THE AUTOMORPHISM GROUP OF A SUPERSINGULAR $K3$ SURFACE WITH ARTIN INVARIANT 1 IN CHARACTERISTIC 3

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Abstract. We present a finite set of generators of the automorphism group of a supersingular $K3$ surface with Artin invariant 1 in characteristic 3.

1. Introduction

To determine the automorphism group $\mathrm{Aut}(Y)$ of a given $K3$ surface Y is an important problem. In this paper, we present a set of generators of the automorphism group of a supersingular $K3$ surface X in characteristic 3 with Artin invariant 1. Our method is computational, and relies heavily on computer-aided calculation. It gives us generators in explicit form, and it can be easily applied to many other $K3$ surfaces by modifying computer programs.

A $K3$ surface defined over an algebraically closed field k is said to be *supersingular* (in the sense of Shioda) if its Picard number is 22. Supersingular $K3$ surfaces exist only when k is of positive characteristic. Let Y be a supersingular $K3$ surface in characteristic $p > 0$, and let S_Y denote its Néron-Severi lattice. Artin [3] showed that the discriminant group of S_Y is a p-elementary abelian group of rank 2σ, where σ is an integer such that $1 \le \sigma \le 10$. This integer σ is called the *Artin invariant* of Y. Ogus [18, 19] proved that a supersingular $K3$ surface with Artin invariant 1 in characteristic p is unique up to isomorphisms (see also [21]).

It is known that the Fermat quartic surface

$$X := \{w^4 + x^4 + y^4 + z^4 = 0\} \subset \mathbb{P}^3$$

defined over an algebraically closed field k of characteristic 3 is a supersingular $K3$ surface with Artin invariant 1 (see [33]). Let

$$h_0 := [\mathcal{O}_X(1)] \in S_X$$

denote the class of the hyperplane section of X. The projective automorphism group $\mathrm{Aut}(X, h_0)$ of $X \subset \mathbb{P}^3$ is equal to the finite subgroup $\mathrm{PGU}_4(\mathbb{F}_9)$ of $\mathrm{PGL}_4(k)$ with order $13,063,680$.

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Let (w, x, y) be the affine coordinates of $\mathbb{P}^3$ with $z = 1$, and let F_{1j} and F_{2j} be polynomials of (w, x, y) with coefficients in

$$\mathbb{F}_9 = \mathbb{F}_3(i) = \{0, \pm 1, \pm i, \pm(1+i), \pm(1-i)\}, \quad \text{where } i := \sqrt{-1},$$

given in Table 1.1.

Proposition 1.1. *For $i = 1$ and 2, the rational map*

$$(w, x, y) \mapsto [F_{i0} : F_{i1} : F_{i2}] \in \mathbb{P}^2$$

induces a morphism $\phi_i : X \to \mathbb{P}^2$ of degree 2.

We denote by

$$X \xrightarrow{\psi_i} Y_i \xrightarrow{\pi_i} \mathbb{P}^2$$

the Stein factorization of $\phi_i : X \to \mathbb{P}^2$, and let $B_i \subset \mathbb{P}^2$ be the branch curve of the finite morphism $\pi_i : Y_i \to \mathbb{P}^2$ of degree 2. Note that Y_i is a normal $K3$ surface, and hence Y_i has only rational double points as its singularities (see [1, 2]). Let $[x_0 : x_1 : x_2]$ be the homogeneous coordinates of $\mathbb{P}^2$.

Proposition 1.2. (1) *The ADE-type of the singularities of Y_1 is $6A_1 + 4A_2$. The branch curve B_1 is defined by $f_1 = 0$, where*

$$\begin{aligned} f_1 := {} & {x_0}^6 + {x_0}^5 x_1 - {x_0}^3 {x_1}^3 - x_0 {x_1}^5 - {x_0}^4 {x_2}^2 \\ & + x_0 {x_1}^3 {x_2}^2 + {x_1}^4 {x_2}^2 + {x_0}^2 {x_2}^4 + {x_1}^2 {x_2}^4 + {x_2}^6. \end{aligned}$$

(2) *The ADE-type of the singularities of Y_2 is $A_1 + A_2 + 2A_3 + 2A_4$. The branch curve B_2 is defined by $f_2 = 0$, where*

$$\begin{aligned} f_2 := {} & {x_0}^5 x_1 + {x_0}^2 {x_1}^4 - {x_0}^4 {x_2}^2 + x_0 {x_1}^3 {x_2}^2 \\ & + {x_1}^4 {x_2}^2 - {x_0}^2 {x_2}^4 - x_0 x_1 {x_2}^4 - {x_1}^2 {x_2}^4 - {x_2}^6. \end{aligned}$$

Our main result is as follows:

Theorem 1.3. *Let $g_i \in \mathrm{Aut}(X)$ denote the involution induced from the deck-transformation of $\pi_i : Y_i \to \mathbb{P}^2$. Then $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ and g_1, g_2.*

See Theorem 7.1 for a more explicit description of the involutions g_1 and g_2.

Let $\mathcal{P}_{S_X}$ denote the connected component of $\{x \in S_X \otimes \mathbb{R} \mid x^2 > 0\}$ that contains h_0. Following Borcherds [4], we prove Theorem 1.3 by calculating a closed chamber D_{S0} in the cone $\mathcal{P}_{S_X}$ with the following properties (see Section 6):

(1) The chamber D_{S0} is invariant under the action of $\mathrm{Aut}(X, h_0)$.
(2) For any nef class $v \in S_X$, there exists $\gamma \in \mathrm{Aut}(X)$ such that $v^\gamma \in D_{S0}$.
(3) For nef classes v, v' in the interior of D_{S0}, there exists $\gamma \in \mathrm{Aut}(X)$ such that $v' = v^\gamma$ if and only if there exists $\tau \in \mathrm{Aut}(X, h_0)$ such that $v' = v^\tau$.

$$
\begin{aligned}
F_{10} = {} & (1+i) + (1+i)\,w + (1-i)\,x - y - (1-i)\,wx - x^2 + i\,wy \\
& + i\,xy - i\,y^2 + (1+i)\,w^3 - i\,w^2x + (1+i)\,wx^2 - i\,x^3 + w^2y \\
& + (1+i)\,wxy + (1+i)\,x^2y - (1-i)\,wy^2 - (1+i)\,xy^2 + i\,y^3 \\
F_{11} = {} & (1-i) - (1+i)\,x - (1-i)\,y - (1-i)\,w^2 - (1-i)\,wx - (1-i)\,x^2 \\
& - (1+i)\,wy - xy - (1+i)\,y^2 - w^3 + (1-i)\,w^2x + wx^2 - i\,x^3 \\
& - (1+i)\,w^2y - (1+i)\,wxy + x^2y - i\,wy^2 - xy^2 + (1-i)\,y^3 \\
F_{12} = {} & (1+i)\,w - i\,x - y - w^2 - wx - i\,x^2 - i\,xy + i\,y^2 + i\,w^3 \\
& - (1+i)\,wx^2 + i\,x^3 - i\,w^2y - wxy + (1-i)\,wy^2 + (1+i)\,y^3
\end{aligned}
$$

$$
\begin{aligned}
F_{20} = {} & -1 - i\,w + (1+i)\,x - y - (1+i)\,w^2 - wx - (1-i)\,x^2 - i\,wy + (1+i)\,xy \\
& - (1-i)\,w^3 + w^2x - wx^2 + x^3 - w^2y + (1-i)\,wxy + x^2y + (1-i)\,wy^2 \\
& + (1-i)\,xy^2 + (1+i)\,y^3 - w^3x - i\,w^2x^2 - wx^3 + w^3y - (1+i)\,w^2xy \\
& - (1-i)\,wxy^2 + x^2y^2 - (1-i)\,wy^3 - (1+i)\,xy^3 - y^4 + (1-i)\,w^3x^2 - i\,x^5 \\
& + (1-i)\,w^3xy + (1+i)\,wx^3y - i\,w^3y^2 + (1+i)\,w^2xy^2 - (1+i)\,wx^2y^2 \\
& + i\,x^3y^2 - w^2y^3 - (1+i)\,wxy^3 - (1-i)\,x^2y^3 + i\,wy^4 + (1-i)\,xy^4 + (1+i)\,y^5 \\
F_{21} = {} & -(1-i) + i\,w + (1-i)\,y - (1+i)\,w^2 + wx + (1+i)\,x^2 + (1+i)\,wy - (1+i)\,xy \\
& - i\,y^2 - w^3 + i\,w^2x + (1+i)\,wx^2 - x^3 - (1+i)\,w^2y - (1-i)\,wxy - (1-i)\,x^2y \\
& - i\,wy^2 - (1+i)\,xy^2 + y^3 - (1-i)\,w^3x - wx^3 + (1-i)\,x^4 + (1-i)\,w^3y + i\,w^2xy \\
& + (1-i)\,wx^2y - i\,x^3y + (1-i)\,w^2y^2 + (1-i)\,wxy^2 - (1+i)\,x^2y^2 + (1-i)\,wy^3 \\
& - i\,xy^3 + i\,y^4 + w^3x^2 + w^2x^3 + (1-i)\,wx^4 - i\,x^5 - i\,w^3xy + w^2x^2y + (1+i)\,wx^3y \\
& + x^4y + w^3y^2 - w^2xy^2 - wx^2y^2 + i\,w^2y^3 + (1+i)\,wxy^3 - i\,wy^4 - i\,xy^4 + y^5 \\
F_{22} = {} & (1-i) - (1+i)\,w - (1+i)\,x - (1-i)\,y + i\,w^2 - (1+i)\,wx - (1-i)\,x^2 + i\,wy \\
& - (1+i)\,xy - w^3 - i\,w^2x - wx^2 + x^3 - (1-i)\,w^2y + wxy + x^2y + (1+i)\,wy^2 \\
& - (1+i)\,xy^2 - y^3 + i\,w^3x - (1-i)\,w^2x^2 - wx^3 - (1+i)\,x^4 + i\,w^3y + w^2xy \\
& + (1-i)\,wx^2y - (1-i)\,w^2y^2 + (1+i)\,wxy^2 + i\,wy^3 + xy^3 + (1-i)\,y^4 - i\,w^3x^2 \\
& - (1+i)\,wx^4 + x^5 - (1-i)\,w^3xy - i\,w^2x^2y + (1+i)\,wx^3y + (1-i)\,x^4y - w^3y^2 \\
& - (1+i)\,w^2xy^2 + i\,wx^2y^2 + i\,x^3y^2 - wxy^3 - (1-i)\,x^2y^3 - wy^4 - xy^4 - y^5
\end{aligned}
$$

TABLE 1.1. Polynomials F_{1j} and F_{2j}

This chamber D_{S0} is bounded by $112+648+5184$ hyperplanes in $\mathcal{P}_{S_X}$. See Proposition 4.5 for the explicit description of these walls. Using D_{S0} and these walls, we can also present a finite set of generators of $\mathrm{O}^+(S_X)$ (see Theorem 8.2).

Vinberg [35] determined the automorphism groups of two complex $K3$ surfaces with Picard number 20 by investigating the orthogonal groups of their Néron-Severi lattices and the associated hyperbolic geometry.

Let L denote an even unimodular lattice of rank 26 with signature $(1, 25)$, which is unique up to isomorphisms by Eichler's theorem. Conway [6] determined the fundamental domain in a positive cone of $L\otimes\mathbb{R}$ under the action of the subgroup of

$\mathrm{O}^+(L)$ generated by the reflections with respect to the vectors of square norm -2. Borcherds [4] applied Conway theory to the investigation of the orthogonal groups of even hyperbolic lattices S primitively embedded in L. Then the first author [15] determined the automorphism group of a generic Jacobian Kummer surface by embedding its Néron-Severi lattice into L and using Conway theory. Keum and the first author [14] applied this method to the Kummer surface of the product of two elliptic curves, Dolgachev and Keum [11] applied it to quartic Hessian surfaces, and Dolgachev and the first author [10] applied it to the supersingular $K3$ surface in characteristic 2 with Artin invariant 1.

Recently, configurations of smooth rational curves on our supersingular $K3$ surface X was studied in [13] with respect to an embedding of S_X into L, and elliptic fibrations on X was classified in [25] by embedding S_X into L.

The new idea introduced in this paper is that, in order to find automorphisms of X necessary to generate $\mathrm{Aut}(X)$, we search for polarizations of degree 2 whose classes are located on the walls of the chamber decomposition of the cone $\mathcal{P}_{S_X}$. The computational tools used in this paper have been developed by the second author for the study [30] of various double plane models of a supersingular $K3$ surface in characteristic 5 with Artin invariant 1. The computational data for this paper is available from the second author's webpage [31].

In [27] and [29], the second author showed that every supersingular $K3$ surface in any characteristic with arbitrary Artin invariant is birational to a double cover of the projective plane. In [28], [32] and [20, 30], projective models of supersingular $K3$ surfaces in characteristic 2, 3 and 5 were investigated, respectively.

This paper is organized as follows. In Section 2, we give a review of the theory of Conway and Borcherds, and investigate chamber decomposition induced on a positive cone of a primitive hyperbolic sublattice S of L. In Section 3, we give explicitly a basis of the Néron-Severi lattice S_X of X, and describe a method to compute the action of $\mathrm{Aut}(X, h_0)$ on S_X. The fact that S_X is generated by the classes of lines in X enables us to calculate projective models of X explicitly. In Section 4, we embed S_X into L, and study the obtained chamber decomposition in detail. In particular, we investigate the walls of the chamber D_{S0} that contains the class h_0. In Section 5, we prove Propositions 1.1 and 1.2, and show that the involutions g_1 and g_2 map h_0 to its mirror images into walls of the chamber D_{S0}. Then we can prove Theorem 1.3 in Section 6. In Section 7, we give another description of the involutions g_i. In the last section, we give a set of generators of $\mathrm{O}^+(S_X)$.

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2. Leech roots

2.1. **Terminologies and notation.** We fix some terminologies and notation about lattices. A *lattice* M is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form

$$(\ , \ )_M : M \times M \to \mathbb{Z}.$$

A submodule N of M is said to be *primitive* if M/N is torsion free. For a submodule N of M, we denote by $N^\perp \subset M$ the submodule defined by

$$N^\perp := \{ \, u \in M \ \mid \ (u, v)_M = 0 \ \text{ for all } \ v \in N \, \},$$

which is primitive by definition. We denote by $\mathrm{O}(M)$ the orthogonal group of M. Throughout this paper, we let $\mathrm{O}(M)$ act on M from *right*. Suppose that M is of rank r. We say that M is *hyperbolic* (resp. *negative-definite*) if the signature of the symmetric bilinear form $(\ , \ )_M$ on $M \otimes \mathbb{R}$ is $(1, r-1)$ (resp. $(0, r)$). We define the *dual lattice* $M^\vee$ of M by

$$M^\vee := \{ \, u \in M \otimes \mathbb{Q} \ \mid \ (u, v)_M \in \mathbb{Z} \ \text{ for all } \ v \in M \, \}.$$

Then M is contained in $M^\vee$ as a submodule of finite index. The finite abelian group $M^\vee/M$ is called the *discriminant group* of M. We say that M is *unimodular* if $M = M^\vee$.

A lattice M is said to be *even* if $(v, v)_M \in 2\mathbb{Z}$ holds for any $v \in M$. The discriminant group $M^\vee/M$ of an even lattice M is naturally equipped with the quadratic form

$$q_M : M^\vee/M \to \mathbb{Q}/2\mathbb{Z}$$

defined by $q_M(u \bmod M) := (u, u)_M \bmod 2\mathbb{Z}$. We call q_M the *discriminant form* of M. The automorphism group of q_M is denoted by $\mathrm{O}(q_M)$. There exists a natural homomorphism $\mathrm{O}(M) \to \mathrm{O}(q_M)$.

Suppose that M is hyperbolic. Then the open subset

$$\{ \, x \in M \otimes \mathbb{R} \ \mid \ (x, x)_M > 0 \, \}$$

of $M \otimes \mathbb{R}$ has two connected components. A *positive cone* of M is one of them. We fix a positive cone $\mathcal{P}$. The *autochronous orthogonal group* $\mathrm{O}^+(M)$ of M is the group of isometries of M that preserve $\mathcal{P}$. Then $\mathrm{O}^+(M)$ is a subgroup of $\mathrm{O}(M)$ with index 2. Note that $\mathrm{O}^+(M)$ acts on $\mathcal{P}$. For a nonzero vector $u \in M \otimes \mathbb{R}$, we denote by $(u)^\perp_M$ the hyperplane of $M \otimes \mathbb{R}$ defined by

$$(u)^\perp_M := \{ \, x \in M \otimes \mathbb{R} \ \mid \ (x, u)_M = 0 \, \}.$$

Let $\mathcal{R}$ be a set of non-zero vectors of $M \otimes \mathbb{R}$, and let

$$\mathcal{H} := \{ \, (u)^\perp_M \ \mid \ u \in \mathcal{R} \, \}$$

be the family of hyperplanes defined by $\mathcal{R}$. Suppose that $\mathcal{H}$ is locally finite in $\mathcal{P}$. Then the closure in $\mathcal{P}$ of each connected component of

$$\mathcal{P} \setminus \left(\mathcal{P} \cap \bigcup_{u \in \mathcal{R}} (u)^{\perp}_{M} \right)$$

is called an $\mathcal{R}$-*chamber*. Let D be an $\mathcal{R}$-chamber. We denote by D° the interior of D. We say that a hyperplane $(u)^{\perp}_{M} \in \mathcal{H}$ *bounds* D, or that $(u)^{\perp}_{M}$ is a *wall* of D, if $(u)^{\perp}_{M} \cap D$ contains a non-empty open subset of $(u)^{\perp}_{M}$. We denote the set of walls of D by

$$\mathcal{W}(D) := \{ \, (u)^{\perp}_{M} \in \mathcal{H} \;\mid\; (u)^{\perp}_{M} \text{ bounds } D \, \}.$$

Suppose that $\mathcal{R}$ is invariant under $u \mapsto -u$. We choose a point $p \in D^{\circ}$, and put

$$\widetilde{\mathcal{W}}(D) := \{ \, u \in \mathcal{R} \;\mid\; (u)^{\perp}_{M} \text{ bounds } D \text{ and } (u, p)_M > 0 \, \},$$

which is independent of the choice of p. It is obvious that D is equal to

$$\{ \, x \in \mathcal{P} \;\mid\; (x, u)_M \geq 0 \;\text{ for all }\; u \in \widetilde{\mathcal{W}}(D) \, \}.$$

2.2. **Conway theory.** We review the theory of Conway [6]. Let L be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphisms by Eichler's theorem (see, for example, [5, Chapter 11, Theorem 1.4]). We choose and fix a positive cone $\mathcal{P}_L$ once and for all. A vector $r \in L$ is called a *root* if the reflection $s_r : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ defined by

$$x \mapsto x - \frac{2(x, r)_L}{(r, r)_L} \cdot r$$

preserves L and $\mathcal{P}_L$, or equivalently, if $(r, r)_L = -2$. We denote by $\mathcal{R}_L$ the set of roots of L, which is invariant under $r \mapsto -r$. Let $W(L)$ denote the subgroup of $\mathrm{O}^{+}(L)$ generated by the reflections s_r associated with all the roots $r \in \mathcal{R}_L$. Then $W(L)$ is a normal subgroup of $\mathrm{O}^{+}(L)$. The family of hyperplanes

$$\mathcal{H}_L := \{ \, (r)^{\perp}_{L} \;\mid\; r \in \mathcal{R}_L \, \}$$

is locally finite in $\mathcal{P}_L$. Hence we can consider $\mathcal{R}_L$-chambers. By definition, each $\mathcal{R}_L$-chamber is a fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

A non-zero primitive vector $w \in L$ is called a *Weyl vector* if $(w, w)_L = 0$, w is contained in the closure of $\mathcal{P}_L$ in $L \otimes \mathbb{R}$, and the negative-definite even unimodular lattice $\langle w \rangle^{\perp} / \langle w \rangle$ of rank 24 has no vectors of square norm -2. Let $w \in L$ be a Weyl vector. We put

$$LR(w) := \{ \, r \in \mathcal{R}_L \;\mid\; (w, r)_L = 1 \, \}.$$

A root in $LR(w)$ is called a *Leech root with respect to* w.

Suppose that w is a non-zero primitive vector of norm 0 contained in the closure of $\mathcal{P}_L$. Then there exists a vector $w' \in L$ such that $(w, w')_L = 1$ and $(w', w')_L = 0$. Let $U \subset L$ denote the hyperbolic sublattice of rank 2 generated by w and w'. By

Niemeier's classification [16] of even definite unimodular lattices of rank 24 (see also [9, Chapter 18]), we see that the condition that $\langle w\rangle^\perp/\langle w\rangle$ have no vectors of square norm -2 is equivalent to the condition that the orthogonal complement $U^\perp$ of U in L be isomorphic to the (negative-definite) Leech lattice Λ. From this fact, we can deduce the following:

Proposition 2.1. *The group $\mathrm{O}^+(L)$ acts on the set of Weyl vectors transitively.*

Proposition 2.2. *Suppose that w is a Weyl vector and that $w' \in L$ satisfies $(w,w')_L = 1$ and $(w',w')_L = 0$. Via an isomorphism $\rho : \Lambda \xrightarrow{\sim} U^\perp$, the map*

$$\lambda \;\mapsto\; -\frac{2+(\lambda,\lambda)_\Lambda}{2}w + w' + \rho(\lambda)$$

induces a bijection from the Leech lattice Λ to the set $LR(w)$.

Using Vinberg's algorithm [34] and the result on the covering radius of the Leech lattice [8], Conway [6] proved the following:

Theorem 2.3. *Let $w \in L$ be a Weyl vector. Then*

$$D_L(w) := \{\, x \in \mathcal{P}_L \;\mid\; (x,r)_L \geq 0 \text{ for all } r \in LR(w) \,\}$$

is an $\mathcal{R}_L$-chamber, and $\widetilde{\mathcal{W}}(D_L(w))$ is equal to $LR(w)$; that is, $(r)_L^\perp$ bounds $D_L(w)$ for any $r \in LR(w)$. The map $w \mapsto D_L(w)$ is a bijection from the set of Weyl vectors to the set of $\mathcal{R}_L$-chambers.

Remark 2.4. Using Proposition 2.2, Conway [6] also showed that the automorphism group $\mathrm{Aut}(D_L(w)) \subset \mathrm{O}^+(L)$ of an $\mathcal{R}_L$-chamber $D_L(w)$ is isomorphic to the group $\cdot\infty$ of *affine* automorphisms of the Leech lattice Λ. Hence $\mathrm{O}^+(L)$ is isomorphic to the split extension of $\cdot\infty$ by $W(L)$.

2.3. **Restriction of $\mathcal{R}_L$-chambers to a primitive sublattice.** Let S be an even hyperbolic lattice of rank $r < 26$ primitively embedded in L. Following Borcherds [4], we explain how the Leech roots of L induce a chamber decomposition on the positive cone

$$\mathcal{P}_S := \mathcal{P}_L \cap (S \otimes \mathbb{R})$$

of $S \otimes \mathbb{R}$.

The orthogonal complement $T := S^\perp$ of S in L is negative-definite of rank $26-r$, and we have

$$S \oplus T \;\subset\; L \;\subset\; S^\vee \oplus T^\vee$$

with $[L : S\oplus T] = [S^\vee \oplus T^\vee : L]$. The projections $L\otimes\mathbb{R} \to S\otimes\mathbb{R}$ and $L\otimes\mathbb{R} \to T\otimes\mathbb{R}$ are denoted by

$$x \mapsto x_S \quad \text{and} \quad x \mapsto x_T,$$

respectively. Note that, if $v \in L$, then $v_S \in S^\vee$ and $v_T \in T^\vee$.

Let $r \in L$ be a root. Then the hyperplane $(r)^{\perp}_{L}$ contains $S \otimes \mathbb{R}$ if and only if $r_S = 0$, or equivalently, $r \in T$. Since T is negative-definite, the set

$$\mathcal{R}_T := \{\ v \in T \ \mid\ (v, v)_T = -2\ \}$$

is finite, and therefore there exist only finite number of hyperplanes $(r)^{\perp}_{L}$ that contain $S \otimes \mathbb{R}$. Suppose that $r_S \neq 0$. If $(r_S, r_S)_S \geq 0$, then either $\mathcal{P}_S$ is entirely contained in the interior of the halfspace

$$\{\ x \in L \otimes \mathbb{R} \ \mid\ (x, r)_L \geq 0\ \}$$

or is disjoint from this halfspace. Hence the hyperplane

$$(r_S)^{\perp}_{S} = (r)^{\perp}_{L} \cap (S \otimes \mathbb{R})$$

of $S \otimes \mathbb{R}$ intersects $\mathcal{P}_S$ if and only if $(r_S, r_S)_S < 0$. We put

$$\begin{aligned} \mathcal{R}_S \quad &:= \quad \{\ r_S \ \mid\ r \in \mathcal{R}_L \ \text{ and } \ (r_S, r_S)_S < 0\ \} \\ &= \quad \{\ r_S \ \mid\ r \in \mathcal{R}_L \ \text{ and } \ (r_S)^{\perp}_{S} \cap \mathcal{P}_S \neq \emptyset\ \}. \end{aligned}$$

Then the associated family of hyperplanes

$$\mathcal{H}_S := \{\ (r_S)^{\perp}_{S} \ \mid\ r_S \in \mathcal{R}_S\ \}$$

is locally finite in $\mathcal{P}_S$, and hence we can consider $\mathcal{R}_S$-chambers in $\mathcal{P}_S$. Note that $\mathcal{R}_S$ is invariant under $r_S \mapsto -r_S$. We investigate the relation between $\mathcal{R}_S$-chambers and $\mathcal{R}_L$-chambers.

If $D_S \subset \mathcal{P}_S$ is an $\mathcal{R}_S$-chamber, then there exists an $\mathcal{R}_L$-chamber $D_L(w) \subset \mathcal{P}_L$ such that $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ holds.

For a given $\mathcal{R}_S$-chamber D_S, the set of $\mathcal{R}_L$-chambers $D_L(w)$ satisfying $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ is in one-to-one correspondence with the set of connected components of

$$(T \otimes \mathbb{R}) \setminus \bigcup_{r \in \mathcal{R}_T} (r)^{\perp}_{T}.$$

Conversely, suppose that an $\mathcal{R}_L$-chamber $D_L(w)$ is given.

Definition 2.5. We say that $D_L(w)$ is S-*nondegenerate* if $D_L(w) \cap (S \otimes \mathbb{R})$ is an $\mathcal{R}_S$-chamber.

By definition, $D_L(w)$ is S-nondegenerate if and only if w satisfies the following two conditions:

(i) There exists $v \in \mathcal{P}_S$ such that $(v, r)_L \geq 0$ holds for any $r \in LR(w)$.
(ii) There exists $v' \in \mathcal{P}_S$ such that $(v', r)_L > 0$ holds for any $r \in LR(w)$ with $(r_S, r_S)_S < 0$.

If $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ is an $\mathcal{R}_S$-chamber, then $\widetilde{\mathcal{W}}(D_S)$ is contained in the image of the set

$$LR(w, S) := \{\ r \in LR(w) \ \mid\ r_S \in \mathcal{R}_S\ \} = \{\ r \in LR(w) \ \mid\ (r_S, r_S)_S < 0\ \}$$

by the projection $L \to S^\vee$. The following proposition shows that D_S is bounded by a finite number of walls if $w_T \neq 0$, and its proof indicates an effective procedure to calculate $LR(w, S)$. (See [30, Section 3] for the details of the necessary algorithms.)

Proposition 2.6. *Let $w \in L$ be a Weyl vector such that $w_T \neq 0$. Then $LR(w, S)$ is a finite set.*

Proof. Since T is negative-definite and $w_T \neq 0$, we have

$$(w_S, w_S)_S = -(w_T, w_T)_T > 0.$$

Suppose that $r \in LR(w)$. Then we have

$$(w_S, r_S)_S + (w_T, r_T)_T = 1, \quad (r_S, r_S)_S + (r_T, r_T)_T = -2.$$

We have $(r_S, r_S)_S < 0$ if and only if $(r_T, r_T)_T > -2$. Since T is negative-definite, the set

$$V_T := \{\ v \in T^\vee \ \mid\ (v, v)_T > -2\ \}$$

is finite. For $v \in V_T$, we put

$$a_v := 1 - (w_T, v)_T, \quad n_v := -2 - (v, v)_T \quad \text{and} \quad A := \{\ (a_v, n_v)\ \mid\ v \in V_T\ \}.$$

For each $(a, n) \in A$, we put

$$V_S(a, n) := \{\ u \in S^\vee \ \mid\ (w_S, u)_S = a,\ (u, u)_S = n\ \}.$$

Since S is hyperbolic and $(w_S, w_S)_S > 0$, the set $V_S(a, n)$ is finite, because $(\ ,\ )_S$ induces on the affine hyperplane

$$\{\ x \in S \otimes \mathbb{R} \ \mid\ (x, w_S)_S = a\ \}$$

of $S \otimes \mathbb{R}$ an inhomogeneous quadratic function whose quadratic part is negative-definite. Then the set $LR(w, S)$ is equal to

$$L\ \cap\ \{\ u + v \ \mid\ v \in V_T,\ u \in V_S(a_v, n_v)\ \},$$

where the intersection is taken in $S^\vee \oplus T^\vee$. □

The notion of $\mathcal{R}_S$-chamber is useful in the study on $\mathrm{O}^+(S)$ because of the following:

Proposition 2.7. *Suppose that the natural homomorphism $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective. Then the action of $\mathrm{O}^+(S)$ preserves $\mathcal{R}_S$. In particular, for an $\mathcal{R}_S$-chamber D_S and an isometry $\gamma \in \mathrm{O}^+(S)$, the image D_S^γ of D_S by γ is also an $\mathcal{R}_S$-chamber. Moreover, if the interior of D_S^γ has a common point with D_S, then $D_S^\gamma = D_S$ holds and γ preserves $\widetilde{\mathcal{W}}(D_S)$.*

Proof. By the assumption $\mathrm{O}(T) \twoheadrightarrow \mathrm{O}(q_T)$, every element $\gamma \in \mathrm{O}^+(S)$ lifts to an element $\tilde{\gamma} \in \mathrm{O}(L)$ that satisfies $\tilde{\gamma}(S) = S$ and $\tilde{\gamma}|_S = \gamma$ (see [17, Proposition 1.6.1]). Since $\tilde{\gamma}$ preserves $\mathcal{R}_L$ and γ preserves $\mathcal{P}_S$, γ preserves $\mathcal{R}_S$. □

3. A basis of the Néron-Severi lattice of X

Recall that $X \subset \mathbb{P}^3$ is the Fermat quartic surface in characteristic 3. From now on, we put

$$S := S_X,$$

which is an even hyperbolic lattice of rank 22 such that $S^\vee/S \cong (\mathbb{Z}/3\mathbb{Z})^2$. We use the affine coordinates w, x, y of $\mathbb{P}^3$ with $z = 1$.

Note that X is the Hermitian surface over $\mathbb{F}_9$ (see [12, Chapter 23]). Hence the number of lines contained in X is 112 (see [24, n. 32] or [26, Corollary 2.22]). Since the indices of these lines are important throughout this paper, we present defining equations of these lines in Table 3.1. (Note that $\ell_i \subset X$ implies that ℓ_i is not contained in the plane $z = 0$ at infinity.) From these 112 lines, we choose the following:

$$\begin{aligned}(3.1)\qquad & \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_9, \ell_{10}, \ell_{11}, \ell_{17},\\ & \qquad \ell_{18}, \ell_{19}, \ell_{21}, \ell_{22}, \ell_{23}, \ell_{25}, \ell_{26}, \ell_{27}, \ell_{33}, \ell_{35}, \ell_{49}.\end{aligned}$$

The intersection matrix N of these 22 lines is given in Table 3.2. Since $\det N = -9$, the classes $[\ell_i] \in S$ of the lines ℓ_i in (3.1) form a basis of S. Throughout this paper, we fix this basis, and write elements of $S \otimes \mathbb{R}$ as row vectors

$$[x_1, \dots, x_{22}]_S.$$

When we use its dual basis, we write

$$[\xi_1, \dots, \xi_{22}]_S^\vee.$$

Since the hyperplane $w + (1 + i) = 0$ cuts out from X the divisor $\ell_1 + \ell_2 + \ell_3 + \ell_4$, the class $h_0 = [\mathcal{O}_X(1)] \in S$ of the hyperplane section is equal to

$$\begin{aligned} h_0 &= [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]_S \\ &= [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]_S^\vee. \end{aligned}$$

As a positive cone $\mathcal{P}_S$ of S, we choose the connected component containing h_0.

From the intersection numbers of the 112 lines, we can calculate their classes $[\ell_i] \in S$.

Remark 3.1. Since these 112 lines are all defined over $\mathbb{F}_9$, every class $v \in S$ is represented by a divisor defined over $\mathbb{F}_9$. More generally, Schütt [23] showed that a supersingular $K3$ surface with Artin invariant 1 in characteristic p has a projective model defined over $\mathbb{F}_{p^2}$, and its Néron-Severi lattice is generated by the classes of divisors defined over $\mathbb{F}_{p^2}$.

Proposition 3.2. *We have*

$$h_0 = \frac{1}{28} \sum_{i=1}^{112} [\ell_i].$$

ℓ_1	$:=$	$\{w+(1+i)=x+(1+i)y=0\}$	ℓ_2	$:=$	$\{w+(1+i)=x+(1-i)y=0\}$
ℓ_3	$:=$	$\{w+(1+i)=x-(1-i)y=0\}$	ℓ_4	$:=$	$\{w+(1+i)=x-(1+i)y=0\}$
ℓ_5	$:=$	$\{w+(1-i)=x+(1+i)y=0\}$	ℓ_6	$:=$	$\{w+(1-i)=x+(1-i)y=0\}$
ℓ_7	$:=$	$\{w+(1-i)=x-(1-i)y=0\}$	ℓ_8	$:=$	$\{w+(1-i)=x-(1+i)y=0\}$
ℓ_9	$:=$	$\{w-(1-i)=x+(1+i)y=0\}$	ℓ_{10}	$:=$	$\{w-(1-i)=x+(1-i)y=0\}$
ℓ_{11}	$:=$	$\{w-(1-i)=x-(1-i)y=0\}$	ℓ_{12}	$:=$	$\{w-(1-i)=x-(1+i)y=0\}$
ℓ_{13}	$:=$	$\{w-(1+i)=x+(1+i)y=0\}$	ℓ_{14}	$:=$	$\{w-(1+i)=x+(1-i)y=0\}$
ℓ_{15}	$:=$	$\{w-(1+i)=x-(1-i)y=0\}$	ℓ_{16}	$:=$	$\{w-(1+i)=x-(1+i)y=0\}$
ℓ_{17}	$:=$	$\{w+iy+i=x+iy-i=0\}$	ℓ_{18}	$:=$	$\{w+iy+i=x-iy+i=0\}$
ℓ_{19}	$:=$	$\{w+iy+i=x+y-1=0\}$	ℓ_{20}	$:=$	$\{w+iy+i=x-y+1=0\}$
ℓ_{21}	$:=$	$\{w+iy-i=x+iy+i=0\}$	ℓ_{22}	$:=$	$\{w+iy-i=x-iy-i=0\}$
ℓ_{23}	$:=$	$\{w+iy-i=x+y+1=0\}$	ℓ_{24}	$:=$	$\{w+iy-i=x-y-1=0\}$
ℓ_{25}	$:=$	$\{w+iy+1=x+iy-1=0\}$	ℓ_{26}	$:=$	$\{w+iy+1=x-iy+1=0\}$
ℓ_{27}	$:=$	$\{w+iy+1=x+y+i=0\}$	ℓ_{28}	$:=$	$\{w+iy+1=x-y-i=0\}$
ℓ_{29}	$:=$	$\{w+iy-1=x+iy+1=0\}$	ℓ_{30}	$:=$	$\{w+iy-1=x-iy-1=0\}$
ℓ_{31}	$:=$	$\{w+iy-1=x+y-i=0\}$	ℓ_{32}	$:=$	$\{w+iy-1=x-y+i=0\}$
ℓ_{33}	$:=$	$\{w-iy+i=x+iy+i=0\}$	ℓ_{34}	$:=$	$\{w-iy+i=x-iy-i=0\}$
ℓ_{35}	$:=$	$\{w-iy+i=x+y+1=0\}$	ℓ_{36}	$:=$	$\{w-iy+i=x-y-1=0\}$
ℓ_{37}	$:=$	$\{w-iy-i=x+iy-i=0\}$	ℓ_{38}	$:=$	$\{w-iy-i=x-iy+i=0\}$
ℓ_{39}	$:=$	$\{w-iy-i=x+y-1=0\}$	ℓ_{40}	$:=$	$\{w-iy-i=x-y+1=0\}$
ℓ_{41}	$:=$	$\{w-iy+1=x+iy+1=0\}$	ℓ_{42}	$:=$	$\{w-iy+1=x-iy-1=0\}$
ℓ_{43}	$:=$	$\{w-iy+1=x+y-i=0\}$	ℓ_{44}	$:=$	$\{w-iy+1=x-y+i=0\}$
ℓ_{45}	$:=$	$\{w-iy-1=x+iy-1=0\}$	ℓ_{46}	$:=$	$\{w-iy-1=x-iy+1=0\}$
ℓ_{47}	$:=$	$\{w-iy-1=x+y+i=0\}$	ℓ_{48}	$:=$	$\{w-iy-1=x-y-i=0\}$
ℓ_{49}	$:=$	$\{w+y+i=x+iy+1=0\}$	ℓ_{50}	$:=$	$\{w+y+i=x-iy-1=0\}$
ℓ_{51}	$:=$	$\{w+y+i=x+y-i=0\}$	ℓ_{52}	$:=$	$\{w+y+i=x-y+i=0\}$
ℓ_{53}	$:=$	$\{w+y-i=x+iy-1=0\}$	ℓ_{54}	$:=$	$\{w+y-i=x-iy+1=0\}$
ℓ_{55}	$:=$	$\{w+y-i=x+y+i=0\}$	ℓ_{56}	$:=$	$\{w+y-i=x-y-i=0\}$
ℓ_{57}	$:=$	$\{w+y+1=x+iy-i=0\}$	ℓ_{58}	$:=$	$\{w+y+1=x-iy+i=0\}$
ℓ_{59}	$:=$	$\{w+y+1=x+y-1=0\}$	ℓ_{60}	$:=$	$\{w+y+1=x-y+1=0\}$
ℓ_{61}	$:=$	$\{w+y-1=x+iy+i=0\}$	ℓ_{62}	$:=$	$\{w+y-1=x-iy-i=0\}$
ℓ_{63}	$:=$	$\{w+y-1=x+y+1=0\}$	ℓ_{64}	$:=$	$\{w+y-1=x-y-1=0\}$
ℓ_{65}	$:=$	$\{w+(1+i)y=x+(1+i)=0\}$	ℓ_{66}	$:=$	$\{w+(1+i)y=x+(1-i)=0\}$
ℓ_{67}	$:=$	$\{w+(1+i)y=x-(1-i)=0\}$	ℓ_{68}	$:=$	$\{w+(1+i)y=x-(1+i)=0\}$
ℓ_{69}	$:=$	$\{w+(1-i)y=x+(1+i)=0\}$	ℓ_{70}	$:=$	$\{w+(1-i)y=x+(1-i)=0\}$
ℓ_{71}	$:=$	$\{w+(1-i)y=x-(1-i)=0\}$	ℓ_{72}	$:=$	$\{w+(1-i)y=x-(1+i)=0\}$
ℓ_{73}	$:=$	$\{w-y+i=x+iy-1=0\}$	ℓ_{74}	$:=$	$\{w-y+i=x-iy+1=0\}$
ℓ_{75}	$:=$	$\{w-y+i=x+y+i=0\}$	ℓ_{76}	$:=$	$\{w-y+i=x-y-i=0\}$
ℓ_{77}	$:=$	$\{w-y-i=x+iy+1=0\}$	ℓ_{78}	$:=$	$\{w-y-i=x-iy-1=0\}$
ℓ_{79}	$:=$	$\{w-y-i=x+y-i=0\}$	ℓ_{80}	$:=$	$\{w-y-i=x-y+i=0\}$
ℓ_{81}	$:=$	$\{w-y+1=x+iy+i=0\}$	ℓ_{82}	$:=$	$\{w-y+1=x-iy-i=0\}$
ℓ_{83}	$:=$	$\{w-y+1=x+y+1=0\}$	ℓ_{84}	$:=$	$\{w-y+1=x-y-1=0\}$
ℓ_{85}	$:=$	$\{w-y-1=x+iy-i=0\}$	ℓ_{86}	$:=$	$\{w-y-1=x-iy+i=0\}$
ℓ_{87}	$:=$	$\{w-y-1=x+y-1=0\}$	ℓ_{88}	$:=$	$\{w-y-1=x-y+1=0\}$
ℓ_{89}	$:=$	$\{w-(1-i)y=x+(1+i)=0\}$	ℓ_{90}	$:=$	$\{w-(1-i)y=x+(1-i)=0\}$
ℓ_{91}	$:=$	$\{w-(1-i)y=x-(1-i)=0\}$	ℓ_{92}	$:=$	$\{w-(1-i)y=x-(1+i)=0\}$
ℓ_{93}	$:=$	$\{w-(1+i)y=x+(1+i)=0\}$	ℓ_{94}	$:=$	$\{w-(1+i)y=x+(1-i)=0\}$
ℓ_{95}	$:=$	$\{w-(1+i)y=x-(1-i)=0\}$	ℓ_{96}	$:=$	$\{w-(1+i)y=x-(1+i)=0\}$
ℓ_{97}	$:=$	$\{w+(1+i)x=y+(1+i)=0\}$	ℓ_{98}	$:=$	$\{w+(1+i)x=y+(1-i)=0\}$
ℓ_{99}	$:=$	$\{w+(1+i)x=y-(1-i)=0\}$	ℓ_{100}	$:=$	$\{w+(1+i)x=y-(1+i)=0\}$
ℓ_{101}	$:=$	$\{w+(1-i)x=y+(1+i)=0\}$	ℓ_{102}	$:=$	$\{w+(1-i)x=y+(1-i)=0\}$
ℓ_{103}	$:=$	$\{w+(1-i)x=y-(1-i)=0\}$	ℓ_{104}	$:=$	$\{w+(1-i)x=y-(1+i)=0\}$
ℓ_{105}	$:=$	$\{w-(1-i)x=y+(1+i)=0\}$	ℓ_{106}	$:=$	$\{w-(1-i)x=y+(1-i)=0\}$
ℓ_{107}	$:=$	$\{w-(1-i)x=y-(1-i)=0\}$	ℓ_{108}	$:=$	$\{w-(1-i)x=y-(1+i)=0\}$
ℓ_{109}	$:=$	$\{w-(1+i)x=y+(1+i)=0\}$	ℓ_{110}	$:=$	$\{w-(1+i)x=y+(1-i)=0\}$
ℓ_{111}	$:=$	$\{w-(1+i)x=y-(1-i)=0\}$	ℓ_{112}	$:=$	$\{w-(1+i)x=y-(1+i)=0\}$

TABLE 3.1. Lines on X

$$\begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2
\end{bmatrix}$$

TABLE 3.2. Gram matrix N of S

Proof. The number of $\mathbb{F}_9$-rational points on X is 280. For each $\mathbb{F}_9$-rational point P of X, the tangent plane $T_{X,P} \subset \mathbb{P}^3$ to X at P cuts out a union of four lines from X. Since each line contains ten $\mathbb{F}_9$-rational points, we have $280\, h_0 = 10 \sum[\ell_i]$. $\square$

As before, we let $\mathrm{O}(S)$ act on S from *right*, so that

$$\mathrm{O}(S) = \{\, T \in \mathrm{GL}_{22}(\mathbb{Z}) \;\mid\; T\, N\, {}^tT = N \,\}.$$

We also let the projective automorphism group $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ act on X from *right*. For each $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$, we can calculate its action τ_* on S by looking at the permutation of the 112 lines induced by τ.

Example 3.3. Consider the projective automorphism

$$\tau \quad : \quad [w:x:y:z] \mapsto [w:x:y:z] \begin{bmatrix} 1 & -1 & -i & -1 \\ -1 & -1+i & -1+i & -1+i \\ -1 & -i & -1-i & 0 \\ i & 0 & -1+i & -1 \end{bmatrix}$$

of X. [1] Then the images ℓ_i^τ of the lines ℓ_i in (3.1) are

$$\begin{aligned}
&\ell_1^\tau = \ell_{60},\ \ \ell_2^\tau = \ell_{31},\ \ \ell_3^\tau = \ell_{105},\ \ \ell_4^\tau = \ell_{95},\ \ \ell_5^\tau = \ell_{92},\ \ \ell_6^\tau = \ell_{30},\\
&\ell_7^\tau = \ell_{76},\ \ \ell_9^\tau = \ell_{110},\ \ \ell_{10}^\tau = \ell_{29},\ \ \ell_{11}^\tau = \ell_{6},\ \ \ell_{17}^\tau = \ell_{20},\ \ \ell_{18}^\tau = \ell_{96},\\
&\ell_{19}^\tau = \ell_{102},\ \ \ell_{21}^\tau = \ell_{13},\ \ \ell_{22}^\tau = \ell_{87},\ \ \ell_{23}^\tau = \ell_{91},\ \ \ell_{25}^\tau = \ell_{108},\ \ \ell_{26}^\tau = \ell_{10},\\
&\ell_{27}^\tau = \ell_{57},\ \ \ell_{33}^\tau = \ell_{52},\ \ \ell_{35}^\tau = \ell_{51},\ \ \ell_{49}^\tau = \ell_{59}.
\end{aligned}$$

Therefore the action τ_* on S is given by $v \mapsto vT_\tau$, where T_τ is the matrix whose row vectors are

$$\begin{aligned}
[\ell_{60}] &= [1,0,0,0,0,0,-1,0,0,0,0,0,0,-1,0,0,0,0,1,0,0,1]_S,\\
[\ell_{31}] &= [1,1,1,1,0,0,0,0,0,0,0,0,-1,0,0,-1,0,0,-1,0,0,0]_S,\\
[\ell_{105}] &= [2,2,2,3,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,-1,0,0,0,-1,0]_S,\\
[\ell_{95}] &= [-3,-2,-2,-3,1,1,2,1,1,1,0,0,1,1,0,1,1,0,0,0,1,-1]_S,\\
[\ell_{92}] &= [-1,0,0,0,0,1,0,0,1,0,0,1,0,0,1,0,0,1,0,-1,-1,-1]_S,\\
[\ell_{30}] &= [1,1,1,1,0,0,0,0,0,0,0,-1,0,0,-1,0,0,-1,0,0,0,0]_S,\\
[\ell_{76}] &= [0,-1,-1,-1,0,-1,0,1,0,1,0,0,0,0,0,0,0,0,0,1,1,1]_S,\\
[\ell_{110}] &= [-1,0,0,-1,0,0,1,0,0,0,0,-1,0,1,0,0,1,0,0,0,1,0]_S,\\
[\ell_{29}] &= [1,1,1,1,0,0,0,0,0,0,-1,0,0,-1,0,0,-1,0,0,0,0,0]_S,\\
[\ell_{6}] &= [0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]_S,\\
[\ell_{20}] &= [1,1,1,1,0,0,0,0,0,0,-1,-1,-1,0,0,0,0,0,0,0,0,0]_S,\\
[\ell_{96}] &= [4,2,3,4,-2,-3,-2,-1,-2,0,1,0,-1,0,-1,-1,-1,-1,-1,2,0,1]_S,\\
[\ell_{102}] &= [-1,-1,-1,-1,1,1,1,0,0,0,0,0,1,0,0,1,0,0,1,-1,0,0]_S,\\
[\ell_{13}] &= [0,1,1,1,-1,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0]_S,\\
[\ell_{87}] &= [-3,-2,-3,-3,2,2,1,1,1,0,0,1,1,0,1,1,1,1,1,-1,0,-1]_S,\\
[\ell_{91}] &= [4,2,3,3,-1,-2,-1,0,-1,0,0,-1,-1,-1,-1,-1,-1,-1,-1,1,0,1]_S,\\
[\ell_{108}] &= [-2,-2,-2,-3,1,0,1,1,0,1,1,0,1,1,0,1,1,0,0,0,1,0]_S,\\
[\ell_{10}] &= [0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0]_S,\\
[\ell_{57}] &= [1,2,1,2,-1,0,-1,-1,0,-1,0,1,0,0,0,-1,0,1,0,0,-1,-1]_S,\\
[\ell_{52}] &= [-1,0,-1,-1,1,1,1,1,1,0,0,0,1,0,0,0,0,0,0,-1,0,-1]_S,\\
[\ell_{51}] &= [1,1,1,2,0,0,-1,-1,0,-1,0,1,0,0,0,0,0,0,0,0,-1,-1]_S,\\
[\ell_{59}] &= [2,1,2,2,-1,-2,0,0,-1,1,0,-1,-1,0,-1,0,-1,-1,-1,1,1,1]_S.
\end{aligned}$$

[1]This matrix was incorrect in the previous version of this paper. We are grateful to Benjamin Church for bringing this error to our attention. This error was corrected in August 2026.

We put the representation

$$\tau \mapsto T_\tau \tag{3.2}$$

of $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ to $\mathrm{O}^+(S)$ in the computer memory. It turns out to be faithful. On the other hand, $\mathrm{Aut}(X, h_0)$ is just the stabilizer subgroup in $\mathrm{Aut}(X)$ of $h_0 \in S$. Therefore we confirm the following fact ([21, Section 8, Proposition 3]):

Proposition 3.4. *The action of* $\mathrm{Aut}(X)$ *on* S *is faithful.*

From now on, we regard $\mathrm{Aut}(X)$ as a subgroup of $\mathrm{O}^+(S)$, and write $v \mapsto v^\gamma$ instead of $v \mapsto v^{\gamma_*}$ for the action γ_* of $\gamma \in \mathrm{Aut}(X)$ on S.

4. Embedding of S into L

Next we embed the Néron-Severi lattice S of X into the even unimodular hyperbolic lattice of rank 26, and calculate the walls of an $\mathcal{R}_S$-chamber.

Let T be the negative-definite root lattice of type $2A_2$. We fix a basis of T in such a way that the Gram matrix is equal to

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

When we use this basis, we write elements of $T \otimes \mathbb{R}$ as $[y_1, y_2, y_3, y_4]_T$, while when we use its dual basis, we write as $[\eta_1, \eta_2, \eta_3, \eta_4]_T^\vee$. Elements of $(S \oplus T) \otimes \mathbb{R}$ are written as

$$[x_1, \dots, x_{22} \mid y_1, \dots, y_4]$$

using the bases of S and T, or as

$$[\xi_1, \dots, \xi_{22} \mid \eta_1, \dots, \eta_4]^\vee$$

using the dual bases of $S^\vee$ and $T^\vee$.

Consider the following vectors of $S^\vee \oplus T^\vee$:

$$\begin{aligned} a_1 &:= \frac{1}{3}[2,2,0,0,0,1,2,2,1,1,2,2,1,1,2,0,0,1,1,0,0,0 \mid 1,2,0,0], \\ a_2 &:= \frac{1}{3}[2,0,2,0,2,1,1,0,2,1,2,1,0,2,2,1,1,0,1,0,0,0 \mid 0,0,1,2]. \end{aligned}$$

We define $\alpha_1, \alpha_2 \in (S \oplus T)^\vee/(S \oplus T)$ by

$$\alpha_1 := a_1 \bmod (S \oplus T), \quad \alpha_2 := a_2 \bmod (S \oplus T).$$

Then α_1 and α_2 are linearly independent in $(S \oplus T)^\vee/(S \oplus T) \cong \mathbb{F}_3^4$. Since

$$q_{S\oplus T}(\alpha_1) = q_{S\oplus T}(\alpha_2) = q_{S\oplus T}(\alpha_1 + \alpha_2) = 0,$$

the vectors α_1 and α_2 generate a maximal isotropic subgroup of $q_{S\oplus T}$. Therefore, by [17, Proposition 1.4.1], the submodule

$$L := (S \oplus T) + \langle a_1 \rangle + \langle a_2 \rangle$$

of $S^\vee \oplus T^\vee$ is an even unimodular overlattice of $S \oplus T$ into which S and T are primitively embedded.

By construction, L is hyperbolic of rank 26. We choose $\mathcal{P}_L$ to be the connected component that contains $\mathcal{P}_S$. Then, by means of the roots of L, we obtain a decomposition of $\mathcal{P}_S$ into $\mathcal{R}_S$-chambers.

The order of $\mathrm{O}(T)$ is 288, while the order of $\mathrm{O}(q_T)$ is 8. It is easy to check that the natural homomorphism $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective. Therefore we obtain the following from Proposition 2.7:

Proposition 4.1. *The action of* $\mathrm{O}^+(S)$ *on* $S \otimes \mathbb{R}$ *preserves* $\mathcal{R}_S$.

We put

$$\begin{aligned} w_0 &:= [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0 \mid -1,-1,-1,-1] \\ &= [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \mid 1,1,1,1]^\vee. \end{aligned}$$

Note that the projection $w_{0S} \in S^\vee$ of w_0 to $S^\vee$ is equal to h_0.

Since $(w_0, w_0)_L = 0$ and $(w_0, h_0)_L > 0$, we see that w_0 is on the boundary of the closure of $\mathcal{P}_L$ in $L \otimes \mathbb{R}$.

Proposition 4.2. *The vector w_0 is a Weyl vector, and the $\mathcal{R}_L$-chamber $D_L(w_0)$ is S-nondegenerate. The $\mathcal{R}_S$-chamber*

$$D_{S0} := D_L(w_0) \cap (S \otimes \mathbb{R})$$

contains $w_{S0} = h_0$ in its interior.

Proof. The only non-trivial part of the first assertion is that $\langle w_0 \rangle^\perp / \langle w_0 \rangle$ has no vectors of square norm -2. We put

$$w_0' := [7,6,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7 \mid 7,5,7,7]^\vee.$$

Then we have $(w_0', w_0')_L = 0$ and $(w_0, w_0')_L = 1$. Let $U \subset L$ be the sublattice generated by w_0 and w_0'. Calculating a basis $\lambda_1, \dots, \lambda_{24}$ of $U^\perp \subset L$, we obtain a Gram matrix of $U^\perp$, which is negative-definite of determinant 1. By the algorithm described in [30, Section 3.1], we verify that there are no vectors of square norm -2 in $U^\perp$.

We show that w_0 satisfies the conditions (i) and (ii) given after Definition 2.5. By Proposition 2.2, in order to verify the condition (i), it is enough to show that the function $Q : U^\perp \to \mathbb{Z}$ given by

$$Q(\lambda) := (h_0, -\frac{2 + (\lambda, \lambda)_L}{2} w_0 + w_0' + \lambda)_L$$

does not take negative values. Using the basis $\lambda_1, \dots, \lambda_{24}$ of $U^\perp$, we can write Q as an inhomogeneous quadratic function of 24 variables. Its quadratic part turns out to be positive-definite. By the algorithm described in [30, Section 3.1], we verify that there exist no vectors $\lambda \in U^\perp$ such that $Q(\lambda) < 0$. Next we show that $w_{0S} = h_0 \in \mathcal{P}_S$ has the property required for v' in the condition (ii), and hence h_0 is contained in the interior of D_{S0}. Note that $w_{0T} = [-1, -1, -1, -1]_T$ is non-zero. Hence we can calculate

$$LR(w_0, S) = \{\, r \in LR(w_0) \mid (r_S, r_S)_S < 0 \,\}$$

by the method described in the proof of Proposition 2.6. Then we can easily show that h_0 satisfies $(h_0, r)_L > 0$ for any $r \in LR(w_0, S)$. □

Remark 4.3. There exist exactly four vectors $\lambda \in U^\perp$ such that $Q(\lambda) = 0$. They correspond to the Leech roots $r \in LR(w_0)$ such that $r = r_T$.

From the surjectivity of $\mathrm{O}(T) \to \mathrm{O}(q_T)$ and Proposition 2.7, we obtain the following:

Corollary 4.4. *The action of* $\mathrm{Aut}(X, h_0)$ *on* $S \otimes \mathbb{R}$ *preserves* D_{S0} *and* $\widetilde{\mathcal{W}}(D_{S0})$.

Proposition 4.5. *The maps* $r \mapsto r_S$ *and* $r_S \mapsto (r_S)^\perp_S$ *induce bijections*

$$LR(w_0, S) \;\cong\; \widetilde{\mathcal{W}}(D_{S0}) \;\cong\; \mathcal{W}(D_{S0}).$$

The action of $\mathrm{Aut}(X, h_0)$ *decomposes* $\widetilde{\mathcal{W}}(D_{S0})$ *into the three orbits*

$$\widetilde{W}_{112} := \widetilde{\mathcal{W}}(D_{S0})_{[1,-2]}, \quad \widetilde{W}_{648} := \widetilde{\mathcal{W}}(D_{S0})_{[2,-4/3]} \quad \textit{and} \quad \widetilde{W}_{5184} := \widetilde{\mathcal{W}}(D_{S0})_{[3,-2/3]}$$

of cardinalities 112*,* 648 *and* 5184*, respectively, where*

$$\widetilde{\mathcal{W}}(D_{S0})_{[a,n]} := \{\, r_S \in \widetilde{\mathcal{W}}(D_{S0}) \mid (r_S, h_0)_S = a,\ (r_S, r_S)_S = n \,\}.$$

The set $\widetilde{W}_{112}$ *coincides with the set of the classes* $[\ell_i]$ *of lines contained in* X*:*

$$\widetilde{W}_{112} = \{[\ell_1], [\ell_2], \dots, [\ell_{112}]\}.$$

The sets $\widetilde{W}_{648}$ *and* $\widetilde{W}_{5184}$ *are the orbits of*

$$\begin{aligned} b_1 &:= \frac{1}{3}[-1,0,-1,0,2,1,1,0,2,1,-1,1,0,-1,-1,1,1,0,1,0,0,0]_S \;\in\; \widetilde{W}_{648}, \quad \textit{and} \\ b_2 &:= \frac{1}{3}[0,1,-1,0,2,0,2,1,1,0,0,-1,2,1,0,1,1,-1,0,0,0,0]_S \;\in\; \widetilde{W}_{5184}, \end{aligned}$$

by the action of $\mathrm{Aut}(X, h_0)$*, respectively.*

Proof. We have calculated the finite set $LR(w_0, S)$ in the proof of Proposition 4.2. We have also stored the classes $[\ell_i]$ of the 112 lines and the action of $\mathrm{Aut}(X, h_0)$ on S in the computer memory. Thus the assertions of Proposition 4.5 are verified by a direct computation, except for the fact that, for any $r \in LR(w_0, S)$, the hyperplane $(r_S)^{\perp}_S$ actually bounds D_{S0}. This is proved by showing that the point

$$p := h_0 - \frac{(h_0, r_S)_S}{(r_S, r_S)_S} r_S$$

on $(r_S)^{\perp}_S$ satisfies $(p, r')_L > 0$ for any $r' \in LR(w_0, S) \setminus \{r\}$. □

Since Proposition 3.2 implies that the interior point h_0 of D_{S0} is determined by $\widetilde{W}_{112}$ and since $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective, we obtain the following from Proposition 2.7:

Corollary 4.6. *For $\gamma \in \mathrm{O}^+(S)$, the following are equivalent:* (i) *the interior of D^{γ}_{S0} has a common point with D_{S0},* (ii) $D^{\gamma}_{S0} = D_{S0}$, (iii) $\widetilde{W}^{\gamma}_{112} = \widetilde{W}_{112}$, (iv) $h^{\gamma}_0 = h_0$, *and* (v) $h^{\gamma}_0 \in D_{S0}$.

In particular, we obtain the following:

Corollary 4.7. *If $\gamma \in \mathrm{Aut}(X)$ satisfies $h^{\gamma}_0 \in D_{S0}$, then γ is in $\mathrm{Aut}(X, h_0)$.*

5. The automorphisms g_1 and g_2

In order to find automorphisms $\gamma \in \mathrm{Aut}(X)$ such that $h^{\gamma}_0 \notin D_{S0}$, we search for polarizations of degree 2 that are located on the walls $(b_1)^{\perp}_S$ and $(b_2)^{\perp}_S$.

We fix terminologies and notation. For a vector $v \in S$, we denote by $\mathcal{L}_v \to X$ a line bundle defined over $\mathbb{F}_9$ whose class is v (see Remark 3.1). We say that a vector $h \in S$ is a *polarization* of degree d if $(h, h)_S = d$ and the complete linear system $|\mathcal{L}_h|$ is nonempty and has no fixed components. If h is a polarization, then $|\mathcal{L}_h|$ has no base-points by [22, Corollary 3.2] and hence defines a morphism

$$\Phi_h : X \to \mathbb{P}^N,$$

where $N = \dim|\mathcal{L}_h|$.

A polynomial in $\mathbb{F}_9[w, x, y]$ is said to be *of normal form* if its degree with respect to w is ≤ 3. For each polynomial $G \in \mathbb{F}_9[w, x, y]$, there exists a unique polynomial $\overline{G}$ of normal form such that

$$G \equiv \overline{G} \mod (w^4 + x^4 + y^4 + 1).$$

We say that $\overline{G}$ is the *normal form* of G. For any $d \in \mathbb{Z}$, the vector space $H^0(X, \mathcal{L}_{dh_0})$ over $\mathbb{F}_9$ is naturally identified with the vector subspace

$$\Gamma(d) := \{\, G \in \mathbb{F}_9[w, x, y] \mid G \text{ is of normal form with total degree} \le d \,\}$$

of $\mathbb{F}_9[w, x, y]$. For an ideal J of $\mathbb{F}_9[w, x, y]$, we put

$$\Gamma(d, J) := \Gamma(d) \cap J.$$

A basis of $\Gamma(d, J)$ is easily obtained by a Gröbner basis of J. Let ℓ_i be a line contained in X. We denote by $I_i \subset \mathbb{F}_9[w, x, y]$ the affine defining ideal of ℓ_i in $\mathbb{P}^3$ (see Table 3.1), and put

$$I_i^{(\nu)} := I_i^{\nu} + (w^4 + x^4 + y^4 + 1) \quad \subset \quad \mathbb{F}_9[w, x, y]$$

for nonnegative integers ν. Suppose that $v \in S$ is written as

$$v = d\, h_0 - \sum_{i=1}^{112} a_i[\ell_i], \tag{5.1}$$

where a_i are nonnegative integers. Then there exists a natural isomorphism

$$H^0(X, \mathcal{L}_v) \cong \Gamma(d, \bigcap_{i=1}^{112} I_i^{(a_i)})$$

with the property that, for

$$v' = d' h_0 - \sum_{i=0}^{112} a'_i[\ell_i] \quad \in \quad S$$

with $a'_i \in \mathbb{Z}_{\geq 0}$, the multiplication homomorphism

$$H^0(X, \mathcal{L}_v) \times H^0(X, \mathcal{L}_{v'}) \to H^0(X, \mathcal{L}_{v+v'})$$

is identified with

$$\Gamma(d, \bigcap I_i^{(a_i)}) \times \Gamma(d', \bigcap I_i^{(a'_i)}) \to \Gamma(d + d', \bigcap I_i^{(a_i + a'_i)})$$

given by $(\overline{G}, \overline{G'}) \mapsto \overline{GG'}$.

Proposition 1.1 in Introduction is an immediate consequence of the following:

Proposition 5.1. *Consider the vectors*

$$\begin{aligned} m_1 &:= [-1, 0, -1, -1, 2, 2, 1, 1, 2, 0, -1, 1, 1, -1, 0, 1, 0, 0, 1, -1, 0, 0]_S \quad \textit{and} \\ m_2 &:= [2, 2, 1, 2, 1, -1, 1, 1, 1, 1, 0, -1, 0, 0, 0, 0, 0, -1, -1, 1, 0, 1]_S \end{aligned}$$

of S. Then each m_i is a polarization of degree 2. If we choose a basis of the vector space $H^0(X, \mathcal{L}_{m_i})$ appropriately, the morphism $\Phi_{m_i} : X \to \mathbb{P}^2$ associated with $|\mathcal{L}_{m_i}|$ coincides with the morphism $\phi_i : X \to \mathbb{P}^2$ given in the statement of Proposition 1.1.

Proof. We have $(m_i, m_i)_S = 2$. By the method described in [30, Section 4.1], we see that m_i is a polarization; namely, we verify that the sets

$$\begin{aligned} &\{\, v \in S \mid (v, v)_S = -2,\ (v, m_i)_S < 0,\ (v, h_0)_S > 0 \,\} \quad \text{and} \\ &\{\, v \in S \mid (v, v)_S = 0,\ (v, m_i)_S = 1 \,\} \end{aligned}$$

are both empty. Since

$$m_1 = 3h_0 - ([\ell_{21}] + [\ell_{22}] + [\ell_{50}] + [\ell_{63}] + [\ell_{65}] + [\ell_{88}]) \quad \text{and} \tag{5.2}$$

$$\begin{aligned} m_2 = 5h_0 - (&[\ell_1] + [\ell_3] + [\ell_6] + [\ell_{18}] + \\ &+ [\ell_{35}] + [\ell_{74}] + [\ell_{90}] + [\ell_{92}] + [\ell_{110}] + [\ell_{111}]), \end{aligned} \tag{5.3}$$

$$
\begin{array}{rcll}
\ell_{37} & \mapsto & [1:1-i:1-i] & (A_1\text{-point}) \\
\ell_{23} & \mapsto & [1:1+i:-(1+i)] & (A_1\text{-point}) \\
\ell_{62} & \mapsto & [1:-(1+i):0] & (A_1\text{-point}) \\
\ell_{102} & \mapsto & [1:-(1-i):0] & (A_1\text{-point}) \\
\ell_{68} & \mapsto & [1:1+i:1+i] & (A_1\text{-point}) \\
\ell_{112} & \mapsto & [1:1-i:-(1-i)] & (A_1\text{-point}) \\
\ell_{49}, \ell_{29} & \mapsto & [1:1:-i] & (A_2\text{-point}) \\
\ell_{73}, \ell_{60} & \mapsto & [1:1:i] & (A_2\text{-point}) \\
\ell_{18}, \ell_{10} & \mapsto & [0:1:-1] & (A_2\text{-point}) \\
\ell_{16}, \ell_{99} & \mapsto & [0:1:1] & (A_2\text{-point})
\end{array}
$$

TABLE 5.1. Lines contracted by $\phi_1 : X \to \mathbb{P}^2$

the vector spaces $H^0(X, \mathcal{L}_{m_1})$ and $H^0(X, \mathcal{L}_{m_2})$ are identified with the subspaces

$$
\begin{array}{rcl}
\Gamma_1 & := & \Gamma(3, I_{21} \cap I_{22} \cap I_{50} \cap I_{63} \cap I_{65} \cap I_{88}) \quad \text{and} \\
\Gamma_2 & := & \Gamma(5, I_1 \cap I_3 \cap I_6 \cap I_{18} \cap I_{35} \cap I_{74} \cap I_{90} \cap I_{92} \cap I_{110} \cap I_{111})
\end{array}
$$

of $\mathbb{F}_9[w, x, y]$, respectively. We calculate a basis of Γ_i by means of Gröbner bases of the ideals I_i. The set $\{F_{i0}, F_{i1}, F_{i2}\}$ of polynomials in Table 1.1 is just a basis of Γ_i thus calculated. $\square$

Remark 5.2. The polarizations m_1 and m_2 in Proposition 5.1 are located on the hyperplanes $(b_1)^{\perp}_S$ and $(b_2)^{\perp}_S$ bounding D_{S0}, respectively, where $b_1 \in \widetilde{W}_{648}$ and $b_2 \in \widetilde{W}_{5184}$ are given in Proposition 4.5.

Proof of Proposition 1.2. The set $\mathrm{Exc}(\phi_i)$ of the classes of (-2)-curves contracted by $\phi_i : X \to \mathbb{P}^2$ is calculated by the method described in [30, Section 4.2]. We first calculate the set

$$R_i^+ := \{\ v \in S \ \mid\ (v, v)_S = -2,\ (v, m_i)_S = 0,\ (v, h_0)_S > 0\ \}.$$

It turns out that every element of R_i^+ is written as a linear combination with coefficients in $\mathbb{Z}_{\geq 0}$ of elements $l \in R_i^+$ such that $(l, h_0)_S = 1$. Hence we have

$$\mathrm{Exc}(\phi_i) = \{\ l \in R_i^+ \ \mid\ (l, h_0)_S = 1\ \}.$$

The ADE-type of the root system $\mathrm{Exc}(\phi_i)$ is equal to $6A_1 + 4A_2$ for $i = 1$ and $A_1 + A_2 + 2A_3 + 2A_4$ for $i = 2$. Thus the assertion on the ADE-type of the singularities of Y_i is proved. Moreover we have proved that all (-2)-curves contracted by $\phi_i : X \to \mathbb{P}^2$ are lines. See Tables 5.1 and 5.2, in which the lines $\ell_{k_1}, \dots, \ell_{k_r}$ contracted by ϕ_i to a singular point P of type A_r are indicated in such an order that $(\ell_{k_\nu}, \ell_{k_{\nu+1}})_S = 1$ holds for $\nu = 1, \dots, r-1$.

The defining equation $f_i = 0$ of the branch curve $B_i \subset \mathbb{P}^2$ is calculated by the method given in [30, Section 5]. We calculate a basis of the vector space

$$\begin{array}{rcll} \ell_{43} & \mapsto & [0:1:0] & (A_1\text{-point}) \\ \ell_{76}, \ell_{94} & \mapsto & [1:-1:0] & (A_2\text{-point}) \\ \ell_{22}, \ell_{49}, \ell_{20} & \mapsto & [1:-1:1] & (A_3\text{-point}) \\ \ell_{7}, \ell_{5}, \ell_{103} & \mapsto & [1:-1:-1] & (A_3\text{-point}) \\ \ell_{10}, \ell_{2}, \ell_{4}, \ell_{91} & \mapsto & [1:0:1] & (A_4\text{-point}) \\ \ell_{33}, \ell_{36}, \ell_{72}, \ell_{83} & \mapsto & [1:0:-1] & (A_4\text{-point}) \end{array}$$

TABLE 5.2. Lines contracted by $\phi_2 : X \to \mathbb{P}^2$

$H^0(X, \mathcal{L}_{3m_i})$ of dimension 11 using (5.2), (5.3) and Gröbner bases of $I_i^{(3)}$. Note that the ten normal forms $M_{i,1}, \dots, M_{i,10}$ of the cubic monomials of F_{i0}, F_{i1}, F_{i2} are contained in $H^0(X, \mathcal{L}_{3m_i})$. We choose a polynomial $G_i \in H^0(X, \mathcal{L}_{3m_i})$ that is not contained in the linear span of $M_{i,1}, \dots, M_{i,10}$. In the vector space $H^0(X, \mathcal{L}_{6m_i})$ of dimension 38, the 39 normal forms of the monomials of $G_i, F_{i0}, F_{i1}, F_{i2}$ of weighted degree 6 with weight $\deg G_i = 3$ and $\deg F_{ij} = 1$ have a non-trivial linear relation. Note that this linear relation is quadratic with respect to G_i. Completing the square and re-choosing G_i appropriately, we confirm that

$$\overline{G_i^2 + f_i(F_{i0}, F_{i1}, F_{i2})} = 0$$

holds. Hence Y_i is defined by $y^2 + f_i(x_0, x_1, x_2) = 0$. □

Remark 5.3. In order to obtain a defining equation of B_i with coefficients in $\mathbb{F}_3$, we have to choose the basis F_{i0}, F_{i1}, F_{i2} of $\Gamma_i = H^0(X, \mathcal{L}_{m_i})$ carefully. See [30, Section 6.10] for the method.

Remark 5.4. The polynomial

$$G_1 = G_{1(0)}(x, y) + G_{1(1)}(x, y)\, w + G_{1(2)}(x, y)\, w^2 + G_{1(3)}(x, y)\, w^3$$

is given in Table 5.3. The polynomial G_2 is too large to be presented in the paper (see [31]).

Proposition 5.5. *Let g_1 and g_2 be the involutions of X defined in Theorem 1.3. Then the action g_{i*} on S is given by $v \mapsto vA_i$, where A_i is the matrix given in Tables 5.4 and 5.5.*

Proof. Recall that $\mathrm{Exc}(\phi_i)$ is the set of the classes of (-2)-curves contracted by $\phi_i : X \to \mathbb{P}^2$. Suppose that $\gamma_1, \dots, \gamma_r \in \mathrm{Exc}(\phi_i)$ are the classes of (-2)-curves that are contracted to a singular point $P \in \mathrm{Sing}(B_i)$ of type A_r. We index them in such a way that $(\gamma_\nu, \gamma_{\nu+1})_S = 1$ holds for $\nu = 1, \dots, r-1$. Then g_{i*} interchanges γ_i and γ_{r+1-i}. Let $V(P) \subset S \otimes \mathbb{Q}$ denote the linear span of the invariant vectors $\gamma_i + \gamma_{r+1-i}$. Then the eigenspace of g_{i*} on $S \otimes \mathbb{Q}$ with eigenvalue 1 is equal to

$$\langle m_i \rangle \oplus \bigoplus_{P \in \mathrm{Sing}(B_i)} V(P),$$

$$
\begin{aligned}
G_{1(0)} = & -(1-i) + (1+i)\,x + (1+i)\,y + i\,x^2 - (1+i)\,xy - (1+i)\,y^2 - xy^2 \\
& + (1+i)\,y^3 - (1-i)\,x^4 - (1+i)\,x^3y - xy^3 - (1-i)\,y^4 - (1-i)\,x^5 - x^3y^2 \\
& - i\,x^2y^3 - (1-i)\,xy^4 + (1+i)\,y^5 - (1-i)\,x^6 + x^5y + i\,x^4y^2 - (1-i)\,x^3y^3 \\
& + (1+i)\,x^2y^4 - i\,xy^5 + (1-i)\,y^6 + (1+i)\,x^7 + x^4y^3 + (1+i)\,x^3y^4 \\
& + i\,xy^6 - i\,y^7 + i\,x^8 + (1+i)\,x^7y + i\,x^6y^2 - i\,x^5y^3 + (1-i)\,x^4y^4 + x^2y^6 \\
& + i\,xy^7 - i\,y^8 - (1+i)\,x^9 - i\,x^8y - (1-i)\,x^7y^2 - (1+i)\,x^6y^3 + i\,x^5y^4 \\
& + (1-i)\,x^4y^5 - (1+i)\,x^3y^6 - (1-i)\,x^2y^7 - (1-i)\,xy^8 - (1+i)\,y^9 \\
G_{1(1)} = & (1-i) + (1-i)\,x - (1+i)\,x^2 - xy + i\,y^2 + x^3 - (1-i)\,x^2y - xy^2 \\
& + (1+i)\,x^4 + (1-i)\,xy^3 - (1-i)\,y^4 - x^5 + x^4y + xy^4 - (1+i)\,y^5 \\
& + (1+i)\,x^5y - (1+i)\,xy^5 + y^6 - i\,x^7 - x^6y + x^5y^2 - (1-i)\,x^4y^3 \\
& + (1-i)\,x^2y^5 + (1-i)\,xy^6 - (1-i)\,y^7 - i\,x^8 + (1+i)\,x^7y - i\,x^6y^2 \\
& - i\,x^5y^3 - (1-i)\,x^4y^4 + (1+i)\,x^3y^5 - (1-i)\,x^2y^6 + (1-i)\,xy^7 + (1-i)\,y^8 \\
G_{1(2)} = & (1-i) - (1+i)\,x - (1+i)\,xy + y^2 - (1-i)\,x^3 - (1+i)\,x^2y - i\,xy^2 \\
& - (1+i)\,y^3 + x^4 - (1+i)\,x^3y + xy^3 - y^4 - x^5 - x^4y - xy^4 - y^5 \\
& - i\,x^6 + x^4y^2 + i\,x^3y^3 + (1+i)\,xy^5 - y^6 - (1-i)\,x^7 - i\,x^6y - i\,x^5y^2 \\
& - i\,x^4y^3 - (1+i)\,x^3y^4 - (1+i)\,x^2y^5 + (1+i)\,xy^6 - (1+i)\,y^7 \\
G_{1(3)} = & (1+i)\,x - (1-i)\,y - (1-i)\,x^2 - (1-i)\,xy + (1+i)\,y^2 - x^3 - x^2y \\
& - xy^2 + y^3 - (1-i)\,x^4 - i\,x^3y + (1-i)\,xy^3 - i\,y^4 + i\,x^5 + x^4y \\
& + (1+i)\,x^3y^2 + (1+i)\,x^2y^3 + (1+i)\,xy^4 + (1+i)\,y^5 - x^6 - (1-i)\,x^5y \\
& + (1+i)\,x^4y^2 + i\,x^3y^3 - (1-i)\,x^2y^4 - (1+i)\,xy^5 + (1+i)\,y^6
\end{aligned}
$$

TABLE 5.3. Polynomial G_1

and the eigenspace with eigenvalue -1 is its orthogonal complement. □

Using the matrix representations A_i of g_{i*}, we verify the following facts:

(1) The eigenspace of g_{i*} with eigenvalue 1 is contained in $(b_i)^{\perp}_{S}$. In particular, we have $b_i^{g_i} = -b_i$.
(2) The vector $h_0^{g_i}$ is equal to the image of h_0 by the reflection into the wall $(b_i)^{\perp}_{S}$, that is $h_0^{g_1} = h_0 + 3\,b_1$ and $h_0^{g_2} = h_0 + 9\,b_2$ hold.

Since $\mathrm{Aut}(X, h_0)$ acts on each of $\widetilde{W}_{648}$ and $\widetilde{W}_{5184}$ transitively, we obtain the following:

Corollary 5.6. *For any $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$, there exists $\tau \in \mathrm{Aut}(X, h_0)$ such that*

$$h_0^{g_i\tau} = h_0 + c_i\, r_S$$

holds, where $i = 1$ and $c_1 = 3$ if $r_S \in \widetilde{W}_{648}$ while $i = 2$ and $c_2 = 9$ if $r_S \in \widetilde{W}_{5184}$.

6. PROOF OF THEOREM 1.3

We denote by

$$G := \langle \mathrm{Aut}(X, h_0), g_1, g_2 \rangle$$

$$\left[\begin{array}{cccccccccccccccccccccc}
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 1 \\
-2 & -1 & -2 & -2 & 2 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\
-1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 3 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-2 & -1 & -2 & -2 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 3 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

TABLE 5.4. The matrix A_1

the subgroup of $\mathrm{Aut}(X)$ generated by $\mathrm{Aut}(X, h_0)$, g_1 and g_2. Note that the action of $\mathrm{Aut}(X)$ on S preserves the set of nef classes.

Theorem 6.1. *If $v \in S$ is nef, there exists $\gamma \in G$ such that $v^\gamma \in D_{S0}$.*

Proof. Let $\gamma \in G$ be an element such that $(v^\gamma, h_0)_S$ attains

$$\min\{(v^{\gamma'}, h_0)_S \,|\, \gamma' \in G\}.$$

We show that $(v^\gamma, r_S)_S \ge 0$ holds for any $r_S \in \widetilde{\mathcal{W}}(D_{S0})$. If $r_S \in \widetilde{W}_{112}$, then $r_S = [\ell_i]$ for some line $\ell_i \subset X$, and hence $(v^\gamma, r_S)_S \ge 0$ holds because v^γ is nef. Suppose that $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$. By Corollary 5.6, there exists $\tau \in \mathrm{Aut}(X, h_0)$ such that $h_0^{g_i \tau} = h_0 + c_i r_S$ holds, where $i = 1$ and $c_1 = 3$ if $r_S \in \widetilde{W}_{648}$ while $i = 2$ and $c_2 = 9$ if $r_S \in \widetilde{W}_{5184}$. Since $\gamma\tau^{-1} g_i \in G$, we have

$$(v^\gamma, h_0)_S \le (v^{\gamma\tau^{-1} g_i}, h_0)_S = (v^\gamma, h_0^{g_i \tau})_S = (v^\gamma, h_0)_S + c_i (v^\gamma, r_S)_S.$$

$$
\left[\begin{array}{cccccccccccccccccccccc}
1 & 1 & -1 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 4 & 0 & 4 & 2 & 2 & 0 & 0 & -2 & 4 & 2 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 4 & 0 & 4 & 2 & 2 & 0 & 0 & -1 & 4 & 2 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
4 & 2 & 3 & 3 & -1 & -2 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 \\
-3 & -1 & -4 & -3 & 4 & 2 & 3 & 2 & 2 & 0 & 0 & 0 & 3 & 1 & 1 & 2 & 2 & 0 & 1 & -1 & 0 & -1 \\
2 & 2 & 1 & 2 & 1 & -2 & 2 & 1 & 0 & 1 & 0 & -2 & 1 & 1 & -1 & 1 & 0 & -2 & -1 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 4 & 1 & 4 & 2 & 2 & 0 & 0 & -2 & 4 & 2 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 2 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & -1 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 2 & 0 & 3 & 2 & 1 & 1 & 0 & -2 & 2 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 1 \\
-2 & -1 & -2 & -2 & 1 & 2 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 2 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 2 & 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \\
4 & 5 & 1 & 4 & 3 & -1 & 2 & 1 & 1 & -1 & 0 & -2 & 3 & 1 & -1 & 1 & 1 & -2 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

TABLE 5.5. The matrix A_2

Therefore $(v^\gamma, r_S)_S \geq 0$ holds. □

The properties (1), (2), (3) of D_{S0} stated in Introduction follow from Corollaries 4.4, 4.6, 4.7 and Theorem 6.1.

Proof of Theorem 1.3. By Corollary 4.7, it is enough to show that, for any $\gamma \in \mathrm{Aut}(X)$, there exists $\gamma' \in G$ such that $h_0^{\gamma\gamma'} \in D_{S0}$ holds. Since h_0^γ is nef, this follows from Theorem 6.1. □

7. The Fermat quartic polarizations for g_1 and g_2

A polarization $h \in S$ of degree 4 is said to be a *Fermat quartic polarization* if, by choosing an appropriate basis of $H^0(X, \mathcal{L}_h)$, the morphism $\Phi_h : X \to \mathbb{P}^3$ associated with $|\mathcal{L}_h|$ induces an automorphism of $X \subset \mathbb{P}^3$. It is obvious that h_0^γ is a Fermat quartic polarization for any $\gamma \in \mathrm{Aut}(X)$. Conversely, if h is a Fermat quartic polarization, then the pull-back of h_0 by the automorphism Φ_h of X is h. Therefore the set of Fermat quartic polarizations is the orbit of h_0 by the action of $\mathrm{Aut}(X)$ on S. Consider the Fermat quartic polarizations

$$\begin{aligned} h_1 &:= h_0^{g_1} = h_0 A_1 = [0,1,0,1,2,1,1,0,2,1,-1,1,0,-1,-1,1,1,0,1,0,0,0]_S, \\ h_2 &:= h_0^{g_2} = h_0 A_2 = [1,4,-2,1,6,0,6,3,3,0,0,-3,6,3,0,3,3,-3,0,0,0,0]_S. \end{aligned}$$

Using the equalities

$$\begin{aligned} (7.1)\quad h_1 &= 6h_0 - ([\ell_3]+[\ell_6]+[\ell_8]+[\ell_{14}]+[\ell_{15}]+[\ell_{17}]+[\ell_{19}]+ \\ &\quad + [\ell_{22}]+[\ell_{31}]+[\ell_{34}]+[\ell_{63}]+[\ell_{70}]+[\ell_{79}]+[\ell_{92}]), \quad \text{and} \\ (7.2)\quad h_2 &= 15h_0 - (3\,[\ell_3]+4\,[\ell_6]+[\ell_{13}]+[\ell_{14}]+3\,[\ell_{18}]+[\ell_{22}]+ \\ &\quad + [\ell_{26}]+[\ell_{27}]+2\,[\ell_{35}]+[\ell_{44}]+2\,[\ell_{50}]+3\,[\ell_{92}]+ \\ &\quad + [\ell_{93}]+[\ell_{106}]+[\ell_{108}]+3\,[\ell_{111}]), \end{aligned}$$

we obtain another description of the involutions g_1 and g_2.

Theorem 7.1. *Let (w,x,y) be the affine coordinates of $\mathbb{P}^3$ with $z=1$, and let*

$$H_{1j}(w,x,y) = H_{1j0}(x,y) + H_{1j1}(x,y)\,w + H_{1j2}(x,y)\,w^2 + H_{1j3}(x,y)\,w^3$$

be polynomials given in Table 7.1. Then the rational map

$$(w,x,y) \mapsto [H_{10} : H_{11} : H_{12} : H_{13}] \in \mathbb{P}^3 \tag{7.3}$$

gives the involution g_1 of X.

Remark 7.2. We have a similar list of polynomials $H_{20}, H_{21}, H_{22}, H_{23}$ that gives the involution g_2. They are, however, too large to be presented in the paper (see [31]).

Proof of Theorem 7.1. We put

$$Z := \{3,6,8,14,15,17,19,22,31,34,63,70,79,92\},$$

which is the set of indices of lines on X that appear in the right-hand side of (7.1). The polynomials $H_{10}, H_{11}, H_{12}, H_{13}$ form a basis of the vector space

$$H^0(X, \mathcal{L}_{h_1}) \cong \Gamma(6, \bigcap_{i\in Z} I_i).$$

(See Section 5 for the notation.) We can easily verify that

$$H_{10}^4 + H_{11}^4 + H_{12}^4 + H_{13}^4 \equiv 0 \mod (w^4+x^4+y^4+1)$$

holds. Hence the rational map (7.3) induces an automorphism g' of X. We prove $g' = g_1$ by showing that the action g'_* of g' on S is equal to the action $v \mapsto vA_1$ of g_1. We homogenize the polynomials H_{1j} to $\tilde{H}_{1j}(w,x,y,z)$ so that g' is given by

$$[w:x:y:z] \mapsto [\tilde{H}_{10} : \tilde{H}_{11} : \tilde{H}_{12} : \tilde{H}_{13}].$$

$$
\begin{aligned}
H_{100} = {} & -1 - (1-i)\,x - (1-i)\,x^2 - (1+i)\,y^2 - i\,x^3 - (1-i)\,xy^2 + x^4 - (1-i)\,x^3y \\
& + (1+i)\,x^2y^2 + (1+i)\,xy^3 - (1-i)\,y^4 - (1-i)\,x^5 + (1+i)\,x^4y - (1-i)\,x^3y^2 - i\,x^2y^3 \\
& - i\,xy^4 + i\,y^5 + i\,x^6 - x^5y - (1+i)\,x^4y^2 - (1+i)\,x^3y^3 + (1+i)\,xy^5 + (1-i)\,y^6 \\
H_{101} = {} & (1+i) + x + (1-i)\,y - i\,x^2 - (1+i)\,xy + i\,y^2 + i\,x^3 - x^2y \\
& - i\,xy^2 + y^3 + x^3y + (1-i)\,x^2y^2 - (1-i)\,xy^3 + (1-i)\,y^4 \\
& + x^5 + i\,x^4y + x^3y^2 - i\,x^2y^3 + (1-i)\,xy^4 - (1-i)\,y^5 \\
H_{102} = {} & i + x + (1+i)\,y + x^2 + (1+i)\,y^2 + (1+i)\,x^3 - x^2y \\
& - (1+i)\,xy^2 + i\,y^3 + (1+i)\,x^4 + (1-i)\,x^2y^2 + xy^3 + (1+i)\,y^4 \\
H_{103} = {} & (1-i) - (1-i)\,x + (1-i)\,y - (1+i)\,x^2 - (1-i)\,xy + (1+i)\,x^3 - (1+i)\,y^3
\end{aligned}
$$

$$
\begin{aligned}
H_{110} = {} & -i + i\,x + y - (1+i)\,x^2 + xy - (1-i)\,y^2 - x^3 - (1-i)\,x^2y + (1+i)\,xy^2 - y^3 \\
& + (1+i)\,x^4 - i\,x^3y - (1-i)\,x^2y^2 + xy^3 + (1-i)\,y^4 - (1+i)\,x^5 + (1-i)\,x^4y + i\,x^2y^3 \\
& - (1+i)\,xy^4 - (1+i)\,y^5 - i\,x^6 + (1+i)\,x^4y^2 + (1+i)\,x^3y^3 + (1-i)\,xy^5 - (1-i)\,y^6 \\
H_{111} = {} & -(1-i) + x + (1+i)\,y - (1+i)\,x^2 - i\,xy - i\,y^2 + (1-i)\,x^3 \\
& - i\,x^2y - y^3 + (1+i)\,x^4 + (1-i)\,x^3y - x^2y^2 + (1+i)\,xy^3 \\
& - i\,y^4 + (1-i)\,x^5 + i\,x^4y + (1-i)\,x^2y^3 + (1-i)\,xy^4 - y^5 \\
H_{112} = {} & -1 + (1+i)\,y + x^2 - (1-i)\,xy - (1+i)\,y^2 - x^2y + (1+i)\,xy^2 \\
& - (1+i)\,y^3 + (1-i)\,x^4 + (1+i)\,x^3y - (1+i)\,x^2y^2 - xy^3 - (1+i)\,y^4 \\
H_{113} = {} & (1+i) - x + y + x^2 - i\,y^2 - (1-i)\,x^3 + i\,x^2y - (1-i)\,xy^2 - i\,y^3
\end{aligned}
$$

$$
\begin{aligned}
H_{120} = {} & (1+i) + (1+i)\,x + (1+i)\,y + (1-i)\,x^2 + y^2 + (1+i)\,x^3 + (1+i)\,x^2y \\
& - i\,xy^2 - y^3 - (1-i)\,x^3y + (1-i)\,x^2y^2 - (1+i)\,xy^3 + (1-i)\,y^4 \\
& + (1-i)\,x^5 - i\,x^4y + (1-i)\,x^3y^2 - (1+i)\,x^2y^3 + i\,xy^4 - y^5 + x^6 \\
& - (1+i)\,x^5y - (1-i)\,x^4y^2 + x^3y^3 - i\,x^2y^4 - (1-i)\,xy^5 + (1-i)\,y^6 \\
H_{121} = {} & i + x + xy - (1+i)\,y^2 + x^3 - (1+i)\,x^2y - (1-i)\,xy^2 + (1+i)\,y^3 + x^4 - (1-i)\,x^3y \\
& - (1-i)\,x^2y^2 + (1+i)\,xy^3 - (1-i)\,y^4 - (1-i)\,x^5 + (1+i)\,x^3y^2 + (1+i)\,x^2y^3 + (1-i)\,y^5 \\
H_{122} = {} & (1-i) - x - (1+i)\,y + i\,x^2 - (1-i)\,xy - (1+i)\,y^2 - x^3 - (1-i)\,xy^2 \\
& - i\,y^3 - (1+i)\,x^4 - (1-i)\,x^3y - (1+i)\,x^2y^2 - xy^3 + (1+i)\,y^4 \\
H_{123} = {} & 1 - (1+i)\,x + (1-i)\,y + x^2 + i\,xy + i\,y^2 - (1+i)\,x^3 + (1-i)\,x^2y - (1+i)\,xy^2 + (1+i)\,y^3
\end{aligned}
$$

$$
\begin{aligned}
H_{130} = {} & -(1-i) + i\,x + (1+i)\,y - (1+i)\,x^2 + (1-i)\,xy + (1-i)\,y^2 + x^3 - (1+i)\,x^2y + i\,xy^2 + i\,y^3 \\
& - (1+i)\,x^4 + i\,x^3y + x^2y^2 - (1+i)\,y^4 + (1+i)\,x^5 - (1-i)\,x^4y + (1-i)\,x^3y^2 - x^2y^3 \\
& - (1+i)\,y^5 - (1+i)\,x^6 - (1-i)\,x^5y - (1+i)\,x^4y^2 + i\,x^3y^3 + i\,x^2y^4 + i\,xy^5 + (1+i)\,y^6 \\
H_{131} = {} & -1 - x + (1+i)\,y - (1-i)\,x^2 + (1+i)\,xy - i\,y^2 - (1+i)\,x^3 - i\,x^2y - xy^2 + i\,y^3 \\
& - x^4 - x^3y + xy^3 - (1+i)\,y^4 - (1+i)\,x^5 + x^4y + (1-i)\,x^3y^2 - i\,x^2y^3 + (1+i)\,xy^4 \\
H_{132} = {} & (1+i) + i\,x + y - x^2 + xy + y^2 + i\,x^3 - (1-i)\,x^2y \\
& - (1+i)\,xy^2 - (1-i)\,x^4 - x^2y^2 - i\,xy^3 - (1-i)\,y^4 \\
H_{133} = {} & i - y + x^2 + (1+i)\,xy - (1-i)\,y^2
\end{aligned}
$$

TABLE 7.1. Polynomials H_{1j}

Let ℓ_k be a line on X whose index k is not in Z. We calculate a parametric representation

$$[u:v] \mapsto [l_{k0}:l_{k1}:l_{k2}:l_{k3}]$$

of ℓ_k in $\mathbb{P}^3$, where u, v are homogeneous coordinates of $\mathbb{P}^1$ and $l_{k\nu}$ are homogeneous linear polynomials of u, v. We put

$$L^{(k)}_{1j} := \tilde{H}_{1j}(l_{k0}, l_{k1}, l_{k2}, l_{k3})$$

for $j = 0, \dots, 3$, which are homogeneous polynomials of u, v. Let $M^{(k)}$ be the greatest common divisor of $L^{(k)}_{10}, L^{(k)}_{11}, L^{(k)}_{12}, L^{(k)}_{13}$ in $\mathbb{F}_9[u, v]$. Then

$$\rho_k \quad : \quad [u:v] \mapsto [L^{(k)}_{10}/M^{(k)} : L^{(k)}_{11}/M^{(k)} : L^{(k)}_{12}/M^{(k)} : L^{(k)}_{13}/M^{(k)}]$$

is a parametric representation of the image of ℓ_k by g'. (If $k \in Z$, then $L^{(k)}_{1j}$ are constantly equal to 0.) Pulling back the defining homogeneous ideal of $\ell_{k'}$ by ρ_k, we can calculate the intersection number $([\ell_k]^{g'}, [\ell_{k'}])_S$. Since the classes $[\ell_k]$ with $k \notin Z$ span $S \otimes \mathbb{Q}$, we can calculate the action g'_* of g' on S, which turns out to be equal to $v \mapsto vA_1$. □

Remark 7.3. The polynomials $H_{10}, H_{11}, H_{12}, H_{13}$ are found by the following method. Let H'_0, H'_1, H'_2, H'_3 be an arbitrary basis of $\Gamma(6, \cap_{i\in Z} I_i) \cong H^0(X, \mathcal{L}_{h_1})$. Then the normal forms of the quartic monomials of H'_0, H'_1, H'_2, H'_3 are subject to a linear relation of the following form (see [24, n. 3] or [30, Theorem 6.11]):

$$\sum_{i,j=0}^{3} a_{ij} \overline{H'_i H'^3_j} = 0,$$

where the coefficients $a_{ij} \in \mathbb{F}_9$ satisfy $a_{ji} = a^3_{ij}$ and $\det(a_{ij}) \neq 0$; that is, the matrix (a_{ij}) is non-singular Hermitian. We search for $B \in \mathrm{GL}_3(\mathbb{F}_9)$ such that

$$(a_{ij}) = B\ {}^t B^{(3)}$$

holds, where $B^{(3)}$ is obtained from B by applying $x \mapsto x^3$ to the entries, and put

$$(H''_0, H''_1, H''_2, H''_3) = (H'_0, H'_1, H'_2, H'_3)B.$$

Then $H''_0, H''_1, H''_2, H''_3$ satisfy

$$H''^4_0 + H''^4_1 + H''^4_2 + H''^4_3 \equiv 0 \mod (w^4 + x^4 + y^4 + 1).$$

Therefore $(w, x, y) \mapsto [H''_0 : H''_1 : H''_2 : H''_3]$ induces an automorphism g'' of X. Using the method described in the proof of Theorem 7.1, we calculate the matrix A'' such that the action g''_* of g'' on S is given by $v \mapsto vA''$. Next we search for $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$ such that $A''T_\tau$ is equal to A_1, where $T_\tau \in \mathrm{O}^+(S)$ is the matrix representation of τ. Then the polynomials

$$(H_{10}, H_{11}, H_{12}, H_{13}) := (H''_0, H''_1, H''_2, H''_3)\tau$$

have the required property.

$$
\begin{bmatrix}
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&-1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&-1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&-1&-1&-2&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
2&1&1&2&-1&-2&-1&-1&-1&0&1&0&0&0&-1&0&0&0&0&1&0&0\\
0&0&0&0&1&1&0&1&1&0&-1&0&0&-1&0&0&0&0&0&-1&0&0\\
2&2&2&2&-1&0&-1&-1&0&-1&0&0&-1&0&0&-1&0&0&0&0&-1&0\\
0&1&1&0&0&1&1&0&0&0&-1&-1&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
2&1&1&2&0&0&-1&0&0&-1&0&1&0&-1&0&-1&-1&0&0&0&-1&0\\
0&0&0&0&0&0&1&0&-1&0&0&-1&0&1&0&1&0&-1&0&0&1&0\\
-3&-2&-2&-3&1&1&1&1&1&1&0&0&1&0&1&1&1&1&1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
4&3&3&4&-1&-1&-1&-1&-1&-1&-1&-1&-1&-1&-1&-1&-1&-1&0&0&0&1
\end{bmatrix}
$$

TABLE 8.1. Frobenius action on S

Remark 7.4. We have calculated the images of the $\mathbb{F}_9$-rational points of X by the morphisms $\psi_i : X \to Y_i$ and $g_i : X \to X$, and confirmed that they are compatible (see [31]).

8. GENERATORS OF $\mathrm{O}^+(S)$

Let $F \in \mathrm{O}^+(S)$ denote the isometry of S obtained from the Frobenius action ϕ of $\mathbb{F}_9$ over $\mathbb{F}_3$ on X. Calculating the action of ϕ on the lines ($\ell_1^\phi = \ell_6, \ell_2^\phi = \ell_5, \ell_3^\phi = \ell_8, \ell_4^\phi = \ell_7, \dots$), we see that F is given $v \mapsto vA_F$, where A_F is the matrix presented in Table 8.1. Since $h_0^F = h_0$, we have $D_{S0}^F = D_{S0}$ by Corollary 4.6.

Proposition 8.1. *The automorphism group* $\mathrm{Aut}(D_{S0}) \subset \mathrm{O}^+(S)$ *of* D_{S0} *is the split extension of* $\langle F \rangle \cong \mathbb{Z}/2\mathbb{Z}$ *by* $\mathrm{Aut}(X, h_0)$.

Proof. Since we have calculated the representation (3.2) of $\mathrm{Aut}(X, h_0)$ into $\mathrm{O}^+(S)$, we can verify that $F \notin \mathrm{Aut}(X, h_0)$. Therefore it is enough to show that the order of $\mathrm{Aut}(D_{S0})$ is equal to 2 times $|\mathrm{Aut}(X, h_0)|$. Since $|\mathrm{PGU}_4(\mathbb{F}_9)|$ is equal to 4 times $|\mathrm{PSU}_4(\mathbb{F}_9)|$, this follows from [13, Lemma 2.1] (see also [7, p. 52]). $\square$

Since $([\ell_1], [\ell_1])_S = -2$, the reflection $s_1 : S \otimes \mathbb{R} \to S \otimes \mathbb{R}$ into the hyperplane $([\ell_1])_S^{\perp}$ is contained in $\mathrm{O}^+(S)$. In the same way as the proof of Theorem 1.3, we obtain the following:

Theorem 8.2. *The autochronous orthogonal group* $\mathrm{O}^+(S)$ *of the Néron-Severi lattice* S *of* X *is generated by* $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$, g_1, g_2, F *and* s_1.

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