\ -14)$	$4A_2+5A_1$	12
$h_1^{[2]} = (30,\ 15,\ -7,\ -2,\ -34,\ -23,\ -45,\ -67,\ -55,\ -42,\ -28,\ -14,$ $-33,\ -22,\ -44,\ -66,\ -54,\ -41,\ -28,\ -14)$	$4A_2+5A_1$	12
$h_1^{[3]} = (43,\ 21,\ -10,\ -3,\ -46,\ -31,\ -62,\ -92,\ -75,\ -57,\ -39,\ -20,$ $-49,\ -33,\ -66,\ -98,\ -80,\ -61,\ -41,\ -21)$	$4A_2+5A_1$	12
$\tilde{h}_1^{(1)} = (45,\ 22,\ -11,\ -3,\ -50,\ -34,\ -67,\ -99,\ -81,\ -62,\ -42,\ -21,$ $-49,\ -33,\ -65,\ -97,\ -79,\ -61,\ -42,\ -21)$	$3A_2+6A_1$	12
$\tilde{h}_1^{(2)} = (43,\ 21,\ -10,\ -3,\ -48,\ -33,\ -65,\ -96,\ -79,\ -60,\ -40,\ -20,$ $-47,\ -32,\ -63,\ -93,\ -76,\ -58,\ -40,\ -20)$	$A_3+4A_2+2A_1$	14
$\tilde{h}_1^{(3)} = (46,\ 23,\ -11,\ -3,\ -50,\ -34,\ -68,\ -100,\ -81,\ -62,\ -42,\ -21,$ $-52,\ -36,\ -70,\ -103,\ -84,\ -64,\ -44,\ -22)$	$5A_2+4A_1$	14
$\tilde{h}_1^{(5)} = (46,\ 23,\ -11,\ -3,\ -52,\ -36,\ -70,\ -104,\ -85,\ -65,\ -44,\ -23,$ $-49,\ -34,\ -67,\ -98,\ -80,\ -62,\ -42,\ -21)$	$2A_3+3A_2+2A_1$	16
$\tilde{h}_1^{(6)} = (76,\ 38,\ -18,\ -5,\ -84,\ -57,\ -112,-167,-136,-103,\ -70,\ -35,$ $-86,\ -59,\ -116,-170,-138,-106,\ -72,\ -36)$	$3A_3+3A_2$	18
$\tilde{h}_1^{(7)} =(106,\ 53,\ -25,\ -7,\ -119,\ -81,\ -159,-235,-192,-146,\ -99,\ -50,$ $-117,\ -81,\ -159,-234,-192,-147,\ -99,\ -51)$	$2A_4+2A_3+A_2$	22
$\tilde{h}_1^{(8)} = (94,\ 47,\ -22,\ -6,\ -104,\ -71,\ -140,-208,-169,-130,\ -88,\ -44,$ $-106,\ -73,\ -143,-211,-173,-132,\ -91,\ -47)$	$2A_4+2A_3+A_2$	22
$\tilde{h}_1^{(9)} =(110,\ 55,\ -26,\ -8,\ -121,\ -84,\ -164,-241,-197,-150,-102,\ -51,$ $-120,\ -80,\ -160,-237,-193,-149,-102,\ -51)$	$2A_5+2A_3$	30
$\tilde{h}_1^{(10)}=(124,\ 62,\ -29,\ -8,\ -138,\ -95,\ -186,-276,-225,-171,-116,\ -58,$ $-139,\ -95,\ -190,-278,-227,-172,-117,\ -59)$	$2A_5+2A_3$	30
$\tilde{h}_1^{(11)}=(217,108,-51,-15,-239,-166,-325,-477,-390,-296,-202,-101,$ $-239,-166,-325,-477,-390,-296,-202,-101)$	$2A_9$	54
$\tilde{h}_1^{(12)}=(250,125,-59,-17,-277,-185,-370,-548,-449,-343,-231,-119,$ $-276,-191,-375,-552,-453,-348,-236,-118)$	$2A_9$	54

TABLE 8.4. The polarizations $h_1^{[i]}$ and $\tilde{h}_1^{(i)}$ of degree 2

mean that $|S| = 2$ and that $\rho_0^{(0)}$ interchanges s_0 and s_1. We denote by $\mathrm{cyc}(a,b,c,d)$ the cyclic matrix

$$\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}.$$

From the set $\mathcal{C}_0(h_\rho)$, we see that $\mathrm{Sing}(Y)$ consists of 6 ordinary nodes, and the group $\langle \rho_0^{(0)} \rangle$ decomposes $\mathrm{Sing}(Y)$ into two orbits $[p_0, p_1, p_2, p_3]$ and $[q_0, q_1]$. From the set $\mathcal{C}_1(h_\rho)$, we see that Y contains exactly 36 lines, and they are decomposed into 9 orbits

$$l_i := [\ell_i, \ell_i', \ell_i'', \ell_i'''] \quad (i = 0, \dots, 8)$$

h	$\mathrm{Sing}(X_h)$	$\langle h, a_2\rangle$
$h_2^{[1]} = (37, 18, -7, -3, -41, -28, -55, -82, -67, -51, -35, -18, -42, -29, -57, -84, -68, -52, -35, -18)$	$5A_2 + 5A_1$	7
$h_2^{[2]} = (25, 12, -5, -2, -27, -18, -36, -54, -44, -34, -23, -12, -28, -19, -38, -56, -46, -35, -24, -12)$	$5A_2 + 5A_1$	7
$h_2^{[3]} = (36, 18, -7, -3, -40, -27, -54, -80, -65, -50, -34, -17, -40, -27, -54, -80, -65, -49, -33, -17)$	$5A_2 + 5A_1$	7
$\tilde{h}_2^{(1)} = (24, 12, -5, -2, -25, -17, -33, -49, -40, -31, -21, -11, -27, -18, -36, -54, -44, -34, -23, -12)$	$2A_3 + 3A_2 + 2A_1$	8
$\tilde{h}_2^{(2)} = (34, 17, -7, -3, -37, -25, -49, -73, -60, -46, -32, -16, -36, -24, -48, -72, -59, -45, -31, -16)$	$3A_3 + 3A_2$	9
$\tilde{h}_2^{(3)} = (65, 32, -12, -6, -70, -48, -94, -140, -114, -87, -60, -30, -73, -49, -97, -145, -118, -91, -62, -32)$	$3A_4 + A_2 + A_1$	12
$\tilde{h}_2^{(4)} = (56, 28, -11, -5, -61, -41, -81, -121, -98, -74, -50, -25, -61, -41, -82, -122, -101, -77, -53, -28)$	$A_5 + 2A_4 + A_3$	13
$\tilde{h}_2^{(5)} = (44, 22, -9, -4, -46, -31, -61, -91, -75, -57, -38, -19, -49, -32, -64, -96, -78, -60, -42, -21)$	$A_5 + 2A_4 + A_3$	13
$\tilde{h}_2^{(6)} = (126, 63, -26, -11, -138, -95, -185, -275, -222, -169, -116, -58, -136, -89, -178, -267, -220, -168, -116, -58)$	$2A_9$	27
$\tilde{h}_2^{(7)} = (145, 70, -28, -13, -160, -110, -215, -320, -260, -200, -135, -70, -160, -105, -210, -315, -255, -195, -130, -65)$	$2A_9$	27

TABLE 8.5. The polarizations $h_2^{[i]}$ and $\tilde{h}_2^{(i)}$ of degree 2

$$\begin{bmatrix}
624 & 312 & -145 & -43 & -720 & -495 & -968 & -1440 & -1164 & -888 & -612 & -306 & -658 & -456 & -890 & -1316 & -1084 & -824 & -564 & -282 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
580 & 290 & -135 & -40 & -668 & -460 & -898 & -1336 & -1080 & -824 & -568 & -284 & -612 & -424 & -828 & -1224 & -1008 & -766 & -524 & -262 \\
1032 & 516 & -240 & -71 & -1188 & -816 & -1596 & -2376 & -1920 & -1464 & -1008 & -504 & -1092 & -756 & -1476 & -2184 & -1800 & -1368 & -936 & -468 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
45 & 22 & -11 & -3 & -52 & -36 & -70 & -104 & -84 & -64 & -44 & -22 & -46 & -32 & -62 & -92 & -76 & -58 & -40 & -20 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
57 & 28 & -13 & -4 & -66 & -45 & -88 & -131 & -106 & -81 & -56 & -28 & -61 & -42 & -82 & -122 & -100 & -76 & -52 & -26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 30 & -14 & -4 & -70 & -48 & -94 & -140 & -113 & -86 & -59 & -30 & -64 & -44 & -86 & -127 & -104 & -79 & -54 & -27 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
44 & 22 & -10 & -3 & -52 & -36 & -70 & -104 & -84 & -64 & -44 & -22 & -46 & -31 & -62 & -92 & -76 & -58 & -40 & -20 \\
16 & 8 & -4 & -1 & -18 & -12 & -24 & -36 & -30 & -23 & -16 & -8 & -17 & -12 & -23 & -34 & -28 & -21 & -14 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
56 & 28 & -13 & -4 & -64 & -44 & -86 & -128 & -104 & -79 & -54 & -27 & -59 & -41 & -80 & -118 & -97 & -74 & -51 & -26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
44 & 22 & -10 & -3 & -52 & -36 & -70 & -104 & -84 & -64 & -44 & -22 & -46 & -32 & -62 & -92 & -76 & -58 & -39 & -20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

TABLE 8.6. The symplectic involution $\sigma_1^{(4)}$

of length 4 by $\langle \rho_0^{(0)} \rangle$. We can choose the element $\ell_i \in l_i$ in such a way that

$$\mathrm{Sing}(Y) \cap \ell_i = \begin{cases} \emptyset & \text{if } i = 0, 1, \\ \{q_0\} & \text{if } i = 2, 3, \\ \{p_0\} & \text{if } i = 4, 5, 6, 7, \\ \{p_0, q_1\} & \text{if } i = 8. \end{cases}$$

$i\backslash j$	0	1	2	3	4	5	6	7	8
0		C_1	C_2	C_3	C_4	C_5	C_4	C_6	C_7
1			C_1	C_8	C_5	C_4	C_6	C_4	C_7
2				C_4	C_3	C_8	C_2	C_3	C_4
3					C_5	C_4	C_4	C_7	C_1
4						C_9	C_4	C_2	C_4
5							C_8	C_4	C_2
6								C_9	C_8
7									C_4

TABLE 9.1. The intersection of lines on the quartic surface model Y of X_0

The intersection pattern of lines in the orbits l_i and l_j is given by the cyclic matrix

$$M_{ij} = \mathrm{cyc}(\langle \tilde{\ell}_i, \tilde{\ell}_j \rangle, \langle \tilde{\ell}_i, \tilde{\ell}'_j \rangle, \langle \tilde{\ell}_i, \tilde{\ell}''_j \rangle, \langle \tilde{\ell}_i, \tilde{\ell}'''_j \rangle),$$

where $\tilde{\ell} \subset X_0$ is the strict transform of a line $\ell \subset Y$. We have

$$M_{ii} = \begin{cases} \mathrm{cyc}(-2,0,1,0) & \text{if } i = 0,1,4,5,6,7, \\ \mathrm{cyc}(-2,1,0,1) & \text{if } i = 2, \\ \mathrm{cyc}(-2,0,0,0) & \text{if } i = 3,8. \end{cases}$$

We put

$$\begin{array}{lll} C_1 := \mathrm{cyc}(0,0,1,0), & C_2 := \mathrm{cyc}(0,0,0,1), & C_3 := \mathrm{cyc}(1,0,0,0), \\ C_4 := \mathrm{cyc}(0,0,0,0), & C_5 := \mathrm{cyc}(1,0,0,1), & C_6 := \mathrm{cyc}(0,1,1,0), \\ C_7 := \mathrm{cyc}(1,1,0,0), & C_8 := \mathrm{cyc}(0,1,0,0), & C_9 := \mathrm{cyc}(0,0,1,1). \end{array}$$

Then the matrices M_{ij} for $i \neq j$ are given in Table 9.1.

9.2. **The double-plane involutions $\tau(h_0^{[i]})$.** The three double-plane involutions $\tau(h_0^{[1]})$, $\tau(h_0^{[2]})$, $\tau(h_0^{[3]})$ of X_0 are conjugate in $\mathrm{Aut}(X_0, a_0)$. Hence there exist a sextic double plane $Y \to \mathbb{P}^2$ and three isomorphisms $\alpha^{[i]} \colon X_0 \xrightarrow{\sim} \widetilde{Y}$ for $i = 1,2,3$ such that $\tau(h_0^{[i]}) = (\alpha^{[i]})^{-1} \circ \tau_Y \circ \alpha^{[i]}$ holds for $i = 1,2,3$, where $\widetilde{Y}$ is the minimal resolution of singularities of Y and τ_Y is the involution of $\widetilde{Y}$ induced by $\mathrm{Gal}(Y/\mathbb{P}^2)$. By a direct calculation, we see that the line configuration of $(X_0, h_0^{[i]})$ is full, and hence, up to finite possibilities, the projective equivalence class of the sextic double plane $Y \to \mathbb{P}^2$ is determined by the line configuration of $(X_0, h_0^{[i]})$. Let $B \subset \mathbb{P}^2$ denote the branch curve of $Y \to \mathbb{P}^2$. From $\mathcal{C}_0(h_0^{[i]})$, we see that $\mathrm{Sing}(B)$ consists of two ordinary cusps q_0, q_1 and seven ordinary nodes $n_0, \ldots, n_6$. The set $\mathcal{C}_1(h_0^{[i]})$ consists of 38 elements, and the action of $\langle \tau(h_0^{[i]}) \rangle$ decomposes $\mathcal{C}_1(h_0^{[i]})$ into the union of 19 orbits of length 2. Hence B does not contain a line as an irreducible component. Therefore B is irreducible, and B has 19 splitting lines. From the intersection pairing between $\mathcal{C}_0(h_0^{[i]})$ and $\mathcal{C}_1(h_0^{[i]})$, we see that, under suitable numbering of ordinary nodes $n_0, \ldots, n_6$, these splitting lines are

$$\ell_{00},\ \ldots,\ \ell_{06},\ \ell_{10},\ \ldots,\ \ell_{16},\ m_{012},\ m_{034},\ m_{056},\ m_{135},\ m_{246},$$

where $\mathrm{Sing}(B) \cap \ell_{ij} = \{q_i, n_j\}$ and $\mathrm{Sing}(B) \cap m_{ijk} = \{n_i, n_j, n_k\}$.

9.3. **The double-plane involution $\tau(\tilde{h}_0^{(1)})$.** Next we examine the double-plane involution $\tau(\tilde{h}_0^{(1)})$ of X_0 that maps the induced chamber $D^{(0)}$ to the induced chamber $D^{(1)}$ adjacent to $D^{(0)}$ across the wall $(v_1)^\perp$, where

$$\begin{aligned} 2v_1 \;=\; (64,\, 32,\, -6,\, -9,\, -72,\, -48,\, -96,\, -142,\, -116,\, -88,\, -60,\, -30,\\ -70,\, -48,\, -94,\, -140,\, -114,\, -86,\, -58,\, -30). \end{aligned}$$

As in the previous subsection, we denote by B the branch curve of the sextic double plane $Y \to \mathbb{P}^2$ associated with the polarization $\tilde{h}_0^{(1)}$ of X_0 given in Table 8.3. By a direct calculation, we see that the line configuration of $(X_0, \tilde{h}_0^{(1)})$ is full, and hence, up to finite possibilities, the projective equivalence class of $Y \to \mathbb{P}^2$ is determined by the line configuration on $(X_0, \tilde{h}_0^{(1)})$. From $\mathcal{C}_0(\tilde{h}_0^{(1)})$, we see that $\mathrm{Sing}(B)$ consists of one ordinary cusp q_0 and eight ordinary nodes $n_0, \dots, n_7$. The set $\mathcal{C}_1(\tilde{h}_0^{(1)})$ consists of 48 elements, and the action of $\langle \tau(\tilde{h}_0^{(1)}) \rangle$ decomposes $\mathcal{C}_1(\tilde{h}_0^{(1)})$ into the union of 24 orbits of length 2. Hence B does not have a line as an irreducible component, and B has 24 splitting lines. We put

$$\mathcal{T} := \{\{0,1,5\},\ \{0,2,6\},\ \{0,3,4\},\ \{1,2,4\},\ \{1,3,7\},\ \{2,5,7\},\ \{3,5,6\},\ \{4,6,7\}\}.$$

Under suitable numbering of the ordinary nodes $n_0, \dots, n_7$, the splitting lines are

$$\ell_{0i}\ \ (i = 0, \dots, 7), \quad \ell_i\ \ (i = 0, \dots, 7), \quad m_{ijk}\ \ (\{i,j,k\} \in \mathcal{T}),$$

where

$$\mathrm{Sing}(B) \cap \ell_{0i} = \{q_0, n_i\}, \quad \mathrm{Sing}(B) \cap \ell_i = \{n_i\}, \quad \mathrm{Sing}(B) \cap m_{ijk} = \{n_i, n_j, n_k\}.$$

Since a triplet of ordinary nodes of B is collinear, we conclude that B is irreducible. Note that, if three ordinary nodes n_i, n_j, n_k are on a line $\ell \subset \mathbb{P}^2$, then ℓ is splitting for B, and hence $\{i,j,k\} \in \mathcal{T}$. Therefore no three of n_0, n_1, n_2, n_3 are collinear. Choosing homogeneous coordinates of $\mathbb{P}^2$ in such a way that

$$n_0 = [1:0:0],\ \ n_1 = [0:1:0],\ \ n_2 = [0:0:1],\ \ n_3 = [1:1:1],$$

we see that

$$n_4 = [0:1:1],\ \ n_5 = [1:\eta:0],\ \ n_6 = [1,0,\bar{\eta}],\ \ n_7 = [1:\eta:1],$$

where η is a root of $z^2 - z + 1 = 0$.

9.4. **The symplectic involution $\sigma_1^{(4)}$.** We examine the symplectic involution $\sigma_1^{(4)}$ on X_1 that maps the induced chamber $D^{(0)}$ to the induced chamber $D^{(4)}$ adjacent to $D^{(0)}$ across the wall $(v_4)^\perp$, where

$$\begin{aligned} 2v_4 \;=\; (44,\, 22,\, -10,\, -3,\, -52,\, -36,\, -70,\, -104,\, -84,\, -64,\, -44,\, -22,\\ -46,\, -32,\, -62,\, -92,\, -76,\, -58,\, -40,\, -20). \end{aligned}$$

Consider the vector

$$\begin{aligned} h_\sigma \;:=\; (60,\, 30,\, -14,\, -4,\, -69,\, -47,\, -92,\, -137,\, -111,\, -85,\, -58,\, -29,\\ -65,\, -45,\, -87,\, -129,\, -106,\, -81,\, -55,\, -28) \end{aligned}$$

of S_1 with $\langle h_\sigma, h_\sigma \rangle = 2$. By Corollary 3.6, we see that h_σ is nef, and by Corollary 7.4, we see that h_σ is a polarization of degree 2. The polarization h_σ is invariant under $\sigma_1^{(4)}$, and hence $\tau(h_\sigma)$ and $\sigma_1^{(4)}$ commute. The symplectic involution $\sigma_1^{(4)}$ induces a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \underset{\bar{\sigma}}{\longrightarrow} & \mathbb{P}^2 \end{array}$$

on the sextic double plane $Y \to \mathbb{P}^2$ associated with h_σ. Let B be the branch curve of $Y \to \mathbb{P}^2$, which is invariant under the action of $\bar{\sigma}$ on $\mathbb{P}^2$. By a direct calculation, we see that the line configuration of (X_1, h_σ) is full, and hence the projective equivalence class of the double plane Y is determined by the line configuration of (X_1, h_σ) up to finite possibilities. From $\mathcal{C}_0(h_\sigma)$, we see that $\mathrm{Sing}(B)$ consists of seven ordinary cusps $q_0, q_1, q_1', q_2, q_2', q_3, q_3'$. In particular, B is irreducible. The involution $\bar{\sigma}$ of $\mathbb{P}^2$ fixes q_0 and interchanges q_i and q_i' for $i = 1, 2, 3$. From $\mathcal{C}_1(h_\sigma)$, we see that B has 10 splitting lines $\ell_0, \ldots, \ell_9$. Under suitable numbering, we have

$$\begin{aligned}
&\ell_0 \cap \mathrm{Sing}(B) = \{q_0, q_1, q_1'\}, && \ell_1 \cap \mathrm{Sing}(B) = \{q_0, q_2, q_3\},\\
&\ell_2 \cap \mathrm{Sing}(B) = \{q_0, q_2', q_3'\}, && \ell_3 \cap \mathrm{Sing}(B) = \{q_1, q_2, q_3'\},\\
&\ell_4 \cap \mathrm{Sing}(B) = \{q_1, q_2'\}, && \ell_5 \cap \mathrm{Sing}(B) = \{q_1', q_2\},\\
&\ell_6 \cap \mathrm{Sing}(B) = \{q_1', q_2', q_3\}, && \ell_7 \cap \mathrm{Sing}(B) = \{q_3, q_3'\},\\
&\ell_8 \cap \mathrm{Sing}(B) = \emptyset, && \ell_9 \cap \mathrm{Sing}(B) = \emptyset.
\end{aligned}$$

The involution $\bar{\sigma}$ fixes ℓ_0 and ℓ_7, and interchanges two lines in the pairs $\{\ell_1, \ell_2\}$, $\{\ell_3, \ell_6\}$, $\{\ell_4, \ell_5\}$ and $\{\ell_8, \ell_9\}$.

9.5. **The double-plane involutions $\tau(\tilde{h}_1^{(11)})$, $\tau(\tilde{h}_1^{(12)})$, $\tau(\tilde{h}_2^{(6)})$, $\tau(\tilde{h}_2^{(7)})$.** These four double-plane involutions have the following common feature. We say that a projective plane curve B of degree 6 is *of type LQ* if the following hold;

(i) B is the union of a line L and an irreducible quintic curve Q,
(ii) L and Q intersect at a point P_0 with intersection multiplicity 5,
(iii) Q is smooth at P_0,
(iv) the singular locus $\mathrm{Sing}(Q)$ of Q consists of a point P_1 of type A_9, and
(v) the line ℓ passing through P_0 and P_1 intersects Q at P_1 with intersection multiplicity 4.

If B is of type LQ, then the ADE-type of $\mathrm{Sing}(B)$ is $2A_9$, and the line ℓ in the condition (v) is splitting for B.

Let h be $\tilde{h}_1^{(11)}$, $\tilde{h}_1^{(12)}$, $\tilde{h}_2^{(6)}$ or $\tilde{h}_2^{(7)}$. We put $k = 1$ if h is $\tilde{h}_1^{(11)}$ or $\tilde{h}_1^{(12)}$, and $k = 2$ if h is $\tilde{h}_2^{(6)}$ or $\tilde{h}_2^{(7)}$, so that $h \in S_k$ and $\tau(h) \in \mathrm{Aut}(X_k)$. The dual graph of the set $\mathcal{C}_0(h)$ is a root system of type $2A_9$. The set $\mathcal{C}_1(h)$ consists of 3 elements, and $\langle \tau(h) \rangle$ decomposes it into the union of two orbits of length 1 and 2. The union of $\mathcal{C}_0(h)$ and $\mathcal{C}_1(h)$ generates a sublattice of rank 19 in S_k. Hence, unfortunately, the line configuration of (X_k, h) is not full. The branch curve of (X_k, h) is of type LQ.

We consider two vectors

$$\begin{aligned}
h' \;:=\; &(172,\ 83,\ -34,\ -15,\ -191,\ -131,\ -257,\ -382,\ -310,\ -238,\ -161,\ -83,\\
&\qquad -189,\ -124,\ -248,\ -372,\ -301,\ -230,\ -154,\ -77),\\
h'' \;:=\; &(183,\ 88,\ -36,\ -16,\ -200,\ -138,\ -269,\ -400,\ -325,\ -250,\ -169,\ -88,\\
&\qquad -204,\ -134,\ -268,\ -401,\ -325,\ -249,\ -166,\ -83)
\end{aligned}$$

in S_2 of square-norm 2. By Corollaries 3.6 and 7.4, we see that they are polarizations of degree 2. We have $\tau(h') = \tau(h'') = \tau(\tilde{h}_2^{(7)})$. Unfortunately again, the line configurations of (X_2, h') and (X_2, h'') are not full. The ADE-type of the singularities of the branch curve of (X_2, h') is $E_6 + A_{11}$, whereas that of (X_2, h'') is $A_{15} + A_3$.

$$
\begin{aligned}
f_1 &:= (1,\ 0,\ 2,\ -1,\ 0,\ 0,\ 0,\ 0,\ -4,\ 0,\ 0,\ 0,\ -4,\ -1,\ -2,\ -8,\ -6,\ -3,\ -4,\ -5)\\
f_2 &:= (0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ -3,\ 0,\ 0,\ 0,\ -3,\ -1,\ -2,\ -6,\ -5,\ -3,\ -3,\ -3)\\
f_3 &:= (0,\ 0,\ 3,\ -1,\ 0,\ 0,\ 0,\ 0,\ -6,\ 0,\ 0,\ 0,\ -4,\ 0,\ 0,\ -8,\ -6,\ -2,\ -4,\ -6)\\
f_4 &:= (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0\ )\\
f_5 &:= (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ )\\
f_6 &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ -1,\ -1,\ 0,\ -1,\ 0,\ 0,\ 0\ )\\
f_7 &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ -1)\\
f_8 &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0,\ 0,\ -1,\ 0,\ -1,\ 0,\ 1\ )\\
f_9 &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ )\\
f_{10} &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0\ )
\end{aligned}
$$

TABLE 10.1. A basis of S_Z

10. The automorphism group of the Enriques surface Z_0

In this section, we compute the automorphism group $\mathrm{Aut}(Z_0)$ of the Enriques surface $Z_0 := X_0/\langle\varepsilon_0^{(0)}\rangle$, and prove Proposition 1.8 and Theorem 1.9.

We put

$$S_0^+ := \{v \in S_0 \,|\, v^{\varepsilon_0^{(0)}} = v\}, \quad S_0^- := \{v \in S_0 \,|\, v^{\varepsilon_0^{(0)}} = -v\}.$$

They are orthogonal complement to each other in S_0. Let $\pi\colon X_0 \to Z_0$ be the universal covering of Z_0 by X_0. Then the pull-back by π identifies the primitive sublattice S_0^+ of S_0 with the lattice $S_Z(2)$. From the matrix representation (Table 8.1) of $\varepsilon_0^{(0)}$, we see that S_0^+ is generated by the vectors $f_1, \ldots, f_{10}$ given in Table 10.1. From now on, we consider $f_1, \ldots, f_{10}$ as a basis of S_Z by π^*. The Gram matrix

$$(\langle f_i, f_j\rangle_Z) = (\langle f_i, f_j\rangle_S/2)$$

of S_Z with respect to this basis is given in Table 10.2.

Note that we have

$$\mathrm{Cen}(\varepsilon_0^{(0)}) = \{\ g \in \mathrm{Aut}(X_0)\ \mid\ (S_0^+)^g = S_0^+\ \}.$$

Hence we have a natural action

$$\psi\colon \mathrm{Cen}(\varepsilon_0^{(0)}) \to \mathrm{O}(S_0^+)$$

of $\mathrm{Cen}(\varepsilon_0^{(0)})$ on S_0^+. With the identifications $\mathrm{O}(S_0^+) \xrightarrow{\sim} \mathrm{O}(S_Z)$ by π^* and $\mathrm{Cen}(\varepsilon_0^{(0)})/\langle\varepsilon_0^{(0)}\rangle \xrightarrow{\sim} \mathrm{Aut}(Z_0)$ by ζ, we see that Proposition 1.8 follows from

$$\mathrm{Ker}\,\psi = \langle\varepsilon_0^{(0)}\rangle. \tag{10.1}$$

Suppose that $g \in \mathrm{Ker}\,\psi$ so that g acts on S_0^+ trivially. Since $\varepsilon_0^{(0)} \in \mathrm{Aut}(X_0, a_0)$, we have $a_0 \in S_0^+$ and hence $a_0^g = a_0$. Consequently, we have $\mathrm{Ker}\,\psi \subset \mathrm{Aut}(X_0, a_0)$. Calculating $\psi(g)$ for the 1440 elements of $\mathrm{Aut}(X_0, a_0)$ by means of their matrix representations, we prove (10.1) and hence Proposition 1.8.

$$\begin{bmatrix}
-54 & -30 & -78 & 0 & 0 & 6 & -5 & 1 & 0 & -2 \\
-30 & -20 & -45 & 0 & 0 & 4 & -3 & 0 & 0 & -1 \\
-78 & -45 & -114 & 0 & 0 & 9 & -7 & 1 & 0 & -3 \\
0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\
6 & 4 & 9 & 0 & 1 & -4 & 1 & 2 & 0 & 1 \\
-5 & -3 & -7 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 2 & 1 & -4 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
-2 & -1 & -3 & 0 & 0 & 1 & 0 & -1 & 1 & -2
\end{bmatrix}$$

TABLE 10.2. The Gram matrix of S_Z

$$\begin{bmatrix}
1076 & 533 & -101 & -148 & -1217 & -817 & -1624 & -2398 & -1955 & -1502 & -1012 & -522 & -1176 & -802 & -1593 & -2352 & -1924 & -1460 & -996 & -500 \\
21 & 10 & -2 & -3 & -23 & -15 & -30 & -45 & -37 & -28 & -19 & -10 & -23 & -16 & -31 & -46 & -38 & -29 & -20 & -10 \\
1224 & 606 & -115 & -168 & -1386 & -930 & -1848 & -2730 & -2226 & -1710 & -1152 & -594 & -1338 & -912 & -1812 & -2676 & -2190 & -1662 & -1134 & -570 \\
1794 & 888 & -168 & -247 & -2028 & -1362 & -2706 & -3996 & -3258 & -2502 & -1686 & -870 & -1962 & -1338 & -2658 & -3924 & -3210 & -2436 & -1662 & -834 \\
73 & 36 & -7 & -10 & -83 & -56 & -111 & -164 & -133 & -102 & -69 & -36 & -79 & -54 & -107 & -158 & -129 & -98 & -67 & -34 \\
20 & 10 & -2 & -3 & -22 & -15 & -29 & -43 & -35 & -27 & -18 & -9 & -21 & -14 & -28 & -42 & -34 & -26 & -18 & -9 \\
66 & 33 & -6 & -9 & -75 & -50 & -100 & -148 & -120 & -92 & -62 & -32 & -73 & -50 & -99 & -146 & -120 & -91 & -62 & -31 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 10 & -2 & -3 & -24 & -16 & -32 & -48 & -39 & -30 & -20 & -10 & -22 & -15 & -30 & -44 & -36 & -27 & -18 & -9 \\
74 & 37 & -7 & -10 & -83 & -56 & -111 & -164 & -134 & -103 & -70 & -36 & -82 & -56 & -111 & -164 & -134 & -102 & -70 & -35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
65 & 32 & -6 & -9 & -74 & -49 & -98 & -145 & -118 & -91 & -62 & -32 & -71 & -48 & -96 & -142 & -116 & -88 & -60 & -30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 11 & -2 & -3 & -27 & -18 & -36 & -53 & -43 & -33 & -22 & -11 & -26 & -18 & -35 & -52 & -42 & -32 & -22 & -11 \\
64 & 32 & -6 & -9 & -72 & -48 & -96 & -142 & -116 & -89 & -60 & -31 & -69 & -47 & -94 & -138 & -113 & -86 & -59 & -30 \\
22 & 11 & -2 & -3 & -25 & -17 & -33 & -49 & -40 & -31 & -21 & -11 & -25 & -17 & -33 & -49 & -40 & -30 & -20 & -10 \\
73 & 36 & -7 & -10 & -82 & -55 & -110 & -162 & -132 & -101 & -68 & -35 & -80 & -54 & -108 & -160 & -131 & -100 & -68 & -34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
65 & 32 & -6 & -9 & -74 & -50 & -99 & -146 & -119 & -92 & -62 & -32 & -70 & -48 & -95 & -140 & -115 & -88 & -60 & -30 \\
8 & 4 & -1 & -1 & -8 & -5 & -10 & -15 & -12 & -9 & -6 & -3 & -10 & -7 & -14 & -20 & -16 & -12 & -8 & -4
\end{bmatrix}$$

TABLE 10.3. The involution $\tau(\tilde{h}_0^{(3)})$

By Remark 8.1, in order to prove the first assertion of Theorem 1.9, it is enough to show that $\zeta(\mathrm{Aut}(X_0, a_0)) = \mathrm{Aut}(X_0, a_0)/\langle \varepsilon_0^{(0)} \rangle$ is a non-splitting extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathfrak{A}_6$ and to calculate the conjugacy classes of this group. Since the symplectic subgroup of $\mathrm{Aut}(X_0, a_0)$ is isomorphic to $\mathfrak{A}_6$, we see that $\zeta(\mathrm{Aut}(X_0, a_0))$ contains a normal subgroup isomorphic to $\mathfrak{A}_6$ as a subgroup of index 2. By direct calculations, we confirm that every element of order 2 of $\zeta(\mathrm{Aut}(X_0, a_0))$ belongs to this normal subgroup. Hence the extension is non-splitting. The conjugacy classes of $\zeta(\mathrm{Aut}(X_0, a_0))$ are calculated as follows:

order	1	2	3	4	4	5	8	8
card.	1	45	80	90	180	144	90	90.

Therefore $\zeta(\mathrm{Aut}(X_0, a_0))$ is isomorphic to M_{10}.

$$\zeta(\rho_0^{(0)}) = \begin{bmatrix} 76 & 40 & -67 & -85 & -56 & -116 & -170 & -108 & -71 & -34 \\ 43 & 23 & -38 & -49 & -32 & -67 & -97 & -62 & -41 & -20 \\ 110 & 58 & -97 & -124 & -82 & -170 & -248 & -158 & -104 & -50 \\ 21 & 10 & -18 & -22 & -15 & -30 & -44 & -27 & -18 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 12 & 5 & -10 & -11 & -8 & -15 & -22 & -14 & -10 & -5 \\ -12 & -5 & 10 & 13 & 9 & 18 & 26 & 16 & 11 & 6 \\ -30 & -15 & 26 & 31 & 22 & 42 & 62 & 39 & 26 & 13 \\ 30 & 15 & -26 & -33 & -23 & -45 & -66 & -41 & -28 & -15 \\ 0 & 0 & 0 & -2 & -1 & -2 & -3 & -2 & -1 & 0 \end{bmatrix}$$

$$\zeta(\tau(h_0^{[1]})) = \begin{bmatrix} 74 & 37 & -64 & -91 & -58 & -121 & -176 & -112 & -74 & -36 \\ 43 & 21 & -37 & -54 & -35 & -73 & -105 & -67 & -45 & -23 \\ 96 & 48 & -83 & -120 & -76 & -160 & -232 & -148 & -98 & -48 \\ 22 & 11 & -19 & -26 & -17 & -34 & -51 & -31 & -21 & -11 \\ 14 & 7 & -12 & -17 & -12 & -23 & -34 & -21 & -14 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -14 & -7 & 12 & 17 & 11 & 21 & 32 & 20 & 13 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -9 & -5 & 8 & 11 & 8 & 15 & 22 & 14 & 9 & 5 \end{bmatrix}$$

$$\zeta(\tau(h_0^{[2]})) = \begin{bmatrix} 109 & 54 & -94 & -130 & -91 & -178 & -258 & -165 & -112 & -59 \\ 79 & 39 & -68 & -93 & -64 & -125 & -183 & -116 & -79 & -42 \\ 166 & 82 & -143 & -198 & -138 & -270 & -392 & -250 & -170 & -90 \\ 34 & 16 & -29 & -37 & -26 & -50 & -74 & -46 & -31 & -16 \\ 43 & 21 & -37 & -49 & -33 & -65 & -96 & -60 & -40 & -20 \\ -13 & -6 & 11 & 17 & 11 & 22 & 33 & 21 & 14 & 7 \\ -21 & -10 & 18 & 23 & 16 & 31 & 46 & 28 & 19 & 10 \\ 0 & 0 & 0 & -2 & -1 & -2 & -4 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\zeta(\tau(h_0^{[3]})) = \begin{bmatrix} 142 & 69 & -122 & -163 & -106 & -215 & -320 & -201 & -138 & -69 \\ 94 & 46 & -81 & -106 & -70 & -141 & -209 & -132 & -90 & -45 \\ 206 & 100 & -177 & -236 & -154 & -312 & -464 & -292 & -200 & -100 \\ 30 & 15 & -26 & -35 & -24 & -48 & -70 & -44 & -30 & -15 \\ 35 & 17 & -30 & -39 & -27 & -53 & -78 & -49 & -34 & -17 \\ -21 & -10 & 18 & 22 & 15 & 30 & 45 & 28 & 20 & 10 \\ -21 & -10 & 18 & 24 & 17 & 33 & 48 & 30 & 20 & 10 \\ -2 & -2 & 2 & 5 & 3 & 6 & 9 & 6 & 4 & 2 \\ 44 & 22 & -38 & -51 & -35 & -68 & -101 & -63 & -43 & -22 \\ -22 & -11 & 19 & 26 & 18 & 35 & 51 & 32 & 22 & 12 \end{bmatrix}$$

$$\zeta(\tau(\tilde{h}_0^{(3)})) = \begin{bmatrix} 581 & 290 & -502 & -666 & -446 & -888 & -1310 & -822 & -554 & -286 \\ 315 & 157 & -272 & -360 & -241 & -479 & -707 & -443 & -299 & -155 \\ 830 & 414 & -717 & -950 & -636 & -1266 & -1868 & -1172 & -790 & -408 \\ 138 & 68 & -119 & -157 & -106 & -210 & -310 & -194 & -131 & -68 \\ 43 & 21 & -37 & -49 & -33 & -65 & -96 & -60 & -40 & -20 \\ -73 & -36 & 63 & 82 & 55 & 110 & 161 & 101 & 68 & 35 \\ -73 & -36 & 63 & 82 & 55 & 109 & 162 & 101 & 68 & 35 \\ 60 & 30 & -52 & -66 & -44 & -88 & -130 & -81 & -55 & -28 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 43 & 21 & -37 & -49 & -32 & -65 & -96 & -60 & -41 & -21 \end{bmatrix}$$

TABLE 10.4. Generators of $\mathrm{Aut}(Z_0)$

The second assertion of Theorem 1.9 is confirmed by a direct calculation from the matrix representation (Table 8.1) of $\varepsilon_0^{(0)}$ and the matrix representation (Table 10.3) of $\tau(\tilde{h}_0^{(3)})$. In fact, we see that every element of the set $\mathrm{Invols}_0^{(3)}$ commutes with $\varepsilon_0^{(0)}$.

In order to prove the third assertion of Theorem 1.9, we consider the positive cone

$$\mathcal{P}(Z_0) := (S_Z \otimes \mathbb{R}) \cap \mathcal{P}(X_0)$$

(0,	0,	0,	0,	0,	0,	0,	0,	1,	0);
(0,	0,	0,	0,	1,	0,	0,	0,	0,	0);
(0,	0,	0,	1,	0,	0,	0,	0,	0,	0);
(10,	6,	−9,	−12,	−8,	−16,	−24,	−15,	−10,	−5);
(13,	6,	−11,	−16,	−11,	−22,	−32,	−20,	−14,	−8);
(13,	6,	−11,	−12,	−8,	−16,	−24,	−15,	−10,	−5);
(14,	7,	−12,	−17,	−12,	−23,	−34,	−21,	−14,	−7);
(14,	7,	−12,	−17,	−11,	−22,	−33,	−21,	−14,	−7);
(14,	7,	−12,	−15,	−10,	−20,	−30,	−18,	−13,	−7);
(14,	7,	−12,	−15,	−10,	−19,	−30,	−18,	−12,	−6);
(21,	10,	−18,	−24,	−16,	−32,	−48,	−30,	−20,	−10);
(21,	10,	−18,	−22,	−15,	−30,	−44,	−27,	−18,	−9);
(22,	11,	−19,	−26,	−17,	−34,	−51,	−31,	−21,	−11);
(22,	11,	−19,	−25,	−17,	−34,	−49,	−31,	−21,	−11);
(22,	11,	−19,	−24,	−16,	−32,	−48,	−30,	−21,	−11);
(22,	11,	−19,	−24,	−16,	−32,	−48,	−30,	−20,	−9);
(26,	12,	−22,	−29,	−19,	−38,	−57,	−35,	−24,	−12);
(30,	15,	−26,	−35,	−24,	−48,	−70,	−44,	−30,	−15);
(30,	15,	−26,	−33,	−23,	−45,	−66,	−42,	−28,	−14);
(30,	15,	−26,	−33,	−23,	−45,	−66,	−41,	−28,	−15);
(30,	15,	−26,	−32,	−21,	−42,	−63,	−39,	−26,	−13);
(34,	16,	−29,	−37,	−26,	−50,	−74,	−46,	−31,	−16);
(35,	17,	−30,	−39,	−27,	−53,	−78,	−49,	−34,	−17);
(35,	17,	−30,	−39,	−26,	−51,	−76,	−47,	−32,	−17);
(42,	20,	−36,	−46,	−31,	−62,	−91,	−57,	−39,	−20);
(43,	21,	−37,	−49,	−33,	−65,	−96,	−60,	−40,	−20);
(43,	21,	−37,	−47,	−32,	−63,	−94,	−58,	−40,	−20);
(44,	22,	−38,	−51,	−35,	−68,	−101,	−63,	−43,	−22);
(44,	22,	−38,	−50,	−33,	−66,	−98,	−61,	−42,	−21);
(44,	22,	−38,	−48,	−33,	−64,	−95,	−59,	−40,	−20);

TABLE 10.5. The orbit $\tilde{o}_0$

(34,	16,	−29,	−38,	−26,	−52,	−76,	−48,	−32,	−16);
(34,	16,	−29,	−36,	−24,	−48,	−72,	−44,	−30,	−16);
(36,	18,	−31,	−42,	−28,	−56,	−84,	−52,	−36,	−18);
(36,	18,	−31,	−40,	−28,	−54,	−80,	−50,	−34,	−18);
(36,	18,	−31,	−40,	−26,	−52,	−78,	−48,	−32,	−16);
(52,	26,	−45,	−60,	−40,	−80,	−118,	−74,	−50,	−26);
(52,	26,	−45,	−58,	−40,	−78,	−116,	−72,	−48,	−24);
(52,	26,	−45,	−56,	−38,	−76,	−112,	−70,	−48,	−24);
(78,	38,	−67,	−88,	−60,	−118,	−174,	−108,	−74,	−38);
(78,	38,	−67,	−86,	−58,	−114,	−170,	−106,	−72,	−36);

TABLE 10.6. The orbit $\tilde{o}_3$

of S_Z that contains an ample class. (Recall that we consider S_Z as a $\mathbb{Z}$-submodule of S_0 by π^*.) We put

$$D_Z^{(0)} := \mathcal{P}(Z_0) \cap D^{(0)},$$

where $D^{(0)}$ is the induced chamber in $N(X_0)$ given in Theorem 1.4. Let

$$\mathrm{pr}_Z \colon S_0 \otimes \mathbb{R} \ \to \ S_Z \otimes \mathbb{R}$$

be the orthogonal projection. Then we have

$$D_Z^{(0)} = \{\ x \in \mathcal{P}(Z_0) \ \mid \ \langle u, x\rangle_Z \geq 0 \ \text{ for any } \ u \in \mathrm{pr}_Z(\Delta(D^{(0)}))\ \}$$

Since the interior point a_0 of $D^{(0)}$ belongs to S_Z, the closed subset $D_Z^{(0)}$ of $\mathcal{P}(Z_0)$ also contains a_0 in its interior, and hence $D_Z^{(0)}$ is a chamber of $\mathcal{P}(Z_0)$. Moreover the finite group $\zeta(\mathrm{Aut}(X_0, a_0))$ acts on $D_Z^{(0)}$. For $v \in \Delta(D^{(0)})$, the hyperplane

$$(\mathrm{pr}_Z(v))^\perp = (v)^\perp \cap \mathcal{P}(Z_0)$$

of $\mathcal{P}(Z_0)$ is a wall of $D_Z^{(0)}$ if and only if the solution of the linear programing to minimize $\langle \mathrm{pr}_Z(v), x\rangle_Z$ under the condition

$$\langle u', x\rangle_Z \geq 0 \ \text{ for all } \ u' \in \mathrm{pr}_Z(\Delta(D^{(0)})) \ \text{ not proporsional to } \mathrm{pr}_Z(v)$$

is unbounded to $-\infty$, where the variable x ranges through $S_Z \otimes \mathbb{Q}$. (See Section 3 of [32]). By this method, we see that the set of primitive outward defining vectors of walls of $D_Z^{(0)}$ consists of 40 vectors, and they are decomposed into the two orbits $\tilde{o}_0$ and $\tilde{o}_3$ of cardinalities 30 and 10 under the action of $\zeta(\mathrm{Aut}(X_0, a_0))$, where

$$\tilde{o}_0 = \{\ 2\,\mathrm{pr}_Z(r) \ \mid \ r \in o_0\ \}, \quad \tilde{o}_3 = \{\ 2\,\mathrm{pr}_Z(v) \ \mid \ v \in o_3\ \}.$$

Here we use the dual basis of S_Z not with respect to $\langle\ ,\ \rangle_S|_{S_Z}$ but with respect to $\langle\ ,\ \rangle_Z$. (Recall that we have $|o_0| = 60$ and $|o_3| = 10$.) The involution $\zeta(\tau(\tilde{h}_0^{(3)}))$ maps $D_Z^{(0)}$ to a chamber of $\mathcal{P}(Z_0)$ adjacent to $D_Z^{(0)}$ across the wall defined by a vector

$$(52, 26, -45, -60, -40, -80, -118, -74, -50, -26)$$

in $\tilde{o}_3$ isomorphically. In particular, the cone $N(Z_0) := \mathcal{P}(Z_0) \cap N(X_0)$ in $\mathcal{P}(Z_0)$ is tessellated by chambers congruent to $D_Z^{(0)}$ under the action of $\mathrm{Aut}(Z_0)$. Thus Theorem 1.9 is proved.

Remark 10.1. The matrix representations of the generators

$$\zeta(\rho_0^{(0)}),\ \ \zeta(\tau(h_0^{[1]})),\ \ \zeta(\tau(h_0^{[2]})),\ \ \zeta(\tau(h_0^{[3]})),\ \ \zeta(\tau(\tilde{h}_0^{(3)}))$$

of $\mathrm{Aut}(Z_0)$ with respect to the basis $f_1, \ldots, f_{10}$ of S_Z are given Table 10.4.

Remark 10.2. The interior point a_0 of $D_Z^{(0)}$ is written as

$$(122, 60, -105, -136, -92, -182, -270, -168, -114, -58)$$

with respect to the basis $f_1, \ldots, f_{10}$ of S_Z. The elements of the orbits $\tilde{o}_0$ and $\tilde{o}_3$ are given in Tables 10.5 and 10.6. By these data and the Gram matrix (Table 10.2) of S_Z, we can completely determine the shape of the chamber $D_Z^{(0)}$.

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