ON EDGE'S CORRESPONDENCE ASSOCIATED WITH $\cdot 222$

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Abstract. We describe explicitly the correspondence of Edge between the set of planes contained in the Fermat cubic 4-fold in characteristic 2, and the set of lattice points T of the Leech lattice Λ_{24} such that $OABT$ is a regular tetrahedron, where O is the origin of Λ_{24}, and A and B are fixed points of Λ_{24} such that OAB is a regular triangle of edge length 2. Using this description, we present Conway's isomorphism from $\mathrm{PSU}(6,4)$ to $\cdot 222$ in terms of matrices.

1. Introduction

In Table 10.4 of Conway and Sloane [1], it is shown that the subgroup $\cdot 222$ of the orthogonal group $\cdot 0 = \mathrm{O}(\Lambda_{24})$ of the Leech lattice Λ_{24} is isomorphic to the simple group $\mathrm{PSU}(6,4)$. In [2], Edge constructed a permutation representation of $\cdot 222$ on a certain set of lattice points of Λ_{24}, and suggested that this representation corresponds to the natural permutation representation of $\mathrm{PSU}(6,4)$ on the set $\mathcal{P}_X$ of linear planes contained in $X \otimes \overline{\mathbb{F}}_2$, where X is the Fermat cubic 4-fold

$$X \,:\, x_1^3 + \cdots + x_6^3 = 0$$

defined over $\mathbb{F}_2$, and $\overline{\mathbb{F}}_2$ is an algebraically closed field of characteristic 2. The purpose of this note is to clarify this correspondence by writing them explicitly. Our idea is based on the investigation in [8] of the lattice of numerical equivalence classes of n-dimensional linear subspaces contained in a $2n$-dimensional Hermite variety. As an application, we write Conway's isomorphism $\mathrm{PSU}(6,4) \cong \cdot 222$ in terms of matrices. It turns out that Conway's isomorphism is closely related to the well-known isomorphism between $\mathrm{PSL}(3,4)$ and the Mathieu group $\mathbb{M}_{21}$ (see Proposition 2.1). For simplicity, we put

$$D := 9196830720 = |\mathrm{PSU}(6,4)| = |\cdot 222|. \tag{1.1}$$

First, we define a geometric object $(\mathcal{P}_X, \nu_{\mathcal{P}})$ in X. The hypersurface $X \otimes \overline{\mathbb{F}}_2 \subset \mathbb{P}^5$ contains exactly 891 linear planes, and they are all defined over $\mathbb{F}_4$ (see Segre [7]). Let $\mathcal{P}_X$ denote the set of all these planes. For $\Pi, \Pi' \in \mathcal{P}_X$, we put

$$\nu_{\mathcal{P}}(\Pi, \Pi') := \Pi \cdot \Pi' = (1 - (-2)^{\dim(\Pi \cap \Pi')+1})/3$$

with the understanding that $\dim(\emptyset) = -1$, where $\Pi \cdot \Pi'$ is the intersection number of the algebraic cycles Π and Π' of X. The second equality in the above formula follows from the excess intersection formula (see Chapter 6.3 of Fulton [3]). For $a \in \overline{\mathbb{F}}_2$, let $a \mapsto \bar{a} := a^2$ denote the Frobenius action over $\mathbb{F}_2$, and for a matrix g with components in $\overline{\mathbb{F}}_2$, let $\bar{g}$ denote the matrix obtained from g by applying $a \mapsto \bar{a}$ to all the components of g. We put

$$\mathrm{GU}(6,4) := \{\, g \in \mathrm{GL}(6, \overline{\mathbb{F}}_2) \;\mid\; g \cdot {}^{\mathrm{T}}\bar{g} = I_6 \,\}, \quad \mathrm{PGU}(6,4) := \mathrm{GU}(6,4)/\mathbb{F}_4^{\times}.$$

This work was supported by JSPS KAKENHI Grant Numbers JP16K13749, JP16H03926.

Then $\mathrm{PGU}(6,4)$ is equal to the projective automorphism group of $X \otimes \overline{\mathbb{F}}_2 \subset \mathbb{P}^5$, and it acts on the set $X(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational points of X. Let $\mathrm{P\Gamma U}(6,4)$ be the subgroup of the full symmetric group of $X(\mathbb{F}_4)$ generated by the permutations induced by the action of $\mathrm{PGU}(6,4)$ and $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. We have

$$|\mathrm{P\Gamma U}(6,4)| = 2\,|\mathrm{PGU}(6,4)| = 6D.$$

Since all $\Pi \in \mathcal{P}_X$ are defined over $\mathbb{F}_4$, we have a natural homomorphism

$$\rho_{\mathcal{P}} \;:\; \mathrm{P\Gamma U}(6,4) \to \mathrm{Aut}(\mathcal{P}_X, \nu_{\mathcal{P}}).$$

Next we define a lattice-theoretic object $(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ in the Leech lattice Λ_{24}. Let O denote the origin of Λ_{24}. For points P, Q of $\Lambda_{24} \otimes \mathbb{R}$, we denote by $|PQ|$ the length $\langle PQ, PQ\rangle^{1/2}$ of the vector

$$PQ := Q - P,$$

where the inner-product $\langle\;,\;\rangle$ on $\Lambda_{24} \otimes \mathbb{R}$ is induced from the symmetric bilinear form of Λ_{24}. The vector $OP = P - O$ is sometimes simply denoted by P. We put

$$\mathcal{A} := \{\, [A,B] \;\mid\; A, B \in \Lambda_{24},\; |OA| = |OB| = |AB| = 2 \,\},$$

whose cardinality $|\mathcal{A}|$ is equal to $4600 \cdot 196560$. Then $\cdot 0$ acts on $\mathcal{A}$ transitively (see Chapter 10 of [1]). Let $[A,B]$ be an arbitrary element of $\mathcal{A}$. We put

$$\mathcal{T}_{AB} := \{\, T \in \Lambda_{24} \;\mid\; |OT| = |AT| = |BT| = 2 \,\}.$$

Let C be the mid-point of an edge of the regular triangle OAB. (Note that C is not a lattice point of Λ_{24}.) For $T, T' \in \mathcal{T}_{AB}$, we put

$$\nu_{\mathcal{T}}(T, T') := \langle CT, CT'\rangle = \langle OT, OT'\rangle - 1.$$

Let Λ'_{AB} denote the sublattice of Λ_{24} generated by the vectors TT', where T and T' run through $\mathcal{T}_{AB}$:

$$\Lambda'_{AB} := \langle\, TT' \;\mid\; T, T' \in \mathcal{T}_{AB} \,\rangle.$$

Then we obviously have a natural homomorphism

$$\tau_{\mathcal{T}} \;:\; \mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}}) \to \mathrm{O}(\Lambda'_{AB}).$$

Edge [2] observed that $|\mathcal{P}_X| = |\mathcal{T}_{AB}| = 891$, and that $\mathcal{P}_X$ and $\mathcal{T}_{AB}$ have many combinatorial properties in common.

Definition 1.1. An *Edge correspondence* is a bijection $\phi\colon \mathcal{P}_X \xrightarrow{\sim} \mathcal{T}_{AB}$ such that $\nu_{\mathcal{T}}(\phi(\Pi), \phi(\Pi')) = \nu_{\mathcal{P}}(\Pi, \Pi')$ holds for any $\Pi, \Pi' \in \mathcal{P}_X$.

Our first result is as follows:

Theorem 1.2. *An Edge correspondence exists.*

Two algebraic cycles Σ, Σ' of codimension 2 on $X \otimes \overline{\mathbb{F}}_2$ are *numerically equivalent* if $\Sigma \cdot \Sigma'' = \Sigma' \cdot \Sigma''$ holds for any algebraic cycle Σ'' of codimension 2 on $X \otimes \overline{\mathbb{F}}_2$. Let $\mathcal{N}_X$ be the lattice of numerical equivalence classes of algebraic cycles of codimension 2 on $X \otimes \overline{\mathbb{F}}_2$. The rank of $\mathcal{N}_X$ is equal to the 4th Betti number $b_4(X) = 23$ of X; that is, X is supersingular (see Shioda and Katsura [10], Tate [11]).

Corollary 1.3. *The lattice $\mathcal{N}_X$ is generated by the numerical equivalence classes of planes $\Pi \in \mathcal{P}_X$, and hence is isomorphic to the lattice generated by the vectors $CT \in \Lambda_{24} \otimes \mathbb{Q}$, where T runs through $\mathcal{T}_{AB}$.*

We denote by $R_{AB} \subset \Lambda_{24}$ the sublattice of rank 2 containing A, B, and put

$$\cdot 222_{AB} := \{g \in \cdot 0 \mid A^g = A,\ B^g = B\}, \quad \circ 222_{AB} := \{g \in \cdot 0 \mid R_{AB}^g = R_{AB}\}.$$

Then $\circ 222_{AB}$ acts on the orthogonal complement $R_{AB}^{\perp}$ of R_{AB} in Λ_{24}. The following lemma is proved in Section 3.1. See [1, Chapter 6] for the definition of laminated lattices.

Lemma 1.4. *The sublattice $\Lambda'_{AB} = \langle TT' \mid T, T' \in \mathcal{T}_{AB}\rangle$ of Λ_{24} is equal to $R_{AB}^{\perp}$, and is isomorphic to the laminated lattice Λ_{22} of rank 22. Moreover, the natural homomorphism*

$$\sigma_{AB} : \circ 222_{AB} \to \mathrm{O}(R_{AB}^{\perp}) = \mathrm{O}(\Lambda'_{AB}) \cong \mathrm{O}(\Lambda_{22})$$

is an isomorphism.

The following theorem gives a geometric explanation of Conway's isomorphism $\mathrm{PSU}(6,4) \cong \cdot 222$ via an Edge correspondence.

Theorem 1.5. *Let $\phi\colon (\mathcal{P}_X, \nu_{\mathcal{P}}) \xrightarrow{\sim} (\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ be an Edge correspondence. Then the composite homomorphism*

$$\mathrm{P\Gamma U}(6,4) \underset{\rho_{\mathcal{P}}}{\longrightarrow} \mathrm{Aut}(\mathcal{P}_X, \nu_{\mathcal{P}}) \underset{\text{by } \phi}{\overset{\sim}{\longrightarrow}} \mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}}) \underset{\tau_{\mathcal{T}}}{\longrightarrow} \mathrm{O}(\Lambda'_{AB}) \underset{\sigma_{AB}^{-1}}{\overset{\sim}{\longrightarrow}} \circ 222_{AB}$$

is injective, has the image of index 2 in $\circ 222_{AB}$, and induces an isomorphism $\mathrm{PSU}(6,4) \cong \cdot 222_{AB}$.

Corollary 1.6. *The homomorphism $\rho_{\mathcal{P}}$ is an isomorphism.*

Corollary 1.7. $\mathrm{O}(\Lambda_{22})/\{\pm I_{22}\} \cong \mathrm{P\Gamma U}(6,4)$.

We prove Theorem 1.2 in Section 2.4 by presenting an example ϕ_0 of the Edge correspondence. Then we write the homomorphism $\mathrm{P\Gamma U}(6,4) \to \circ 222_{AB}$ in Theorem 1.5 for $\phi = \phi_0$ in terms of matrices (Theorem 3.2). Using these matrices, we prove Theorem 1.5 and Corollaries in Section 3.4. Since we state our results explicitly, most of them can be checked by direct computation. The computational data is available from the author's webpage [9]. For the computation, we used `GAP` [4].

Thanks are due to Professor Tetsuji Shioda and Professor Shigeru Mukai for discussions, and to Professor Ivan Cheltsov for inviting me to write this paper.

2. Constructing an Edge correspondence

2.1. **Notation.** Let S be a finite set. For a subset $W \subset S$, let $v_W\colon S \to \mathbb{F}_2$ denote the function such that $v_W^{-1}(1) = W$. By $W \mapsto v_W$, we equip the power set 2^S of S with a structure of the $\mathbb{F}_2$-vector space. We thus regard a linear binary code on S as a non-empty subset of 2^S closed under the symmetric difference.

2.2. **Fixing a basis of Λ_{24}.** In order to be explicit, we fix a basis of the Leech lattice Λ_{24}. Let $M := \{1, 2, \ldots, 24\}$ be the set of positions of the MOG (Miracle Octad Generators, see Chapter 11 of [1]) indexed by the following diagram.

1	5	9	13	17	21
2	6	10	14	18	22
3	7	11	15	19	23
4	8	12	16	20	24

Let $\mathbb{Z}^M$ be the $\mathbb{Z}$-module of functions from M to $\mathbb{Z}$, and we equip $\mathbb{Z}^M$ with the inner product

$$(x_1y_1 + \cdots + x_{24}y_{24})/8.$$

We define Λ_{24} to be the sublattice of $\mathbb{Z}^M$ generated by the row vectors of the matrix B_Λ in Figure 4.12 of [1], with the scalar multiplication $1/\sqrt{8}$ removed. Each point of Λ_{24} is written as a row vector with respect to the standard basis of $\mathbb{Z}^M$. Hence each element of $\cdot 0$ acts on Λ_{24} from the *right*, and is expressed by a 24×24 orthogonal matrix g with components in $\mathbb{Q}$ whose action on $\mathbb{Z}^M \otimes \mathbb{Q}$ preserves $\Lambda_{24} \subset \mathbb{Z}^M \otimes \mathbb{Q}$; that is, we have

$$\cdot 0 = \{\, g \in \mathrm{GL}_{24}(\mathbb{Q}) \;\mid\; g \cdot {}^{\mathrm{T}}g = I_{24},\;\; B_\Lambda \cdot g \cdot B_\Lambda^{-1} \in \mathrm{GL}_{24}(\mathbb{Z}) \,\}.$$

2.3. **Two binary codes of length** 21. Let $\mathbb{P}^2$ be a projective plane over $\mathbb{F}_2$. We denote by S the set $\mathbb{P}^2(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational points of $\mathbb{P}^2$. Let $\mathcal{H} \subset 2^S$ be the linear code generated by the codewords $\ell(\mathbb{F}_4) \subset S$ of weight 5, where ℓ runs through the set of $\mathbb{F}_4$-rational lines on $\mathbb{P}^2$, and $\ell(\mathbb{F}_4)$ is the set of $\mathbb{F}_4$-rational points of ℓ. Then $\mathcal{H}$ is of dimension 10. Let $\mathrm{P\Gamma L}(3,4)$ be the subgroup of the full symmetric group of S generated by the permutations induced by the actions of $\mathrm{PGL}(3,4)$ and $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. It is obvious that the group $\mathrm{P\Gamma L}(3,4)$ of order 120960 acts on the linear code $\mathcal{H}$.

Let $\mathcal{C}_{24} \subset 2^M$ be the extended binary Golay code defined by MOG (see Chapter 11 of [1]). It is well-known that the automorphism group of $\mathcal{C}_{24}$ is the Mathieu group $\mathbb{M}_{24}$. We put $M' := \{1, 2, \dots, 21\} \subset M$, and let $\mathrm{pr}_{21} \colon 2^M \to 2^{M'}$ be the natural projection. We then put

$$\mathcal{C}'_{21} := \{\, w \in \mathcal{C}_{24} \;\mid\; w(22) = w(23) = w(24) \,\}, \quad \mathcal{C}_{21} := \mathrm{pr}_{21}(\mathcal{C}'_{21}).$$

The following well-known fact explains the isomorphism between $\mathrm{PSL}(3,4)$ and the Mathieu group $\mathbb{M}_{21} = \{\sigma \in \mathbb{M}_{24} \mid n^\sigma = n \text{ for } n = 22, 23, 24\}$.

Proposition 2.1. (1) *The binary codes $\mathcal{H}$ and $\mathcal{C}_{21}$ are isomorphic. The weight distribution of these codes is*

$$0^1\, 5^{21}\, 8^{210}\, 9^{280}\, 12^{280}\, 13^{210}\, 16^{21}\, 21^1.$$

(2) *The automorphism group of $\mathcal{H}$ is equal to* $\mathrm{P\Gamma L}(3,4)$. *In particular, there exist exactly* $|\mathrm{P\Gamma L}(3,4)| = 120960$ *isomorphisms between $\mathcal{H}$ and $\mathcal{C}_{21}$.*

2.4. **Proof of Theorem 1.2.** The condition $\nu_{\mathcal{P}}(\Pi, \Pi') = \nu_{\mathcal{T}}(\phi(\Pi), \phi(\Pi'))$ required for an Edge correspondence $\phi \colon \mathcal{P}_X \xrightarrow{\sim} \mathcal{T}_{AB}$ is equivalent to the condition that the third $\Longleftrightarrow$ in each row of the following table should hold:

$\dim(\Pi \cap \Pi')$		$\lvert\Pi(\mathbb{F}_4) \cap \Pi'(\mathbb{F}_4)\rvert$		$\Pi \cdot \Pi'$		$\langle \phi(\Pi), \phi(\Pi') \rangle$
-1	$\Longleftrightarrow$	0	$\Longleftrightarrow$	0	$\Longleftrightarrow$	1
0	$\Longleftrightarrow$	1	$\Longleftrightarrow$	1	$\Longleftrightarrow$	2
1	$\Longleftrightarrow$	5	$\Longleftrightarrow$	-1	$\Longleftrightarrow$	0
2	$\Longleftrightarrow$	21	$\Longleftrightarrow$	3	$\Longleftrightarrow$	4.

We construct an example ϕ_0 of the Edge correspondence explicitly. Let $\omega \in \mathbb{F}_4$ be a root of $x^2 + x + 1 = 0$, and let $\Pi_0, \Pi_\infty \in \mathcal{P}_X$ be the planes specified by three points on them as follows:

$$\begin{aligned}
\Pi_0 &= \langle\; (1:0:0:\omega:0:0),\;\; (0:1:0:0:\omega:0),\;\; (0:0:1:0:0:\omega) \;\rangle,\\
\Pi_\infty &= \langle\; (1:0:0:\bar\omega:0:0),\;\; (0:1:0:0:\bar\omega:0),\;\; (0:0:1:0:0:\bar\omega) \;\rangle.
\end{aligned}$$

Note that $\nu_{\mathcal{P}}(\Pi_0, \Pi_\infty) = 0$. Then there exist exactly 21 planes $\Pi_s \in \mathcal{P}_X$ such that

$$\dim(\Pi_0 \cap \Pi_s) = 1, \quad \dim(\Pi_\infty \cap \Pi_s) = -1.$$

Let $S = \{\Pi_1, \dots, \Pi_{21}\}$ be the set of these 21 planes. The mapping $\Pi_s \mapsto \Pi_s \cap \Pi_0$ establishes a bijection from S to the set $\Pi_0^\vee(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational lines on Π_0, where $\Pi_0^\vee$ is the dual plane of Π_0. Via this bijection and $\Pi_0^\vee \cong \mathbb{P}^2$, the notation $S = \{\Pi_1, \dots, \Pi_{21}\}$ is compatible with the notation $S = \mathbb{P}^2(\mathbb{F}_4)$ in Section 2.3. For each $\mathbb{F}_4$-rational point P of Π_0, the codeword

$$\ell_P := \{\ \Pi_s \in S \ \mid \ P \in \Pi_s\ \} \in 2^S$$

is of weight 5. Each codeword ℓ_P is an $\mathbb{F}_4$-rational line of $S = \Pi_0^\vee(\mathbb{F}_4)$, and these 21 codewords ℓ_P generate a linear code $\mathcal{H}_X \subset 2^S$ of dimension 10.

Recall that each vector of Λ_{24} is written as a row vector with respect to the standard basis of $\mathbb{Z}^M$. Since $\cdot 0$ acts on $\mathcal{A}$ transitively (see Chapter 10 of [1]), we can assume without loss of generalities that the fixed lattice points $A, B \in \Lambda_{24}$ are

$$A = (0^{21}, 4, 0, -4), \quad B = (0^{21}, 0, 4, -4), \tag{2.1}$$

where 0^{21} is the constant map to 0 from $M' = \{1, \dots, 21\} \subset M$, and the last three coordinates indicates the values at $22, 23, 24 \in M$. In the following, we will use this choice of A and B. It is easy to calculate the set $\mathcal{T}_{AB}$. It turns out that $\mathcal{T}_{AB}$ consists of the following 891 vectors:

one element of type	$(0^{21}, 4, 4, 0)$	(type 0),
42 elements of type	$(0^{20}(\pm 4)^1, 0, 0, -4)$	(type 1),
336 elements of type	$(0^{16}(\pm 2)^5, 2, 2, -2)$	(type 2),
512 elements of type	$((\pm 1)^{21}, 1, 1, -3)$	(type 3),

where, for example, $0^{16}(\pm 2)^5$ indicates a map from M' to $\{0, \pm 2\}$ whose fiber over 0 is of cardinality 16. These vectors are described more precisely as follows.

- For a vector of type 1, the position of $(\pm 4)^1$ is arbitrary in M' and the sign is also arbitrary.
- For a vector of type 2, the positions of $(\pm 2)^5$ form a subset W of M' such that $W \cup \{22, 23, 24\}$ is an octad of the Golay code $\mathcal{C}_{24}$, and the number of minus sign is odd.
- For a vector of type 3, the positions of minus sign in $(\pm 1)^{21}$ form a subset W of M' that is a code word of $\mathcal{C}_{24}$ disjoint from $\{22, 23, 24\}$.

Let $T_0 \in \mathcal{T}_{AB}$ and $T_\infty \in \mathcal{T}_{AB}$ be the lattice points

$$T_0 = (0^{21}, 4, 4, 0), \quad T_\infty = (1^{21}, 1, 1, -3).$$

Note that $\nu_{\mathcal{T}}(T_0, T_\infty) = 0$. The set of all $T \in \mathcal{T}_{AB}$ satisfying

$$\langle T_0, T \rangle = 0, \quad \langle T_\infty, T \rangle = 1$$

is equal to the set of the points

$$T_i = (0^{20}(-4), 0, 0, -4),$$

where -4 is located at the ith position for $i = 1, \dots, 21$. By $T_i \mapsto i$, we identify this set with $M' = \{1, \dots, 21\}$. A point $T \in \mathcal{T}_{AB}$ satisfies $\langle T, T_0 \rangle = 2$ if and only if T is of type 2 above, and when this is the case, the codeword

$$F_T := \{\ T_i \in M' \ \mid \ \langle T, T_i \rangle \in \{0, 2\}\ \} \ \in \ 2^{M'}.$$

is equal to the set of positions j such that the jth component of T is ± 2. Hence, by the definition of $\mathcal{C}_{21}$, the binary code generated by these F_T is equal to $\mathcal{C}_{21}$.

By Proposition 2.1, there exist exactly 120960 bijections $\varphi\colon S \xrightarrow{\sim} M'$ that induce $\mathcal{H}_X \cong \mathcal{C}_{21}$. We choose the following isomorphism φ_0. As noted above, we identify $S = \{\Pi_1, \dots, \Pi_{21}\}$ with $\Pi_0^\vee(\mathbb{F}_4)$. We use $(x_4 : x_5 : x_6)$ as the homogeneous coordinates of Π_0, and let $[\xi_4 : \xi_5 : \xi_6]$ be the homogeneous coordinates of $\Pi_0^\vee$ dual to $(x_4 : x_5 : x_6)$. Then the map $\varphi_0\colon S = \Pi_0^\vee(\mathbb{F}_4) \xrightarrow{\sim} M'$ given by the diagram below induces $\mathcal{H}_X \cong \mathcal{C}_{21}$.

$[0:1:0]$	$[1:0:0]$	$[1:\bar\omega:0]$	$[1:\omega:0]$	$[1:1:0]$	$[0:0:1]$
$[0:1:1]$	$[1:0:1]$	$[1:\bar\omega:\omega]$	$[1:\omega:\bar\omega]$	$[1:1:1]$	
$[0:1:\omega]$	$[1:0:\omega]$	$[1:\bar\omega:\bar\omega]$	$[1:\omega:1]$	$[1:1:\omega]$	
$[0:1:\bar\omega]$	$[1:0:\bar\omega]$	$[1:\bar\omega:1]$	$[1:\omega:\omega]$	$[1:1:\bar\omega]$	

We define ϕ_0 on the subset $\{\Pi_0, \Pi_\infty\} \cup S$ of $\mathcal{P}_X$ by $\phi_0(\Pi_0) = T_0$, $\phi(\Pi_\infty) = T_\infty$, and $\phi_0|S = \varphi_0$. Then it is a matter of simple calculation to show that, for each plane $\Pi \in \mathcal{P}_X \setminus (\{\Pi_0, \Pi_\infty\} \cup S)$, there exists a unique point $\phi_0(\Pi) \in \mathcal{T}_{AB}$ that satisfies $\nu_{\mathcal{P}}(\Pi, \Pi') = \nu_{\mathcal{T}}(\phi_0(\Pi), \phi_0(\Pi'))$ for all $\Pi' \in \{\Pi_0, \Pi_\infty\} \cup S$, and that the map $\phi_0\colon \mathcal{P}_X \to \mathcal{T}_{AB}$ thus constructed is an Edge correspondence.

We describe the vector representation $(x_1, \dots, x_{24}) \in \mathbb{Z}^M$ of $\phi_0(\Pi) \in \mathcal{T}_{AB}$ for each $\Pi \in \mathcal{P}_X$ in the table below, where $\delta_0 = \dim(\Pi \cap \Pi_0)$, $\delta_\infty = \dim(\Pi \cap \Pi_\infty)$, and, for example, the entry $280 : (-1)^{12}1^9$ means that there exist exactly 280 planes $\Pi \in \mathcal{P}_X$ with $\delta_0 = \delta_\infty = -1$, and that they are mapped to the lattice points of the form $((-1)^{12}1^9, 1, 1, -3)$.

$\delta_\infty \backslash \delta_0$	-1	0	1	2
-1	$280 : (-1)^{12}1^9$	$210 : 0^{16}(-2)^3 2^2$	$21 : 0^{20}(-4)$	$1 : 0^{21}$
0	$210 : (-1)^8 1^{13}$	$105 : 0^{16}(-2)2^4$	$21 : 0^{20}4$	
1	$21 : (-1)^{16}1^5$	$21 : 0^{16}(-2)^5$		
2	$1 : 1^{21}$			
(x_{22}, x_{23}, x_{24})	$(1, 1, -3)$	$(2, 2, -2)$	$(0, 0, -4)$	$(4, 4, 0)$

Remark 2.2. The above construction of ϕ_0 involves seemingly *ad hoc* choices of $\Pi_0, \Pi_\infty, T_0, T_\infty$, and the code isomorphism φ_0. In Remark 3.4, we show that these choices can be made arbitrarily for the construction of an Edge correspondence.

3. The representation of $\mathrm{P\Gamma U}(6, 4)$ in the Leech lattice

3.1. **Proof of Lemma 1.4.** We recall the notion of *discriminant forms* due to Nikulin [5]. Let L be an even lattice; that is, $\langle v, v\rangle \in 2\mathbb{Z}$ holds for all $v \in L$. Then L is naturally embedded into the dual lattice $L^\vee := \{x \in L \otimes \mathbb{Q} \mid \langle x, v\rangle \in \mathbb{Z} \text{ for all } v \in L\}$ as a submodule of finite index. We call $D_L := L^\vee/L$ the *discriminant group* of L. By $q_L(x \bmod L) := \langle x, x\rangle \bmod 2\mathbb{Z}$, we obtain a finite quadratic form

$$q_L \,:\, D_L \to \mathbb{Q}/2\mathbb{Z},$$

which is called the *discriminant form* of L. In the following, we denote by $\mathrm{O}(q_L)$ the automorphism group of the finite quadratic form $q_L\colon D_L \to \mathbb{Q}/2\mathbb{Z}$, which we let act on q_L from the right, and by

$$\eta\colon \mathrm{O}(L) \to \mathrm{O}(q_L)$$

the natural homomorphism.

It is obvious that $\Lambda'_{AB} = \langle TT' \mid T, T' \in \mathcal{T}_{AB} \rangle$ is contained in $R^{\perp}_{AB}$. We can calculate that the orders of the discriminant groups of Λ'_{AB} and of $R^{\perp}_{AB}$ are both 12. Hence $\Lambda'_{AB} = R^{\perp}_{AB}$ holds. By the choice of A and B in (2.1), the sublattice $\Lambda'_{AB} = R^{\perp}_{AB}$ is the section of Λ_{24} by the linear subspace of $\Lambda_{24} \otimes \mathbb{R}$ defined by $x_{22} = x_{23} = x_{24}$. By the definition of Λ_{22} in Figure 6.2 of [1], we obtain $\Lambda'_{AB} = \Lambda_{22}$. (This fact has been already proved in [8].)

The orthogonal group $\mathrm{O}(R_{AB})$ is isomorphic to the dihedral group of order 12. Indeed, there exist exactly 6 vectors of square norm 4 in R_{AB}, and they form a regular hexagon. By direct calculation, we see the following:

(3.1) The natural homomorphism $\eta\colon \mathrm{O}(R_{AB}) \to \mathrm{O}(q_{R_{AB}})$ is an isomorphism.

By Nikulin [5], the even unimodular overlattice Λ_{24} of the orthogonal direct sum $R^{\perp}_{AB} \oplus R_{AB}$ defines an anti-isometry

$$\zeta_\Lambda \,:\, q_{R^{\perp}_{AB}} \xrightarrow{\sim} -q_{R_{AB}},$$

and that $\circ 222_{AB}$ is identified with

$$\{\ (g_1, g_2) \in \mathrm{O}(R^{\perp}_{AB}) \times \mathrm{O}(R_{AB}) \ \mid\ \eta(g_1) \circ \zeta_\Lambda = \zeta_\Lambda \circ \eta(g_2)\ \}.$$

By (3.1), we see that the first projection $\sigma_{AB}\colon \circ 222_{AB} \to \mathrm{O}(R^{\perp}_{AB})$ is an isomorphism. Thus Lemma 1.4 is proved.

Remark 3.1. The proof above indicates how to calculate $\sigma^{-1}_{AB}\colon \mathrm{O}(R^{\perp}_{AB}) \to \circ 222_{AB}$. Let $g_1 \in \mathrm{O}(R^{\perp}_{AB})$ be given. We calculate $\eta(g_1) \in \mathrm{O}(q_{R^{\perp}_{AB}})$, and transplant $\eta(g_1)$ to $\eta(g_2) \in \mathrm{O}(q_{R_{AB}})$ via ζ_Λ. Then there exists a unique isometry $g_2 \in \mathrm{O}(R_{AB})$ that induces $\eta(g_2)$. The action of (g_1, g_2) on $R^{\perp}_{AB} \oplus R_{AB}$ preserves the overlattice Λ_{24}, and hence induces $g \in \mathrm{O}(\Lambda_{24})$, which belongs to $\circ 222_{AB}$ by definition.

3.2. **Computation of $\mathrm{P\Gamma U}(6,4) \to \circ 222_{AB}$.** For an Edge correspondence ϕ, we denote by

$$\Psi_\phi \,:\, \mathrm{P\Gamma U}(6,4) \to \circ 222_{AB}$$

the composite homomorphism in Theorem 1.5. We present $\Psi_0 := \Psi_{\phi_0}$ in terms of matrices for the Edge correspondence ϕ_0 constructed in Section 2.4.

We let $\mathrm{PGU}(6,4)$ act on $\mathbb{P}^5$ from the right. Taylor [12] gave a generating set $\{\alpha', \beta'\}$ of the group

$$\mathrm{GU}(6,4)' := \{\ g \in \mathrm{GL}(6,4) \ \mid\ g \cdot J_6 \cdot {}^{\mathrm{T}}\bar{g} = J_6\ \},$$

where J_6 denotes the Hermitian form $x_1\bar{x}_6 + x_2\bar{x}_5 + \cdots + x_6\bar{x}_1$ on $\mathbb{P}^5$ defined over $\mathbb{F}_2$. From this result, we see that $\mathrm{GU}(6,4)$ is generated by the following two elements of order 3 and 10, respectively.

$$\alpha := \begin{bmatrix} \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{bmatrix}, \quad \beta := \begin{bmatrix} \bar{\omega} & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \omega & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \bar{\omega} & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \omega \end{bmatrix}.$$

Let $\gamma \in \mathrm{P\Gamma U}(6,4)$ denote the Frobenius action of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Then $\mathrm{P\Gamma U}(6,4)$ is generated by α, β, γ. By direct calculation, we obtain the following:

Theorem 3.2. *Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the three matrices in Figure 3.1 . Then the homomorphism Ψ_0 is given by $\alpha \mapsto \tilde{\alpha}, \beta \mapsto \tilde{\beta}, \gamma \mapsto \tilde{\gamma}$.*

$$
\frac{1}{4}\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1
\end{bmatrix}
$$

$$
\frac{1}{8}\begin{bmatrix}
1 & 1 & 3 & -1 & -1 & 1 & 1 & 3 & -1 & 3 & -1 & -1 & 3 & 1 & -1 & 1 & 1 & 3 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 3 & -1 & -1 & 1 & -3 & -1 & -1 & -1 & 3 & -1 & -1 & 1 & 3 & 1 & -3 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
3 & 3 & -3 & -3 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -3 & 1 & 1 & 1 \\
-1 & -1 & -3 & 1 & -3 & -1 & -1 & 1 & 1 & 1 & 1 & -3 & 1 & 3 & 1 & -1 & -1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -3 & 1 & 3 & -1 & 1 & 1 & 1 & -3 & 1 & -3 & -1 & 1 & -1 & -1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 3 & -3 & -3 & 3 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 3 & -1 & -1 & -1 \\
-1 & 3 & 1 & 1 & 1 & 3 & -1 & 1 & 1 & -3 & 1 & 1 & 1 & 3 & 1 & -1 & 3 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
-3 & 1 & -1 & -1 & -1 & -3 & 1 & -1 & -1 & -1 & -1 & 3 & -1 & 1 & 3 & 1 & 1 & 3 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -3 & 3 & -3 & 3 & -1 & 1 & -1 & 1 & -3 & 1 & 1 & 1 \\
-3 & 1 & -1 & -1 & -1 & 1 & -3 & -1 & -1 & 3 & -1 & -1 & -1 & 1 & -1 & -3 & 1 & -1 & -3 & -1 & -1 & -1 & -1 & -1 \\
1 & -3 & -1 & -1 & 3 & 1 & 1 & -1 & -1 & -1 & 3 & -1 & -1 & 1 & -1 & -3 & 1 & 3 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -3 & 1 & -1 & -1 & -3 & 1 & 1 & 1 & -3 & 1 & -1 & 1 & 3 & 3 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
3 & -1 & 1 & 1 & -3 & -1 & -1 & 1 & -3 & 1 & 1 & 1 & -3 & -1 & 1 & -1 & 3 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -3 & 1 & -1 & 3 & 1 & -3 & 1 & 1 & 1 & 1 & 3 & 1 & -1 & -1 & -3 & -1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 3 & 1 & 1 & 1 & -1 & -1 & -3 & -3 & 1 & 1 & 1 & 1 & -1 & -3 & -1 & -1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -3 & -3 & -3 & -3 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -3 & 1 & 1 & 1 \\
1 & -3 & -1 & -1 & -1 & 1 & -3 & -1 & -1 & -1 & -1 & 3 & 3 & 1 & -1 & 1 & 1 & -1 & 1 & 3 & -1 & -1 & -1 & -1 \\
3 & -1 & 1 & 1 & 1 & -1 & -1 & -3 & 1 & 1 & -3 & 1 & 1 & 3 & 1 & -1 & -1 & 1 & -1 & -3 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -3 & -3 & -1 & -1 & 1 & 1 & -3 & 1 & 1 & 1 & -1 & -3 & -1 & -1 & 1 & -1 & -3 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -3 & 3 & -3 & 3 & 3 & -1 & -1 & -1 \\
-1 & -1 & -3 & 1 & 1 & 3 & -1 & 1 & -3 & 1 & 1 & 1 & 1 & -1 & 1 & 3 & -1 & 1 & -1 & -3 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 5 & -3 & -3 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -3 & 5 & -3 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -3 & -3 & 5
\end{bmatrix}
$$

$$
\frac{1}{8}\begin{bmatrix}
1 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\
1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\
-3 & -3 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
-3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\
-3 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\
-3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & -3 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & -3 & -3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 5 & -3
\end{bmatrix}
$$

FIGURE 3.1. $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$

3.3. **Proof of Corollary 1.3.** Let $\mathcal{L}_X$ be the sublattice of $\mathcal{N}_X$ generated by the numerical equivalence classes of the planes $\Pi \in \mathcal{P}_X$. Theorem 1.2 implies that $\mathcal{L}_X$ is isomorphic to the lattice generated by the vectors $CT \in \Lambda_{24} \otimes \mathbb{Q}$, where C is the mid-point of an edge of OAB, and T runs through $\mathcal{T}_{AB}$. We show that $\mathcal{L}_X = \mathcal{N}_X$. A direct calculation shows that the discriminant group $D_{\mathcal{L}_X}$ of $\mathcal{L}_X$ is isomorphic to $\mathbb{F}_2^2$, and that the discriminant form $q_{\mathcal{L}_X} \colon D_{\mathcal{L}_X} \to \mathbb{Q}/\mathbb{Z}$ is given by

$$\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Note that $\mathrm{PGU}(6,4)$ acts on $\mathcal{L}_X \hookrightarrow \mathcal{N}_X$ equivariantly. If $\mathcal{L}_X \neq \mathcal{N}_X$, then $\mathcal{N}_X/\mathcal{L}_X$ would be a non-zero $\mathrm{PGU}(6,4)$-invariant isotropic subspace of $q_{\mathcal{L}_X}$. However, by looking at the action of $\alpha \in \mathrm{PGU}(6,4)$ on $D_{\mathcal{L}_X}$, we conclude that there exists no such isotropic subspace.

3.4. **Proof of Theorem 1.5 and Corollaries 1.6 and 1.7.** By Lemma 1.4, the homomorphism $\tau_{\mathcal{T}}$ is regarded as a homomorphism to $\mathrm{O}(R_{AB}^{\perp})$.

Lemma 3.3. *The natural homomorphism* $\tau_{\mathcal{T}} \colon \mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}}) \to \mathrm{O}(R_{AB}^{\perp})$ *is injective, and the image does not contain* $-I_{22}$.

Proof. Let $\mathcal{S}_0$ be the set of pairs $[T, T']$ with $T, T' \in \mathcal{T}_{AB}$ and $\langle T, T' \rangle = 0$. We have $|\mathcal{S}_0| = 891 \cdot 42$. Let $\mathrm{pr}_{R^{\perp}} : \Lambda_{24} \otimes \mathbb{Q} \to R_{AB}^{\perp} \otimes \mathbb{Q}$ be the orthogonal projection. We consider the map

$$s_0 \,:\, [T, T'] \mapsto \mathrm{pr}_{R^{\perp}}(TT')$$

from $\mathcal{S}_0$ to $R_{AB}^{\perp}$. By direct calculation, we see that s_0 is injective, and its image spans $R_{AB}^{\perp} \otimes \mathbb{Q}$. Hence an automorphism of $(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ belonging to the kernel of $\tau_{\mathcal{T}}$ acts on $\mathcal{S}_0$ trivially. Therefore $\tau_{\mathcal{T}}$ is injective. If $-I_{22}$ were in the image of $\tau_{\mathcal{T}}$, then there would exist an automorphism of $(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ that acts on $\mathcal{S}_0$ as $[T, T'] \mapsto [T', T]$, which is absurd. □

Proof of Theorem 1.5 and Corollaries 1.6 and 1.7. First we prove the assertion of Theorem 1.5 for Ψ_0; that is, we are in the case where $\phi = \phi_0$. Since $\lambda^6 = 1$ for all $\lambda \in \mathbb{F}_4^{\times}$, it follows that $\det \colon \mathrm{GL}(6,4) \to \mathbb{F}_4^{\times}$ factors as $\mathrm{GL}(6,4) \to \mathrm{PGL}(6,4) \to \mathbb{F}_4^{\times}$. Hence we have a homomorphism

$$\det{}' \,:\, \mathrm{P\Gamma U}(6,4) = \mathrm{PGU}(6,4) \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \to \mathbb{F}_4^{\times} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2),$$

whose kernel is $\mathrm{PSU}(6,4)$. We consider the composite homomorphism

$$\psi_0 \,:\, \mathrm{P\Gamma U}(6,4) \underset{\Psi_0}{\longrightarrow} \circ 222_{AB} \longrightarrow \mathrm{O}(R_{AB}),$$

where $\circ 222_{AB} \to \mathrm{O}(R_{AB})$ is the natural homomorphism. Note that the kernel of $\circ 222_{AB} \to \mathrm{O}(R_{AB})$ is $\cdot 222_{AB}$. We denote elements of $\mathrm{O}(R_{AB})$, which acts on R_{AB} from the right, by the matrices with respect to the basis OA and OB. Using the three matrices $\Psi_0(\alpha) = \tilde{\alpha}, \Psi_0(\beta) = \tilde{\beta}, \Psi_0(\gamma) = \tilde{\gamma}$ and the algorithm in Remark 3.1, we obtain the following:

	α	β	γ
$\det'$	$\bar{\omega}$	1	Frobenius
ψ_0	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

Since $\det'(\beta) = 1$ and $\Psi_0(\beta) = \tilde{\beta} \neq 1$, we see that $\mathrm{PSU}(6,4) \not\subset \operatorname{Ker}\Psi_0$. Since $\mathrm{PSU}(6,4)$ is simple, we have $\mathrm{PSU}(6,4) \cap \operatorname{Ker}\Psi_0 = 1$. Hence $\det'$ embeds $\operatorname{Ker}\Psi_0$ into $\mathbb{F}_4^\times \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. The table above also shows that the mapping

$$\bar{\omega} \mapsto \psi_0(\alpha), \quad 1 \mapsto \psi_0(\beta), \quad \text{Frobenius} \mapsto \psi_0(\gamma)$$

induces an injective homomorphism

$$i : \mathbb{F}_4^\times \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \hookrightarrow \mathrm{O}(R_{AB})$$

such that ψ_0 factors as

$$\mathrm{P\Gamma U}(6,4) \underset{\det'}{\longrightarrow} \mathbb{F}_4^\times \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \underset{i}{\hookrightarrow} \mathrm{O}(R_{AB}).$$

Therefore $\operatorname{Ker}\Psi_0$ is trivial. Plesken and Pohst [6] showed that $|\mathrm{O}(\Lambda_{22})| = 12D$, where D is given in (1.1). By Lemma 1.4, we have $|\circ 222_{AB}| = 12D$. By Lemma 3.3, we see that $\operatorname{Im}\Psi_0$ does not contain -1 and hence $|\operatorname{Im}\Psi_0| \le 6D = |\mathrm{P\Gamma U}(6,4)|$. Therefore $\rho_{\mathcal{P}}$ is an isomorphism, $|\operatorname{Im}\Psi_0| = 6D$, and $\circ 222_{AB}$ is generated by $\operatorname{Im}\Psi_0$ and -1. Moreover Ψ_0 induces an isomorphism from $\mathrm{PSU}(6,4)$ to $\cdot 222_{AB}$. Thus the assertion of Theorem 1.5 for Ψ_0 and Corollaries 1.6 and 1.7 are proved.

Let ϕ be an arbitrary Edge correspondence. Since $\rho_{\mathcal{P}}$ is an isomorphism, the homomorphism Ψ_ϕ differs from Ψ_0 only by an inner automorphism of $\mathrm{P\Gamma U}(6,4)$. Hence Theorem 1.5 holds for ϕ. □

Remark 3.4. We prove the assertion made in Remark 2.2. We put

$$\Xi := \{\, [\Pi, \Pi'] \;\mid\; \Pi, \Pi' \in \mathcal{P}_X,\ \dim(\Pi \cap \Pi') = -1 \,\}.$$

We have $|\Xi| = 891 \cdot 512$. For each $[\Pi, \Pi'] \in \Xi$, we put

$$S_{[\Pi,\Pi']} := \{\, \Pi'' \in \mathcal{P}_X \;\mid\; \dim(\Pi \cap \Pi'') = 1,\ \dim(\Pi' \cap \Pi'') = -1 \,\}$$

Let $\mathcal{H}_{[\Pi,\Pi']} \subset 2^{S_{[\Pi,\Pi']}}$ be the linear code generated by the codewords

$$\ell_P := \{\, \Pi'' \in S_{[\Pi,\Pi']} \;\mid\; P \in \Pi'' \,\},$$

where P runs through $\Pi(\mathbb{F}_4)$. We can easily prove that $\mathrm{P\Gamma U}(6,4)$ acts on Ξ transitively. Let $G_{[\Pi,\Pi']}$ be the stabilizer subgroup of $[\Pi, \Pi'] \in \Xi$ in $\mathrm{P\Gamma U}(6,4)$, whose order is $6D/891/512 = 120960 = |\mathrm{P\Gamma L}(3,4)|$. Then $G_{[\Pi,\Pi']}$ acts on $S_{[\Pi,\Pi']}$ and on $\mathcal{H}_{[\Pi,\Pi']}$. We can also prove by direct calculation that the action of $G_{[\Pi,\Pi']}$ on $S_{[\Pi,\Pi']}$ is faithful (see also [8]). Hence $G_{[\Pi,\Pi']} \to \mathrm{Aut}(\mathcal{H}_{[\Pi,\Pi']}) \cong \mathrm{P\Gamma L}(3,4)$ is an isomorphism. Since $\mathrm{P\Gamma U}(6,4) \cong \mathrm{Aut}(\mathcal{P}_X, \nu_{\mathcal{P}}) \cong \mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$, it follows that $\mathrm{Aut}(\mathcal{T}_{AB}, \nu_{\mathcal{T}})$ acts on the set

$$\phi_0(\Xi) = \{\, [T, T'] \;\mid\; T, T' \in \mathcal{T}_{AB},\ \langle T, T' \rangle = 1 \,\}$$

transitively. Therefore, in the construction of the Edge correspondence ϕ, the choices of $\Pi_0, \Pi_\infty, T_0, T_\infty$, and the code isomorphism φ_0 can be made arbitrarily.

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