THE AUTOMORPHISM GROUPS OF CERTAIN SINGULAR $K3$ SURFACES AND AN ENRIQUES SURFACE: COMPUTATIONAL DATA

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We explain the contents of the computational data for the paper

[*] I. Shimada: The automorphism groups of certain singular $K3$ surfaces and an Enriques surface

that are presented in the author's web-page

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html` .

The numbers of Tables, Theorems, etc. and various notations are according to the paper [*].

- In the file `GramS.txt`, we present the Gram matrix $\texttt{GramS}[\texttt{k}]$ of the Néron-Severi lattice S_k of X_k with respect to the basis given in Corollary 3.5.
- In the file `ample.txt`, we present the ample vector $\texttt{ample}[\texttt{k}] = a_k \in S_k$ that appears in Theorem 1.2 and Table 5.2.
- In the file `AutXa.txt`, we present all the elements of the finite group $\texttt{AutXa}[\texttt{k}] = \mathrm{Aut}(X_k, a_k)$ as a list of 20×20 matrices in $\mathrm{O}(S_k)$.
- In the file `walls.txt`, we present the orbit decomposition $\texttt{walls}[\texttt{k}]$ of the set $\Delta(D^{(0)})$ of outward primitive defining vectors of walls of the induced chamber $D^{(0)} \subset N(X_k)$ by the action of $\mathrm{Aut}(X_k, a_k)$. Each element of $\texttt{walls}[\texttt{k}]$ is an orbit, which is a list of outward primitive defining vectors of walls. The orbits are listed in the same order as in Table 1.1.
- In the file `Invols.txt`, we present the set $\texttt{Invols}[\texttt{k},\texttt{i}] = \mathrm{Invols}_k^{(i)}$ of involutions , where $i = 0, \dots, 12$ for $k = 0$, $i = 0, \dots, 12$ for $k = 1$, and $i = 0, \dots, 7$ for $k = 2$. Each $\texttt{Invols}[\texttt{k},\texttt{i}]$ is a list of $[\texttt{M}, \texttt{type}]$, where `M` is the 20×20 matrix that represents the involution, and $\texttt{type} \in \{\texttt{\_symp}, \texttt{\_Enr}, \texttt{\_rat}\}$ is the type of the involution. For $i > 0$, the involutions in $\texttt{Invols}[\texttt{k},\texttt{i}] = \mathrm{Invols}_k^{(i)}$

Partially supported by JSPS Grants-in-Aid for Scientific Research (C) No.25400042 .

map $D^{(0)}$ to the induced chamber $D^{(i)}$ adjacent to $D^{(0)}$ along the wall defined by the *first element* in the corresponding orbit in the list walls[k] of orbits.

- In the file `EnriquesInvol.txt`, we present the matrix representation of the Enriques involution `EnriquesInvol` $:= \varepsilon_0^{(0)} \in \mathrm{Aut}(X_0, a_0)$ (Table 8.1).
- In the file `NonSympOrder4.txt`, we present the matrix representation of the purely non-symplectic automorphism `NonSympOrder4` $:= \rho_0^{(0)} \in \mathrm{Aut}(X_0, a_0)$ of order 4 (Table 8.2).
- In the file `hrhodata.txt`, we present the data [hrho, nodes, lines], where hrho $= h_\rho$ is the polarization of degree 4 of X_0 invariant under $\rho_0^{(0)}$ (see Section 9.1), nodes is the orbit decomposition of the set of classes of smooth rational curves that are contracted to the 8 nodes of the quartic surface Y under the action of $\langle \rho_0^{(0)} \rangle$ (an orbit $[p_0, p_1, p_2, p_3]$ of length 4 and an orbit $[q_0, q_1]$ of length 2), and lines is the orbit decomposition of the set of classes of smooth rational curves that are mapped to lines of Y isomorphically under the action of $\langle \rho_0^{(0)} \rangle$ (9 orbits $l_i = [\ell_i, \ell'_s, \ell''_i, \ell'''_i]$ of length 4).
- In the file `SympInvol14.txt`, we present the matrix representation of the symplectic involution `SympInvol14` $:= \sigma_1^{(4)} \in \mathrm{Invols}_1^{(4)}$ (Table 8.6).
- In the file `hsigmadata.txt`, we present the data [hsigma, M, cusps, lines], where hsigma $= h_\sigma$ is the polarization of degree 2 of X_1 invariant under $\sigma_1^{(4)}$ (see Section 9.4), M is the matrix representation of the double-plane involution $\tau(h_\sigma)$, cusps is the list of the 7 pairs of classes of two smooth rational curves that are contracted to a singular point of Y (ordered as $q_0, q_1, q'_1, q_2, q'_2, q_3, q'_3$), and lines is the list of the 10 pairs of classes of two smooth rational curves that are mapped isomorphically to a splitting line of B (ordered as $\ell_0, \dots, \ell_9$).

For a polarization h of degree 2, the set $\mathcal{C}_0(h)$ of the classes of smooth rational curves contracted by $X \to X_h$ is decomposed into the union of indecomposable root systems, and each indecomposable root system is presented in the form of

$$[\text{ the } ADE\text{-type}, \ \text{ the list of roots in this system }],$$

so that the ADE-type of $\mathrm{Sing}\, X_h$ is easily obtained.

- In the file `threeinvols_AutXa.txt`, we present the data threeinvols_AutXa[k] of the three involutions $\tau(h_k^{[1]})$, $\tau(h_k^{[2]})$, $\tau(h_k^{[3]})$ in $\mathrm{Aut}(X_k, a_k)$. The data threeinvols_AutXa[k] consists of three quartets [h, M, C0, C1], where h $= h_k^{[i]}$

is the polarization of degree 2 in Tables 8.3, 8.4, 8.5, `M` is the matrix representation of the double-plane involution $\tau(h_k^{[i]})$, `C0` is the list $\mathcal{C}_0(h_k^{[i]})$ formatted as above, and `C1` is the list $\mathcal{C}_1(h_k^{[i]})$.

- In the file `adjacentAutX.txt`, we present the data `adjacentAutX`[`k`, `i`] of the automorphisms $\tau(\tilde{h}_k^{(i)})$ in $\mathrm{Aut}(X_k)$, where $i = 1, \dots, 12$ for $k = 0$, $i = 1, \dots, 3, 5, \dots, 12$ for $k = 1$, and $i = 1, \dots, 7$ for $k = 2$. The data `adjacentAutX`[`k`, `i`] is a quartet [`h`, `M`, `C0`, `C1`], where $\mathtt{h} = \tilde{h}_k^{(i)}$ is the polarization of degree 2 in Tables 8.3, 8.4, 8.5, `M` is the matrix representation of the double-plane involution $\tau(\tilde{h}_k^{(i)})$, `C0` is the list $\mathcal{C}_0(\tilde{h}_k^{(i)})$ formatted as above, and `C1` is the list $\mathcal{C}_1(\tilde{h}_k^{(i)})$.

We put

$$\mathrm{Aut}(X_k, a_k)' := \langle \tau(h_k^{[1]}), \tau(h_k^{[2]}), \tau(h_k^{[3]}) \rangle,$$

which is of index 2 (resp. 1) for $k = 0$ (resp. $k = 1, 2$) in $\mathrm{Aut}(X_k, a_k)$, and

$$H_k := \begin{cases} \mathfrak{A}_6 \times \{\pm 1\} & \text{if } k = 0, \\ \mathrm{PGL}_2(\mathbb{F}_9) & \text{if } k = 1, 2. \end{cases}$$

Then we have an isomorphism $\phi\colon \mathrm{Aut}(X_k, a_k)' \xrightarrow{\sim} H_k$ (see Section 8).

- In the file `Isom.txt`, we present the list `Isom`[`k`] of the pairs [`M`, `phi`(`M`)], where `M` is an element of $\mathrm{Aut}(X_k, a_k)'$ and `phi`(`M`) is its image by the isomorphism ϕ. An element of $G_0 = \mathfrak{A}_6 \times \{\pm 1\}$ is written in the form $[[a_1, a_2, a_3, a_4, a_5, a_6], \pm 1]$, where $[a_1, a_2, a_3, a_4, a_5, a_6]$ is the permutation
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix}.$$
An element of $G_1 = G_2 = \mathrm{PGL}_2(\mathbb{F}_9)$ is written in the form of the matrix $[[a, b], [c, d]]$, where $a, b, c, d \in \mathbb{F}_9 = \mathbb{F}_3(\sqrt{2})$, that represents
$$z \mapsto \frac{az + b}{cz + d}.$$

For Borcherds method, we put the following data:

- In the file `GramL.txt`, we present the Gram matrix of L_{26} with respect to the basis (5.1).
- In the file `w0.txt`, we present the Weyl vector w_0.
- In the file `EmbS.txt`, we present the embeddings $\varepsilon_k\colon S_k \hookrightarrow L_{26}$. The map $v \mapsto v \cdot \mathtt{EmbS}[\mathtt{k}]$ gives the embedding $\varepsilon_k\colon S_k \hookrightarrow L_{26}$.

- In the file `EmbR.txt`, we present the embeddings $R_k \hookrightarrow L_{26}$. The map $v \mapsto v \cdot \texttt{EmbR}[\texttt{k}]$ gives the embedding of the orthogonal complement R_k of S_k in L_{26}, where the basis of R_k form the standard root system of type $2A_2 + 2A_1$ (for $k = 0$), $A_3 + A_2 + A_1$ (for $k = 1$), $A_4 + A_2$ (for $k = 2$).

We also present the following data for the calculation of the automorphism group $\mathrm{Aut}(Z_0)$ of the Enriques surface $Z_0 := X_0/\varepsilon_0^{(0)}$.

- In the file `EmbZ.txt`, we present a basis of the submodule S_0^+ of S_0, which is identified with S_Z by π^* up to the multiplicative factor 2 on the intersection pairing. The rows of the 10×20 matrix `EmbZ` are the basis $f_1, \dots, f_{10}$ of Table 10.1.
- In the file `GramSZ.txt`, we present the Gram matrix `GramSZ` of the lattice $(S_Z, \langle\ ,\ \rangle_Z)$ with respect to the basis $f_1, \dots, f_{10}$.
- In the file `prZMat.txt`, we present the 20×10 matrix `prZMat` such that $v \mapsto v \cdot \texttt{prZMat}$ gives the orthogonal projection $\mathrm{pr}_Z \colon S_0 \otimes \mathbb{R} \to S_Z \otimes \mathbb{R}$.
- In the file `gensAutZ.txt`, we present the matrix presentations of the generators

$$\begin{aligned}
\zeta(\rho_0^{(0)}) \qquad & (\texttt{zetaNonSympOrder4}),\\
\zeta(\tau(h_0^{[1]})) \qquad & (\texttt{zetainvol001}),\\
\zeta(\tau(h_0^{[2]})) \qquad & (\texttt{zetainvol002}),\\
\zeta(\tau(h_0^{[3]})) \qquad & (\texttt{zetainvol003}),\\
\zeta(\tau(\tilde{h}_0^{(3)})) \qquad & (\texttt{zetaadjinvol03}).
\end{aligned}$$

- In the file `DZ0.txt`, the data of the chamber $D_Z^{(0)}$ is given. We present the interior point

$$\texttt{aZ} = [122, 60, -105, -136, -92, -182, -270, -168, -114, -58].$$

  We also give the lists `tildeorbit0` and `tildeorbit3` of the vectors in the orbits $\tilde{o}_0$ and $\tilde{o}_3$.

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