VANISHING CYCLES, THE GENERALIZED HODGE CONJECTURE AND GRÖBNER BASES

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Abstract. Let X be a general complete intersection of a given multi-degree in a complex projective space. Suppose that the anti-canonical line bundle of X is ample. Using the cylinder homomorphism associated with the family of complete intersections contained in X, we prove that the vanishing cycles in the middle homology group of X are represented by topological cycles whose support is contained in a proper Zariski closed subset $T \subset X$ of certain codimension. In some cases, we can find such a Zariski closed subset T with codimension equal to the upper bound obtained from the Hodge structure of the middle cohomology group of X by means of Gröbner bases. Hence a consequence of the generalized Hodge conjecture is verified in these cases.

1. Introduction

There are only few non-trivial examples that can be used as supporting evidence for the generalized Hodge conjecture formulated by Grothendieck [8]. In this paper, we deal with complete intersections of small multi-degrees in a complex projective space, and prove, in some cases, a consequence of the generalized Hodge conjecture for these complete intersections by means of cylinder homomorphisms.

We work over the complex number field $\mathbb{C}$. Let X be a general complete intersection of multi-degree $\mathbf{a} = (a_1, \dots, a_r)$ in $\mathbb{P}^n$ with $\min(\mathbf{a}) \geq 2$. Suppose that X is Fano, that is, the total degree $\sum_{i=1}^r a_i$ of X is less than or equal to n. We put

$$m := \dim X = n - r \quad \text{and} \quad k := \Big[\frac{1}{\max(\mathbf{a})}\Big(n - \sum_{i=1}^{r} a_i\Big)\Big] + 1,$$

where [ ] denotes the integer part. It is known that the Hodge structure of the middle cohomology group $H^m(X, \mathbb{Q})$ of X satisfies the following ([6, Exposé XI, Corollaire 2.8]):

$$H^{\nu, m-\nu}(X) = 0 \iff 0 \leq \nu < k \ \text{ or } \ 0 \leq m - \nu < k. \tag{1.1}$$

If the generalized Hodge conjecture is true, then there should exist a Zariski closed subset T of X with codimension k such that the inclusion $T \hookrightarrow X$ induces a surjective homomorphism $H_m(T, \mathbb{Q}) \twoheadrightarrow H_m(X, \mathbb{Q})$.

We will try to verify this consequence of the generalized Hodge conjecture by means of *cylinder homomorphisms*. Let $\mathbf{b} = (b_1, \dots, b_s)$ be another sequence of integers satisfying $\min(\mathbf{b}) \geq 1$ and $r < s < n$. We denote by $F_{\mathbf{b}}(X)$ the scheme parameterizing all complete intersections of multi-degree $\mathbf{b}$ in $\mathbb{P}^n$ that are contained

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in X, and by $Z_{\mathbf{b}}(X) \subset X \times F_{\mathbf{b}}(X)$ the universal family with

$$\begin{array}{ccc} Z_{\mathbf{b}}(X) & \xrightarrow{\alpha_X} & X \\ {\scriptstyle \pi_X}\downarrow & & \\ F_{\mathbf{b}}(X) & & \end{array} \tag{1.2}$$

being the diagram of the projections. We put $l := n - s$. Suppose that $F_{\mathbf{b}}(X)$ is non-empty, and that $m > 2l$ holds. Since π_X is proper and flat of relative dimension l, we have a homomorphism

$$H_{m-2l}(F_{\mathbf{b}}(X), \mathbb{Z}) \rightarrow H_m(Z_{\mathbf{b}}(X), \mathbb{Z})$$

that maps a homology class $[\tau] \in H_{m-2l}(F_{\mathbf{b}}(X), \mathbb{Z})$ represented by a topological $(m - 2l)$-cycle τ in $F_{\mathbf{b}}(X)$ to the homology class $[\pi_X^{-1}(\tau)] \in H_m(Z_{\mathbf{b}}(X), \mathbb{Z})$ represented by the topological m-cycle $\pi_X^{-1}(\tau)$ in $Z_{\mathbf{b}}(X)$. We define a homomorphism

$$\psi_{\mathbf{b}}(X) : H_{m-2l}(F_{\mathbf{b}}(X), \mathbb{Z}) \rightarrow H_m(X, \mathbb{Z})$$

by $\psi_{\mathbf{b}}(X)([\tau]) := \alpha_{X*}([\pi_X^{-1}(\tau)])$, and call $\psi_{\mathbf{b}}(X)$ *the cylinder homomorphism associated with the family* $\pi_X : Z_{\mathbf{b}}(X) \rightarrow F_{\mathbf{b}}(X)$.

It was remarked in [18] that there exists a Zariski closed subset T of X with codimension $\geq l$ such that the image of the homomorphism $H_m(T, \mathbb{Q}) \rightarrow H_m(X, \mathbb{Q})$ induced from the inclusion $T \hookrightarrow X$ contains $\operatorname{Im} \psi_{\mathbf{b}}(X) \otimes \mathbb{Q}$. (See also Corollary 5.4 of this paper.) Therefore, in view of the generalized Hodge conjecture, it is an interesting problem to find a sequence $\mathbf{b}$ with l as large as possible (hopefully $l = k$) such that the cylinder homomorphism $\psi_{\mathbf{b}}(X)$ has a "big" image.

Our Main Theorem, which will be stated in §2, gives us a sufficient condition on $(n, \mathbf{a}, \mathbf{b})$ for the image of $\psi_{\mathbf{b}}(X)$ to contain the *module of vanishing cycles*

$$V_m(X, \mathbb{Z}) := \operatorname{Ker}(H_m(X, \mathbb{Z}) \rightarrow H_m(\mathbb{P}^n, \mathbb{Z})).$$

This sufficient condition can be checked by means of Gröbner bases. Combining Main Theorem with a theorem of Debarre and Manivel [5, Théorème 2.1] about the variety of linear subspaces contained in a general complete intersection, we also give a simple numerical condition on $(n, \mathbf{a}, \mathbf{b})$ that is sufficient for $\operatorname{Im} \psi_{\mathbf{b}}(X) \supseteq V_m(X, \mathbb{Z})$ to hold (Theorem 7.2). In many cases, our method yields $\mathbf{b}$ with l larger than any previously known results, and sometimes we can verify the consequence of the generalized Hodge conjecture. See §8 for the examples.

After the work of Clemens and Griffiths [2] on the family of lines in a cubic threefold, many authors have studied the cylinder homomorphisms of type $\psi_{\mathbf{b}}(X)$, and proved that the image contains the vanishing cycles ([1], [3], [4], [10], [11], [12], [13], [14], [15], [16], [19], [21]). Our method provides us with a unified proof and a generalization of these results.

This paper is organized as follows. In §2, we state Main Theorem. In §3, we study a connection between vanishing cycles and cylinder homomorphisms in general setting. Theorem 3.1 in this section is essentially same as the result of [17]. However we present a complete and improved proof for readers' convenience. In §4, we construct the universal family of the families $Z_{\mathbf{b}}(X) \rightarrow F_{\mathbf{b}}(X)$ over the scheme parameterizing all complete intersections of multi-degree $\mathbf{a}$ in $\mathbb{P}^n$, which is a Zariski open subset of a Hilbert scheme, and studies its properties. Combining the results in §3 and §4, we prove Main Theorem in §5. In §6, we explain a method for checking the conditions on $(n, \mathbf{a}, \mathbf{b})$ required by Main Theorem by means of Gröbner bases.

In §7, an application of the theorem of Debarre and Manivel is presented. Examples are investigated in relation to the generalized Hodge conjecture in §8.

Conventions. (1) We work over $\mathbb{C}$. A point of a scheme means a $\mathbb{C}$-valued point unless otherwise stated. (2) For an analytic space X or a scheme X over $\mathbb{C}$, let T_pX denote the Zariski tangent space to X at a point p of X. (3) The multi-degree of a complete intersection is always denoted in the *non-decreasing* order.

2. Statement of Main Theorem

We fix an integer $n \geq 4$. Let $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_s)$ be sequences of integers satisfying

$$2 \leq a_1 \leq \dots \leq a_r, \quad 1 \leq b_1 \leq \dots \leq b_s \quad \text{and} \quad r < s < n. \tag{2.1}$$

We put

$$m := n - r \quad \text{and} \quad l := n - s.$$

We denote by $H_{n,\mathbf{a}}$ the scheme parameterizing all complete intersections of multi-degree $\mathbf{a}$ in $\mathbb{P}^n$. For a point t of $H_{n,\mathbf{a}}$, we denote by X_t the corresponding complete intersection. Let $S_{n,\mathbf{a}}$ denote the Zariski closed subset of $H_{n,\mathbf{a}}$ parameterizing all singular complete intersections. It is well-known that $S_{n,\mathbf{a}}$ is an irreducible hypersurface of $H_{n,\mathbf{a}}$, and that, if u is a general point of $S_{n,\mathbf{a}}$, then X_u has only one singular point p. For $t \in H_{n,\mathbf{a}}$, we denote by $F_{\mathbf{b}}(X_t)$ the scheme parameterizing all complete intersections of multi-degree $\mathbf{b}$ in $\mathbb{P}^n$ that are contained in X_t as subschemes. If $m > 2l$ and $F_{\mathbf{b}}(X_t) \neq \emptyset$, then we have the cylinder homomorphism

$$\psi_{\mathbf{b}}(X_t) \;:\; H_{m-2l}(F_{\mathbf{b}}(X_t), \mathbb{Z}) \;\to\; H_m(X_t, \mathbb{Z}).$$

We put $t_a := \mathrm{Card}\{\, i \mid a_i = a_r \,\}$ and $t_b := \mathrm{Card}\{\, j \mid b_j = a_r \,\}$.

Main Theorem. *Suppose that the following inequalities are satisfied*:

$$a_i \geq b_i \quad (i = 1, \dots, r), \quad a_r \geq b_s, \quad \textit{and} \tag{2.2}$$

$$m - 2l \geq t_b - t_a, \quad m > 2l. \tag{2.3}$$

Suppose also that, for a general point u of $S_{n,\mathbf{a}}$, there exists a complete intersection of multi-degree $\mathbf{b}$ in $\mathbb{P}^n$ that is contained in X_u, passing through the unique singular point p of X_u, and smooth at p. Then, for a general point t of $H_{n,\mathbf{a}}$, the scheme $F_{\mathbf{b}}(X_t)$ is non-empty, and the image of the cylinder homomorphism $\psi_{\mathbf{b}}(X_t)$ contains the module of vanishing cycles $V_m(X_t, \mathbb{Z})$.

Remark 2.1. In Proposition 4.15, we will give several conditions equivalent to the second condition of Main Theorem. One of them can be tested easily by means of Gröbner bases, as will be explained in §6.

3. Vanishing cycles and a cylinder homomorphism

In this section, we work in the category of complex analytic spaces and holomorphic maps. We study in general setting the problem when the image of a cylinder homomorphism contains a given vanishing cycle. For the detail of the classical theory of vanishing cycles, we refer to [9].

Let $\varphi : Y \to \Delta$ be a proper surjective holomorphic map from a smooth irreducible complex analytic space of dimension $m + 1 \geq 2$ to the open unit disk $\Delta \subset \mathbb{C}$. For a point $a \in \Delta$, we denote by Y_a the fiber $\varphi^{-1}(a)$. Suppose that φ has only one critical point p, that p is on the central fiber Y_0, and that the Hessian

$$H : T_pY \times T_pY \to \mathbb{C}$$

of φ at p is non-degenerate. We put $\Delta^\times := \Delta \setminus \{0\}$. For any $\varepsilon \in \Delta^\times$, the kernel of the homomorphism $H_m(Y_\varepsilon, \mathbb{Z}) \to H_m(Y, \mathbb{Z})$ induced from the inclusion $Y_\varepsilon \hookrightarrow Y$ is generated by the *vanishing cycle* $[\Sigma_\varepsilon] \in H_m(Y_\varepsilon, \mathbb{Z})$ associated to the non-degenerate critical point p of φ.

Let $\varrho : F \to \Delta$ be a surjective holomorphic map from a smooth irreducible complex analytic space F of dimension k to the unit disk, and let W be a reduced closed analytic subspace of $Y \times_\Delta F$ such that the projection $\varpi : W \to F$ is flat of relative dimension $l > 0$. Since φ is proper, so is ϖ. Let $\gamma : W \to Y$ be the projection onto the first factor. We obtain the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\gamma} & Y \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ F & \xrightarrow[\varrho]{} & \Delta. \end{array} \tag{3.1}$$

For $u \in F$, the fiber $\varpi^{-1}(u)$ can be regarded as a closed l-dimensional analytic subspace of $Y_{\varrho(u)}$ by γ. For $a \in \Delta$, we put $F_a := \varrho^{-1}(a)$ and $W_a := \varpi^{-1}(F_a)$. Then we obtain a family of l-dimensional closed analytic subspaces of Y_a:

$$\begin{array}{ccc} W_a & \longrightarrow & Y_a \\ {\scriptstyle\varpi_a}\downarrow & & \\ F_a. & & \end{array} \tag{3.2}$$

Since the restriction $\varpi_a : W_a \to F_a$ of ϖ to W_a is proper and flat of relative dimension l, we have the cylinder homomorphism

$$\psi_a : H_{m-2l}(F_a, \mathbb{Z}) \to H_m(Y_a, \mathbb{Z})$$

associated with the family (3.2) for any $a \in \Delta$.

Theorem 3.1. *We assume $m > 2l > 0$.*

(1) *Suppose that there exists a point q of W_0 such that $\gamma(q)$ is the critical point p of φ, that ϖ is smooth at q, and that γ is an immersion at q. Then $k = \dim F$ is less than or equal to $m - 2l + 1$.*

(2) *Suppose moreover that $k = m - 2l + 1$ holds. Then $\varpi(q)$ is a critical point of ϱ, and the Hessian of ϱ at $\varpi(q)$ is non-degenerate. Let ε be a point of $\Delta^\times$ with $|\varepsilon|$ small enough, and let $[\sigma_\varepsilon] \in H_{m-2l}(F_\varepsilon, \mathbb{Z})$ be the vanishing cycle associated to the non-degenerate critical point $\varpi(q)$ of ϱ. If the vanishing cycle $[\Sigma_\varepsilon] \in H_m(Y_\varepsilon, \mathbb{Z})$ is not a torsion element, then $\psi_\varepsilon([\sigma_\varepsilon])$ is equal to $[\Sigma_\varepsilon]$ up to sign.*

Proof. (1) Let $U_{W,q}$ be a small open connected neighborhood of q in W. We can assume that ϖ is smooth at every point of $U_{W,q}$, and that γ embeds $U_{W,q}$ into Y. We put

$$o := \varpi(q) \quad \text{and} \quad Z := \varpi^{-1}(o).$$

Then $\gamma(U_{W,q} \cap Z)$ and $\gamma(U_{W,q})$ are smooth locally closed analytic subsets of Y passing through p. Let T_1 and T_2 be the Zariski tangent spaces to $\gamma(U_{W,q} \cap Z)$ and $\gamma(U_{W,q})$ at p, respectively. We have $T_1 \subseteq T_2 \subseteq T_pY$ and $\dim T_1 = l$, $\dim T_2 = k + l$. We will show that T_1 and T_2 are orthogonal with respect to the Hessian H of φ at p. Let v be an arbitrary vector of T_1. Since the structure $\varpi | U_{W,q} : U_{W,q} \to F$ of the smooth fibration on $U_{W,q}$ is carried over to $\gamma(U_{W,q})$, there exists a holomorphic vector field $\tilde{v}$ defined in a small open neighborhood $U_{Y,p}$ of p in Y such that $\tilde{v}_p$ is equal to v, and that, if $q' \in U_{W,q}$ satisfies $\gamma(q') \in U_{Y,p}$, then $\tilde{v}_{\gamma(q')}$ is tangent to the smooth locally closed analytic subset $\gamma(U_{W,q} \cap \varpi^{-1}(\varpi(q')))$ of Y. Since the diagram (3.1) is commutative, the function φ is constant on $\gamma(\varpi^{-1}(\varpi(q')))$ for any $q' \in U_{W,q}$, and hence the holomorphic function $\tilde{v}(\varphi)$ is constantly zero on $\gamma(U_{W,q}) \cap U_{Y,p}$, which means that the following holds for any $w \in T_2$:

$$H(w, v) := w(\tilde{v}(\varphi)) = 0.$$

Thus T_1 is contained in the orthogonal complement $T_2^{\perp}$ of T_2 with respect to H. Since H is non-degenerate, we have

$$l = \dim T_1 \leq \dim T_pY - \dim T_2 = (m+1) - (k+l).$$

Therefore we obtain $k \leq m + 1 - 2l$.

(2) From now on, we assume $k = m + 1 - 2l$. Then we have $T_1 = T_2^{\perp}$. Hence H induces a non-degenerate symmetric bilinear form

$$H' : T_2/T_1 \times T_2/T_1 \to \mathbb{C}.$$

Since ϖ is smooth at q, there is a local holomorphic section $s : U_{F,o} \to W$ of ϖ defined in a small open neighborhood $U_{F,o}$ of $o = \varpi(q)$ in F such that $s(o) = q$. We take $U_{F,o}$ so small that $s(U_{F,o}) \subset U_{W,q}$ holds. Let S be the image of $\gamma \circ s$, which is a smooth locally closed analytic subset of Y passing through p, and let T_3 be the Zariski tangent space to S at p. We have $T_2 = T_1 \oplus T_3$. It follows from the non-degeneracy of H' that the restriction $H|T_3 : T_3 \times T_3 \to \mathbb{C}$ of H to T_3 is also non-degenerate. Since $\gamma \circ s$ yields an isomorphism from $U_{F,o}$ to S, and ϱ coincides on $U_{F,o}$ with

$$U_{F,o} \xrightarrow{\gamma \circ s} S \xrightarrow{\varphi|S} \Delta,$$

the point o is a critical point of ϱ. Moreover, the Hessian of ϱ at o is equal to $H|T_3$ via the isomorphism $(d\,(\gamma \circ s))_o : T_oF \xrightarrow{\sim} T_3$, and hence is non-degenerate.

We will describe the holomorphic maps in the diagram (3.1) in terms of local coordinates. Let t be the coordinate on Δ. There exist local analytic coordinates $x = (x_1, \ldots, x_k)$ on F with the center o such that ϱ is given by

$$\varrho^* t = x_1^2 + \cdots + x_k^2. \tag{3.3}$$

Since ϖ is smooth at q, there exists a local analytic coordinate system $(w, w') = (w_1, \ldots, w_k, w'_1, \ldots, w'_l)$ on W with the center q such that ϖ is given by

$$\varpi^* x_i = w_i \qquad (i = 1, \ldots, k). \tag{3.4}$$

Since γ is an immersion at q, there exist local analytic coordinates $(y, y', y'') = (y_1, \ldots, y_k, y'_1, \ldots, y'_l, y''_1, \ldots, y''_l)$ on Y with the center p such that γ is given by

$$\begin{cases} \gamma^* y_i = w_i & (i = 1, \ldots, k), \\ \gamma^* y'_j = w'_j & (j = 1, \ldots, l), \\ \gamma^* y''_j = 0 & (j = 1, \ldots, l). \end{cases} \tag{3.5}$$

(Note that $\dim Y$ is equal to $m + 1 = k + 2l$.) Then the locally closed analytic subset $\gamma(U_{W,q})$ of Y is defined by $y''_1 = \cdots = y''_l = 0$ locally around p. From the commutativity of the diagram (3.1), it follows that $\varphi^* t$ and $y_1^2 + \cdots + y_k^2$ coincide on $\gamma(U_{W,q})$. Therefore, in a small neighborhood of p, the function $\varphi^* t$ is written as follows:

$$y_1^2 + \cdots + y_k^2 + a_1 y_1'' + \cdots + a_l y_l'',$$

where $a = (a_1, \dots, a_l)$ is a system of holomorphic functions defined locally around p. Since p is a critical point of φ, we have $a_1(p) = \cdots = a_l(p) = 0$. The non-degeneracy of the Hessian H of φ at p implies that the $l \times l$ matrix $\big(\partial a_i / \partial y_j'(p)\big)_{i,j=1,\dots,l}$ is non-degenerate. Hence (y, a, y'') is another local analytic coordinate system on Y with the center p. We replace y' with a. Then we have

$$\varphi^* t = y_1^2 + \cdots + y_k^2 + y_1' y_1'' + \cdots + y_l' y_l''. \tag{3.6}$$

We can make coordinate transformation on w' according to the coordinate transformation on y' so that (3.5) remains valid. We put

$$\begin{cases} z_i := y_i & (i = 1, \dots, k), \\ z_{k+j} := (y_j' + y_j'')/2 & (j = 1, \dots, l), \\ z_{k+l+j} := \sqrt{-1}\,(y_j' - y_j'')/2 & (j = 1, \dots, l). \end{cases} \tag{3.7}$$

Then we have

$$\varphi^* t = z_1^2 + \cdots + z_{m+1}^2. \tag{3.8}$$

Let η be a sufficiently small positive real number, and let B_η be the closed ball in Y defined by

$$|z_1|^2 + \cdots + |z_{m+1}|^2 \le \eta.$$

Let ε be a positive real number that is small enough compared with η. Let s be a real number satisfying $0 < s \le \varepsilon$. The closed subset

$$Y_s \cap B_\eta \;=\; \{\, (z_1, \dots, z_{m+1}) \mid |z_1|^2 + \cdots + |z_{m+1}|^2 \le \eta,\ z_1^2 + \cdots + z_{m+1}^2 = s \,\}$$

of $Y_s = \varphi^{-1}(s)$ is homeomorphic to the total space

$$E := \{\ (u, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \ \mid\ \|u\| = 1,\ \|v\| \le 1,\ u \perp v\ \}$$

of the unit disk tangent bundle $\tau : E \to S^m$ of the m-dimensional sphere $S^m := \{u \in \mathbb{R}^{m+1} \mid \|u\| = 1\}$, where the projection τ is given by $\tau(u, v) = u$. We identify S^m with the zero section of $\tau : E \to S^m$. The homeomorphism $h_s : Y_s \cap B_\eta \xrightarrow{\sim} E$ is written explicitly as follows:

$$u = \frac{\mathfrak{Re}(z)}{\|\,\mathfrak{Re}(z)\|}, \qquad v = \sqrt{\frac{2}{\eta - s}}\,\mathfrak{Im}(z). \tag{3.9}$$

Its inverse $h_s^{-1} : E \xrightarrow{\sim} Y_s \cap B_\eta$ is given by the following:

$$z = \sqrt{s + \Big(\frac{\eta - s}{2}\Big)\|v\|^2} \cdot u + \sqrt{-\Big(\frac{\eta - s}{2}\Big)} \cdot v. \tag{3.10}$$

The sphere $S^m \subset E$ is mapped by h_s^{-1} to the closed submanifold

$$\Sigma_s := \left\{ \ (z_1, \dots, z_{m+1}) \in Y \ \middle|\ \begin{array}{l} z_1^2 + \cdots + z_{m+1}^2 = s, \\ \mathfrak{Im}(z_i) = 0 \ \ (i = 1, \dots, m+1) \end{array} \right\}$$

of Y_s. With an orientation, this topological m-cycle Σ_s represents the vanishing cycle $[\Sigma_s] \in H_m(Y_s, \mathbb{Z})$, which generates the kernel of the homomorphism $H_m(Y_s, \mathbb{Z}) \to H_m(Y, \mathbb{Z})$ induced from $Y_s \hookrightarrow Y$.

For $s \in (0, \varepsilon]$, let σ_s denote the $(m-2l)$-dimensional sphere contained in $F_s = \varrho^{-1}(s)$ defined by

$$\sigma_s := \left\{ (x_1, \dots, x_k) \in F \;\middle|\; \begin{array}{l} x_1^2 + \cdots + x_k^2 = s, \\ \mathfrak{Im}(x_i) = 0 \;\; (i = 1, \dots, k) \end{array} \right\}.$$

With an orientation, this topological $(m-2l)$-cycle σ_s represents the vanishing cycle $[\sigma_s] \in H_{m-2l}(F_s, \mathbb{Z})$ associated to the non-degenerate critical point o of ϱ. Since ϖ is proper and flat of relative dimension l, the inverse image $\varpi^{-1}(\sigma_s)$ of the oriented sphere σ_s can be considered as a topological m-cycle in $W_s = \varpi^{-1}(F_s)$. The image $\psi_\varepsilon([\sigma_\varepsilon])$ of $[\sigma_\varepsilon] \in H_{m-2l}(F_\varepsilon, \mathbb{Z})$ by the cylinder homomorphism $\psi_\varepsilon : H_{m-2l}(F_\varepsilon, \mathbb{Z}) \to H_m(Y_\varepsilon, \mathbb{Z})$ is represented by the topological m-cycle

$$\gamma | \varpi^{-1}(\sigma_\varepsilon) \,:\, \varpi^{-1}(\sigma_\varepsilon) \;\to\; Y_\varepsilon.$$

Since the sphere σ_ε bounds an $(m-2l+1)$-dimensional closed ball in F, the topological m-cycle $\gamma|\varpi^{-1}(\sigma_\varepsilon)$ is a boundary of a topological $(m+1)$-chain in Y; that is, $\psi_\varepsilon([\sigma_\varepsilon])$ belongs to the kernel of $H_m(Y_\varepsilon, \mathbb{Z}) \to H_m(Y, \mathbb{Z})$. Hence there exists an integer c such that the following holds in $H_m(Y_\varepsilon, \mathbb{Z})$:

$$\psi_\varepsilon([\sigma_\varepsilon]) = c\,[\Sigma_\varepsilon]. \tag{3.11}$$

We will show that, if $[\Sigma_\varepsilon]$ is not a torsion element in $H_m(Y_\varepsilon, \mathbb{Z})$, then c is ± 1.

We put

$$Y_{[0,\varepsilon]} := \varphi^{-1}([0, \varepsilon]) = \bigcup_{s \in [0,\varepsilon]} Y_s.$$

For any closed subset A of $Y_{[0,\varepsilon]}$, we set

$$A^\sharp := A \setminus (A \cap B_\eta^\circ), \quad A^\flat := A \cap B_\eta \quad \text{and} \quad \partial^B A := A \cap \partial B_\eta,$$

where B_η° is the interior of the closed ball B_η, and ∂B_η is the boundary of B_η. The sharp $\sharp$ means "outside the ball", and the flat $\flat$ means "inside the ball". The explicit descriptions (3.9) and (3.10) of the homeomorphism $h_s : Y_s^\flat \xrightarrow{\sim} E$ for $s \in (0, \varepsilon]$ show that the restriction $h_s | \partial^B Y_s : \partial^B Y_s \xrightarrow{\sim} \partial E$ of h_s to $\partial^B Y_s$ can be extended to a homeomorphism from

$$\partial^B Y_0 = \{\ (z_1, \dots, z_{m+1}) \ \mid\ |z_1|^2 + \cdots + |z_{m+1}|^2 = \eta,\ z_1^2 + \cdots + z_{m+1}^2 = 0\ \}$$

to $\partial E = \{(u, v) \in E \mid \|v\| = 1\}$ smoothly. We denote these homeomorphisms by

$$\partial^B h_s \,:\, \partial^B Y_s \xrightarrow{\sim} \partial E \quad (s \in [0, \varepsilon]).$$

The homeomorphism $\partial^B h_0 : \partial^B Y_0 \xrightarrow{\sim} \partial E$ is given by the following:

$$u = \sqrt{2/\eta}\ \mathfrak{Re}(z), \quad v = \sqrt{2/\eta}\ \mathfrak{Im}(z), \quad \text{and} \quad z = \sqrt{\eta/2}\,(u + \sqrt{-1}v).$$

Putting these homeomorphisms $\partial^B h_s$ $(s \in [0, \varepsilon])$ together, we obtain a trivialization

$$\partial^B h \,:\, \partial^B Y_{[0,\varepsilon]} \xrightarrow{\sim} \partial E \times [0, \varepsilon]$$

of the restriction $\varphi | \partial^B Y_{[0,\varepsilon]} : \partial^B Y_{[0,\varepsilon]} \to [0, \varepsilon]$ of φ to $\partial^B Y_{[0,\varepsilon]}$ over $[0, \varepsilon]$. Let

$$\partial^B f \,:\, \partial^B Y_{[0,\varepsilon]} \xrightarrow{\sim} \partial^B Y_\varepsilon \times [0, \varepsilon]$$

be the trivialization of $\varphi | \partial^B Y_{[0,\varepsilon]}$ obtained by composing $\partial^B h$ and $(\partial^B h_\varepsilon \times \mathrm{id})^{-1}$. Since the only critical point p of φ is not contained in $Y^\sharp_{[0,\varepsilon]}$, we can show by Ehresmann's fibration theorem for the manifolds with boundaries that the trivialization

$\partial^B f$ extends to a trivialization

$$(f^\sharp, \partial^B f) : (Y^\sharp_{[0,\varepsilon]}, \partial^B Y_{[0,\varepsilon]}) \xrightarrow{\sim} (Y^\sharp_\varepsilon, \partial^B Y_\varepsilon) \times [0, \varepsilon] \tag{3.12}$$

of $\varphi | Y^\sharp_{[0,\varepsilon]} : Y^\sharp_{[0,\varepsilon]} \to [0, \varepsilon]$ in such a way that the restriction of $(f^\sharp, \partial^B f)$ to the fiber over ε is the identity map. For $s \in [0, \varepsilon]$, let

$$(f^\sharp_s, \partial^B f_s) : (Y^\sharp_s, \partial^B Y_s) \xrightarrow{\sim} (Y^\sharp_\varepsilon, \partial^B Y_\varepsilon)$$

denote the restriction of $(f^\sharp, \partial^B f)$ to the fiber over s.

We put

$$C_s := \gamma(\varpi^{-1}(\sigma_s)) \subset Y_s.$$

When s approaches 0, this closed subset C_s degenerates into $C_0 := \gamma(\varpi^{-1}(o))$, which is an l-dimensional closed analytic subset of Y_0. We decompose $\varpi^{-1}(\sigma_s)$ into the union of $\varpi^{-1}(\sigma_s)^{(\sharp)}$ and $\varpi^{-1}(\sigma_s)^{(\flat)}$, where

$$\begin{aligned} \varpi^{-1}(\sigma_s)^{(\sharp)} &:= \varpi^{-1}(\sigma_s) \setminus (\gamma^{-1}(B^\circ_\eta) \cap \varpi^{-1}(\sigma_s)) \quad \text{and} \\ \varpi^{-1}(\sigma_s)^{(\flat)} &:= \gamma^{-1}(B_\eta) \cap \varpi^{-1}(\sigma_s). \end{aligned}$$

Since η and ε are small enough, and W is a subspace of $Y \times F$, we have

$$\varpi^{-1}(\sigma_s)^{(\flat)} = W \cap (B_\eta \times \sigma_s) \quad \subset \quad U_{W,q} \tag{3.13}$$

for all $s \in [0, \varepsilon]$, where $U_{W,q}$ is the open neighborhood of q in W that was introduced at the beginning of the proof. Recalling that γ embeds $U_{W,q}$ into Y, we see that the map γ yields a homeomorphism from $\varpi^{-1}(\sigma_\varepsilon)^{(\flat)}$ to $C^\flat_\varepsilon$. By definition, γ maps $\varpi^{-1}(\sigma_\varepsilon)^{(\sharp)}$ to $C^\sharp_\varepsilon$. We then define a closed subset $\widetilde{C}_\varepsilon$ of Y_ε by

$$\widetilde{C}_\varepsilon := C^\flat_\varepsilon \cup \Big(\bigcup_{s \in [0,\varepsilon]} \partial^B f_s(\partial^B C_s) \Big) \cup f^\sharp_0(C^\sharp_0). \tag{3.14}$$

Note that we have

$$\partial^B \widetilde{C}_\varepsilon = \bigcup\nolimits_{s \in [0,\varepsilon]} \partial^B f_s(\partial^B C_s) \quad \text{and} \quad \widetilde{C}^\flat_\varepsilon = C^\flat_\varepsilon \cup \partial^B \widetilde{C}_\varepsilon, \quad \widetilde{C}^\sharp_\varepsilon = \partial^B \widetilde{C}_\varepsilon \cup f^\sharp_0(C^\sharp_0).$$

Using the trivialization $(f^\sharp, \partial^B f)$, we can "squeeze" the topological m-cycle $\gamma | \varpi^{-1}(\sigma_\varepsilon) : \varpi^{-1}(\sigma_\varepsilon) \to Y_\varepsilon$ outside the ball so that the image is contained in $\widetilde{C}_\varepsilon$. More precisely, we can construct a homotopy from $\gamma | \varpi^{-1}(\sigma_\varepsilon) : \varpi^{-1}(\sigma_\varepsilon) \to Y_\varepsilon$ to a continuous map $\beta : \varpi^{-1}(\sigma_\varepsilon) \to Y_\varepsilon$ with the following properties:

(β-1) The image $\beta(\varpi^{-1}(\sigma_\varepsilon))$ of β coincides with $\widetilde{C}_\varepsilon$.

(β-2) The homotopy is stationary on $\varpi^{-1}(\sigma_\varepsilon)^{(\flat)}$. In particular, β yields a homeomorphism from $\varpi^{-1}(\sigma_\varepsilon)^{(\flat)}$ to the first piece $C^\flat_\varepsilon$ of the decomposition (3.14).

(β-3) The image $\beta(\varpi^{-1}(\sigma_\varepsilon)^{(\sharp)})$ of $\varpi^{-1}(\sigma_\varepsilon)^{(\sharp)}$ by β is contained in $\widetilde{C}^\sharp_\varepsilon$.

(See Figure 3.1.) Since $\psi_\varepsilon([\sigma_\varepsilon])$ is represented by $\gamma | \varpi^{-1}(\sigma_\varepsilon)$, it is also represented by the topological m-cycle β.

From (3.13) and (3.3), (3.4), (3.5), (3.7), we see that $C^\flat_s$ ($s \in [0, \varepsilon]$) is given in terms of the local coordinate system z by the following:

$$\begin{cases} |z_1|^2 + \cdots + |z_{m+1}|^2 \le \eta, \\ z_1^2 + \cdots + z_k^2 = s, \\ \mathfrak{Im}(z_i) = 0 \quad (i = 1, \ldots, k), \\ y''_j = z_{k+j} + \sqrt{-1} z_{k+l+j} = 0 \quad (j = 1, \ldots, l). \end{cases}$$

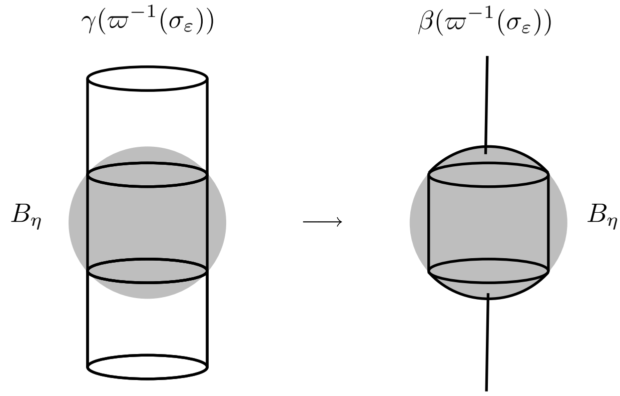

FIGURE 3.1. Homotopy from $\gamma|\varpi^{-1}(\sigma_\varepsilon)$ to β

For $s \in [0, \varepsilon]$, let G_s be the closed subset of E defined by the following equations:

$$\begin{cases} (\,2s + (\eta - s)\|v\|^2\,)(\,u_1^2 + \cdots + u_k^2\,) = 2s, \\ v_1 = \cdots = v_k = 0, \\ v_{k+j} = -g_s(\|v\|) \cdot u_{k+l+j} \quad (j = 1, \dots, l), \\ v_{k+l+j} = g_s(\|v\|) \cdot u_{k+j} \quad (j = 1, \dots, l), \end{cases}$$

where

$$g_s(\|v\|) := \sqrt{\frac{2s}{\eta - s} + \|v\|^2}.$$

Then, for $s \in (0, \varepsilon]$, the homeomorphism $h_s : Y_s^\flat \,\tilde{\to}\, E$ maps $C_s^\flat$ to G_s. In particular, the first piece $C_\varepsilon^\flat$ of the decomposition (3.14) of $\widetilde{C}_\varepsilon$ is mapped homeomorphically to G_ε by h_ε. It is easy to check that, for any $s \in [0, \varepsilon]$ (including $s = 0$), the homeomorphism $\partial^B h_s : \partial^B Y_s \,\tilde{\to}\, \partial E$ maps $\partial^B C_s$ to $G_s \cap \partial E$. We put

$$T_\varepsilon := \{\; u \in S^m \mid u_1^2 + \cdots + u_k^2 < 2\varepsilon/(\eta + \varepsilon) \;\},$$

and let T_ε^- be the closure of T_ε. We can easily check that the projection $\tau : E \to S^m$ induces a homeomorphism from G_ε to $S^m \setminus T_\varepsilon$, and that, for any $s \in [0, \varepsilon]$, $G_s \cap \partial E$ is contained in $\tau^{-1}(T_\varepsilon^-) \cap \partial E$. In particular, the second piece $\partial^B \widetilde{C}_\varepsilon$ of the decomposition (3.14) is mapped by $\partial^B h_\varepsilon$ into $\tau^{-1}(T_\varepsilon^-) \cap \partial E$.

Let a be a point of $S^m \setminus T_\varepsilon^-$. Then the closed subset $h_\varepsilon^{-1}(\tau^{-1}(a))$ of $Y_\varepsilon^\flat$ intersects the first piece $C_\varepsilon^\flat$ of the decomposition (3.14) at only one point, which is in the interior of $C_\varepsilon^\flat$, and the intersection is transverse. Moreover, $h_\varepsilon^{-1}(\tau^{-1}(a))$ is disjoint from the second piece $\partial^B \widetilde{C}_\varepsilon$ of the decomposition (3.14). The third piece $f_0^\sharp(C_0^\sharp)$ is a topological $2l$-cycle in $(Y_\varepsilon^\sharp, \partial^B Y_\varepsilon)$, because C_0 is a topological $2l$-cycle in Y_0.

If $[\Sigma_\varepsilon] \in H_m(Y_\varepsilon, \mathbb{Z})$ is zero, then $\psi_\varepsilon([\sigma_\varepsilon]) = 0$ by (3.11) and hence there is nothing to prove. Suppose that $[\Sigma_\varepsilon]$ is not zero and not a torsion element. Then there exists a homology class $[\Theta] \in H_m(Y_\varepsilon, \mathbb{Z})$ such that the intersection number $[\Sigma_\varepsilon] \cdot [\Theta]$ of $[\Sigma_\varepsilon]$ and $[\Theta]$ in Y_ε is not zero. In order to show that the integer c in (3.11) is ± 1, it is enough to prove the following:

$$\psi_\varepsilon([\sigma_\varepsilon]) \cdot [\Theta] = \pm[\Sigma_\varepsilon] \cdot [\Theta]. \tag{3.15}$$

Multiplying $[\Theta]$ by a positive integer if necessary, we can assume that $[\Theta]$ is represented by a compact oriented m-dimensional differentiable submanifold Θ of Y_ε ([20]). By the elementary transversality theorem (see, for example, [7]), we can move Θ in Y_ε in such a way that the following hold:

(Θ-1) The closed subset $h_\varepsilon(\Theta^\flat)$ of E is a union of finite number of fibers of $\tau : E \to S^m$ over points in $S^m \setminus T_\varepsilon^-$.
(Θ-2) The topological m-cycle $\Theta^\sharp$ of $(Y_\varepsilon^\sharp, \partial^B Y_\varepsilon)$ is disjoint from the topological $2l$-cycle $f_0^\sharp(C_0^\sharp)$. Here we use the assumption $m > 2l$.

From (Θ-1) and (Θ-2), the points $\Theta \cap \widetilde{C}_\varepsilon$ are contained in the interior of the first piece $C_\varepsilon^\flat$ of the decomposition (3.14) of $\widetilde{C}_\varepsilon$, and the intersections are all transverse. Moreover, the total intersection number of Θ and $\widetilde{C}_\varepsilon$ is equal to that of Θ and Σ_ε up to sign, because both of them are equal, up to sign, to the number of fibers of τ constituting $h_\varepsilon(\Theta^\flat)$ (counted with signs according to the orientation). Combining these with the properties (β-1)-(β-3) of the topological m-cycle β, we see that $[\beta] \cdot [\Theta] = \pm[\Sigma_\varepsilon] \cdot [\Theta]$. We have seen that $\psi_\varepsilon([\sigma_\varepsilon])$ is represented by β. Thus we obtain (3.15). □

4. The universal family

In this section, we will construct the universal family of the incidence varieties of complete intersections in a complex projective space $\mathbb{P}^n$.

First we fix some notation. Let

$$R = \bigoplus_{d=0}^{\infty} R_d := \mathbb{C}[x_0, \dots, x_n]$$

be the polynomial ring of $n+1$ variables with coefficients in $\mathbb{C}$ graded by the degree d of polynomials. We set $R_d := 0$ for $d < 0$. Let M be a graded R-module. We denote by M_d the vector space consisting of homogeneous elements of M with degree d. For an integer k, let $M(k)$ be the R-module M with grading shifted by $M(k)_d := M_{k+d}$. For another graded R-module N, let $\mathrm{Hom}(M, N)_0$ denote the vector space of degree-preserving homomorphisms from M to N. Let $\mathbf{c} = (c_1, \dots, c_t)$ be a sequence of positive integers. We assume $t < n$. Let us define the graded free R-module $M_{\mathbf{c}}$ by

$$M_{\mathbf{c}} := \bigoplus_{i=1}^{t} R(c_i).$$

An element of $M_{\mathbf{c}}$ is written as a column vector. Let $f = (f_1, \dots, f_t)^T$ be an element of $(M_{\mathbf{c}})_0 = \oplus R_{c_i}$, where f_i is a homogeneous polynomial of degree c_i. We denote by J_f the homogeneous ideal of R generated by $f_1, \dots, f_t$. There exists a Zariski open dense subset $(M_{\mathbf{c}})_0^{ci}$ of the vector space $(M_{\mathbf{c}})_0$ consisting of all $f \in (M_{\mathbf{c}})_0$ such that the ideal J_f defines a complete intersection of multi-degree $\mathbf{c}$ in $\mathbb{P}^n = \mathrm{Proj}\, R$. For $f \in (M_{\mathbf{c}})_0^{ci}$, let $Y_{\langle f \rangle}$ denote the complete intersection defined by J_f. It is well-known that, for any integer ν, the dimension of the vector space

$$H^0(Y_{\langle f \rangle}, \mathcal{O}(\nu)) = ((R/J_f)(\nu))_0$$

is independent of the choice of $f \in (M_{\mathbf{c}})_0^{ci}$.

Let $H_{n,\mathbf{c}}$ denote the scheme parameterizing all complete intersections of multi-degree $\mathbf{c}$ in $\mathbb{P}^n$. It is well-known that $H_{n,\mathbf{c}}$ is a smooth irreducible quasi-projective scheme. For an element $f \in (M_{\mathbf{c}})_0^{ci}$, let $\langle f\rangle$ denote the point of $H_{n,\mathbf{c}}$ corresponding to the complete intersection $Y_{\langle f\rangle}$. We have a surjective morphism

$$q_{\mathbf{c}} : (M_{\mathbf{c}})_0^{ci} \twoheadrightarrow H_{n,\mathbf{c}}$$

that maps f to $\langle f\rangle$. Let $\mathcal{Y}_{\mathbf{c}} \subset \mathbb{P}^n \times H_{n,\mathbf{c}}$ be the universal family of complete intersections of multi-degree $\mathbf{c}$ in $\mathbb{P}^n$ with $\phi_{\mathbf{c}} : \mathcal{Y}_{\mathbf{c}} \to H_{n,\mathbf{c}}$ and $\tau_{\mathbf{c}} : \mathcal{Y}_{\mathbf{c}} \to \mathbb{P}^n$ the projections.

Proposition 4.1. (1) *The morphism $q_{\mathbf{c}}$ is smooth.* (2) *The morphism $\tau_{\mathbf{c}}$ is smooth. In particular, $\mathcal{Y}_{\mathbf{c}}$ is smooth.*

Proof. (1) The Zariski tangent space to $H_{n,\mathbf{c}}$ at $\langle f\rangle$ is given by

$$T_{\langle f\rangle}H_{n,\mathbf{c}} = H^0(Y_{\langle f\rangle}, \mathcal{N}_{Y_{\langle f\rangle}/\mathbb{P}^n}) = (M_{\mathbf{c}}/J_f M_{\mathbf{c}})_0, \tag{4.1}$$

where $\mathcal{N}_{Y_{\langle f\rangle}/\mathbb{P}^n}$ is the normal sheaf of $Y_{\langle f\rangle}$ in $\mathbb{P}^n$, which is isomorphic to $\oplus_{i=1}^t \mathcal{O}(c_i)$. By (4.1) and $T_f(M_{\mathbf{c}})_0^{ci} \cong (M_{\mathbf{c}})_0$, the linear map $(dq_{\mathbf{c}})_f : T_f(M_{\mathbf{c}})_0^{ci} \to T_{\langle f\rangle}H_{n,\mathbf{c}}$ is identified with the quotient homomorphism $(M_{\mathbf{c}})_0 \twoheadrightarrow (M_{\mathbf{c}}/J_f M_{\mathbf{c}})_0$. Hence $q_{\mathbf{c}}$ is smooth.

(2) Let $P = (p, \langle f\rangle)$ be a point of $\mathcal{Y}_{\mathbf{c}}$, where p is a point of $Y_{\langle f\rangle}$, and let I_p be the homogeneous ideal of R defining the point p. The kernel of $(d\tau_{\mathbf{c}})_P : T_P\mathcal{Y}_{\mathbf{c}} \to T_p\mathbb{P}^n$ is mapped isomorphically to a subspace of $T_{\langle f\rangle}H_{n,\mathbf{c}}$ by $(d\phi_{\mathbf{c}})_P : T_P\mathcal{Y}_{\mathbf{c}} \to T_{\langle f\rangle}H_{n,\mathbf{c}}$. This subspace coincides with the subspace $(I_pM_{\mathbf{c}}/J_fM_{\mathbf{c}})_0$ of $(M_{\mathbf{c}}/J_fM_{\mathbf{c}})_0$ under the identification (4.1). Since $\dim(M_{\mathbf{c}}/I_pM_{\mathbf{c}})_0 = t$, $\dim \operatorname{Ker}(d\tau_{\mathbf{c}})_P$ is equal to $\dim H_{n,\mathbf{c}} - t = \dim \mathcal{Y}_{\mathbf{c}} - n$ for any point $P \in \mathcal{Y}_{\mathbf{c}}$. □

Let $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_s)$ be two sequences of integers satisfying (2.1). Instead of $\mathcal{Y}_{\mathbf{a}}$ and $\mathcal{Y}_{\mathbf{b}}$, we denote by

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\tau} & \mathbb{P}^n \\ \phi\downarrow & & \\ H_{n,\mathbf{a}} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{Z} & \xrightarrow{\tau'} & \mathbb{P}^n \\ \phi'\downarrow & & \\ H_{n,\mathbf{b}} & & \end{array} \tag{4.2}$$

the universal families over $H_{n,\mathbf{a}}$ and $H_{n,\mathbf{b}}$. For $f \in (M_{\mathbf{a}})_0^{ci}$ and $g \in (M_{\mathbf{b}})_0^{ci}$, we denote by $X_{\langle f\rangle}$ and $Z_{\langle g\rangle}$ the complete intersections corresponding to $\langle f\rangle \in H_{n,\mathbf{a}}$ and $\langle g\rangle \in H_{n,\mathbf{b}}$, respectively.

An element h of $\operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ is expressed by an $r \times s$ matrix (h_{ij}) with $h_{ij} \in R_{a_i - b_j}$. When $g \in (M_{\mathbf{b}})_0$ is fixed, the image of the linear map $\operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0 \to (M_{\mathbf{a}})_0$ given by $h \mapsto h(g)$ coincides with $(J_gM_{\mathbf{a}})_0$. The following proposition is then obvious:

Proposition 4.2. *The following three conditions on the pair (f, g) of $f \in (M_{\mathbf{a}})_0^{ci}$ and $g \in (M_{\mathbf{b}})_0^{ci}$ are equivalent*:

(i) *$X_{\langle f\rangle}$ contains $Z_{\langle g\rangle}$ as a subscheme,*
(ii) *f is contained in $(J_gM_{\mathbf{a}})_0$, and*
(iii) *there exists an element $h \in \operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ such that $f = h(g)$.* □

Let $\mathcal{F}_{\mathbf{b},\mathbf{a}}$ be the contravariant functor from the category of locally noetherian schemes over $\mathbb{C}$ to the category of sets that associates to a locally noetherian scheme $S \to \operatorname{Spec}\mathbb{C}$ the set of pairs (Z_S, X_S), where $Z_S \subset \mathbb{P}^n \times S$ and $X_S \subset \mathbb{P}^n \times S$ are families of complete intersections in $\mathbb{P}^n$ with multi-degrees $\mathbf{b}$ and $\mathbf{a}$, respectively, parameterized by S such that Z_S is a subscheme of X_S. This functor $\mathcal{F}_{\mathbf{b},\mathbf{a}}$ is represented by a closed subscheme $F_{\mathbf{b},\mathbf{a}}$ of $H_{n,\mathbf{b}} \times H_{n,\mathbf{a}}$. (The scheme $F_{\mathbf{b},\mathbf{a}}$ may possibly be empty.) We denote the projections by $\rho' : F_{\mathbf{b},\mathbf{a}} \to H_{n,\mathbf{b}}$ and $\rho : F_{\mathbf{b},\mathbf{a}} \to H_{n,\mathbf{a}}$. The universal family over $F_{\mathbf{b},\mathbf{a}}$ is the pair $(\widetilde{\mathcal{Z}}, \widetilde{\mathcal{X}})$ of $\widetilde{\mathcal{Z}} := \mathcal{Z} \times_{H_{n,\mathbf{b}}} F_{\mathbf{b},\mathbf{a}}$ and $\widetilde{\mathcal{X}} := \mathcal{X} \times_{H_{n,\mathbf{a}}} F_{\mathbf{b},\mathbf{a}}$. We denote by $\pi : \widetilde{\mathcal{Z}} \to F_{\mathbf{b},\mathbf{a}}$ and $\beta : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ the natural projections. We also denote by $\alpha : \widetilde{\mathcal{Z}} \to \mathcal{X}$ the composite of the closed immersion $\widetilde{\mathcal{Z}} \hookrightarrow \widetilde{\mathcal{X}}$ and the natural projection $\widetilde{\mathcal{X}} \to \mathcal{X}$. Thus we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
\mathbb{P}^n & \stackrel{\tau'}{\longleftarrow} & \mathcal{Z} & \stackrel{\beta}{\longleftarrow} & \widetilde{\mathcal{Z}} & \stackrel{\alpha}{\longrightarrow} & \mathcal{X} & \stackrel{\tau}{\longrightarrow} & \mathbb{P}^n \\
 & & {\scriptstyle\phi'}\downarrow & \square & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\phi} & & \\
 & & H_{n,\mathbf{b}} & \underset{\rho'}{\longleftarrow} & F_{\mathbf{b},\mathbf{a}} & \underset{\rho}{\longrightarrow} & H_{n,\mathbf{a}}, & &
\end{array} \tag{4.3}
$$

in which $\tau \circ \alpha = \tau' \circ \beta$ holds. A point of $\widetilde{\mathcal{Z}}$ is a triple

$$(p, \langle g \rangle, \langle f \rangle) \in \mathbb{P}^n \times H_{n,\mathbf{b}} \times H_{n,\mathbf{a}}$$

that satisfies $p \in Z_{\langle g \rangle} \subset X_{\langle f \rangle}$. The projection π maps $(p, \langle g \rangle, \langle f \rangle)$ to $(\langle g \rangle, \langle f \rangle) \in F_{\mathbf{b},\mathbf{a}}$, and the morphism α maps $(p, \langle g \rangle, \langle f \rangle)$ to $(p, \langle f \rangle) \in \mathcal{X}$.

The right square of the diagram (4.3) is the universal family of the families (1.2) of complete intersections of multi-degree $\mathbf{b}$ contained in complete intersections of multi-degree $\mathbf{a}$. Remark that the linear automorphism group $PGL(n+1)$ of $\mathbb{P}^n$ acts on the diagram (4.3).

Remark 4.3. If $(n, \mathbf{a}, \mathbf{b})$ satisfies the first inequality $a_i \geq b_i$ $(i = 1, \dots, r)$ of the condition (2.2) in Main Theorem, then $F_{\mathbf{b},\mathbf{a}}$ is non-empty. Indeed, we choose linear forms $\ell_1, \dots, \ell_r, \ell'_1, \dots, \ell'_s \in R_1$ generally. We define $g \in (M_{\mathbf{b}})_0^{ci}$ by $g_j := {\ell'_j}^{b_j}$. Since $a_i \geq b_i$, we can define $f \in (M_{\mathbf{a}})_0^{ci}$ by $f_i := {\ell'_i}^{b_i} {\ell_i}^{a_i - b_i}$. Then $(\langle g \rangle, \langle f \rangle)$ is a point of $F_{\mathbf{b},\mathbf{a}}$.

From now on to the end of this section, we assume that $F_{\mathbf{b},\mathbf{a}}$ is non-empty. We define a vector space U with a natural morphism $\nu : U \to (M_{\mathbf{a}})_0$ by

$$U := (M_{\mathbf{b}})_0 \times \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0 \quad \text{and} \quad \nu(g, h) := h(g).$$

We then put

$$U^{ci} := ((M_{\mathbf{b}})_0^{ci} \times \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0) \cap \nu^{-1}((M_{\mathbf{a}})_0^{ci}).$$

Note that U^{ci} is a Zariski open subset of U, and hence is irreducible. By Proposition 4.2, the map

$$\sigma(g, h) := (\langle g \rangle, \langle h(g) \rangle)$$

defines a surjective morphism $\sigma : U^{ci} \twoheadrightarrow F_{\mathbf{b},\mathbf{a}}$, which makes the following diagram commutative:

$$
\begin{array}{ccc}
U^{ci} & \stackrel{\nu|U^{ci}}{\longrightarrow} & (M_{\mathbf{a}})_0^{ci} \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle q_{\mathbf{a}}} \\
F_{\mathbf{b},\mathbf{a}} & \underset{\rho}{\longrightarrow} & H_{n,\mathbf{a}}.
\end{array}
$$

In particular, the scheme $F_{\mathbf{b},\mathbf{a}}$ is irreducible.

Proposition 4.4. *The morphism $\rho' : F_{\mathbf{b},\mathbf{a}} \to H_{n,\mathbf{b}}$ is smooth.*

Proof. For a non-negative integer k, we set $A_k := \mathbb{C}[t]/(t^{k+1})$, and for a scheme T over $\mathbb{C}$, we denote by $T(A_k)$ the set of A_k-valued points of T. Suppose we are given $\langle g\rangle^{[k+1]} \in H_{n,\mathbf{b}}(A_{k+1})$ and $(\langle g\rangle^{[k]}, \langle f\rangle^{[k]}) \in F_{\mathbf{b},\mathbf{a}}(A_k)$ satisfying $\langle g\rangle^{[k]} = \langle g\rangle^{[k+1]} \bmod t^{k+1}$. It is enough to show that $(\langle g\rangle^{[k]}, \langle f\rangle^{[k]})$ extends to an element $(\langle g\rangle^{[k+1]}, \langle f\rangle^{[k+1]})$ of $F_{\mathbf{b},\mathbf{a}}(A_{k+1})$ over the given point $\langle g\rangle^{[k+1]} \in H_{n,\mathbf{b}}(A_{k+1})$. Since $q_{\mathbf{a}} : (M_{\mathbf{a}})_0^{ci} \twoheadrightarrow H_{n,\mathbf{a}}$ and $q_{\mathbf{b}} : (M_{\mathbf{b}})_0^{ci} \twoheadrightarrow H_{n,\mathbf{b}}$ are smooth, there exist

$$g^{[k+1]} \in (M_{\mathbf{b}})_0 \otimes_{\mathbb{C}} A_{k+1} \quad \text{and} \quad f^{[k]} \in (M_{\mathbf{a}})_0 \otimes_{\mathbb{C}} A_k$$

that satisfy $q_{\mathbf{b}}(g^{[k+1]}) = \langle g\rangle^{[k+1]}$ and $q_{\mathbf{a}}(f^{[k]}) = \langle f\rangle^{[k]}$. We put

$$g^{[k]} := g^{[k+1]} \bmod t^{k+1} \in (M_{\mathbf{b}})_0 \otimes_{\mathbb{C}} A_k,$$

which satisfies $\langle g^{[k]}\rangle = \langle g\rangle^{[k]}$. By the definition of $F_{\mathbf{b},\mathbf{a}}$, the ideal $J_{g^{[k]}}$ of $R \otimes_{\mathbb{C}} A_k$ generated by the components of $g^{[k]}$ contains the ideal $J_{f^{[k]}}$. Hence there exists $h^{[k]} \in \operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0 \otimes_{\mathbb{C}} A_k$ such that $f^{[k]} = h^{[k]}(g^{[k]})$ holds. Let $h^{[k+1]}$ be any element of $\operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0 \otimes_{\mathbb{C}} A_{k+1}$ satisfying $h^{[k+1]} \bmod t^{k+1} = h^{[k]}$. We put

$$f^{[k+1]} := h^{[k+1]}(g^{[k+1]}) \in (M_{\mathbf{a}})_0 \otimes_{\mathbb{C}} A_{k+1}.$$

Since being a complete intersection is an open condition on defining polynomials, the ideal $J_{f^{[k+1]}}$ of $R \otimes_{\mathbb{C}} A_{k+1}$ defines a family of complete intersections of multi-degree $\mathbf{a}$ over $\operatorname{Spec} A_{k+1}$. Thus $(\langle g^{[k+1]}\rangle, \langle f^{[k+1]}\rangle)$ is the hoped-for A_{k+1}-valued point of $F_{\mathbf{b},\mathbf{a}}$. □

Corollary 4.5. (1) *The scheme $F_{\mathbf{b},\mathbf{a}}$ is smooth.* (2) *The morphism $\beta : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is smooth. In particular, $\widetilde{\mathcal{Z}}$ is smooth.* □

Let (g, h) be a point of U^{ci}. We have the following natural identifications of vector spaces:

$$H^0(Z_{\langle g\rangle}, \mathcal{N}_{Z_{\langle g\rangle}/\mathbb{P}^n}) = T_{\langle g\rangle} H_{n,\mathbf{b}} = (M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0, \tag{4.4}$$

$$H^0(Z_{\langle g\rangle}, \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n}|Z_{\langle g\rangle}) = (M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0, \tag{4.5}$$

$$H^0(X_{\langle h(g)\rangle}, \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n}) = T_{\langle h(g)\rangle} H_{n,\mathbf{a}} = (M_{\mathbf{a}}/J_{h(g)} M_{\mathbf{a}})_0. \tag{4.6}$$

The restriction map $\mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n} \to \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n}|Z_{\langle g\rangle}$ of coherent sheaves induces, via (4.6), a linear map

$$\zeta' : T_{\langle h(g)\rangle} H_{n,\mathbf{a}} \to H^0(Z_{\langle g\rangle}, \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n}|Z_{\langle g\rangle}).$$

Under the identifications (4.6) and (4.5), the linear map ζ' is identified with the natural quotient homomorphism

$$(M_{\mathbf{a}}/J_{h(g)} M_{\mathbf{a}})_0 \twoheadrightarrow (M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0.$$

In particular, ζ' is surjective. On the other hand, since $Z_{\langle g\rangle}$ is a subscheme of $X_{\langle h(g)\rangle}$, there is a natural homomorphism

$$\mathcal{N}_{Z_{\langle g\rangle}/\mathbb{P}^n} \to \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n}|Z_{\langle g\rangle}$$

of coherent sheaves over $Z_{\langle g\rangle}$, which induces, via (4.4), a linear map

$$\zeta : T_{\langle g\rangle} H_{n,\mathbf{b}} \to H^0(Z_{\langle g\rangle}, \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n}|Z_{\langle g\rangle}).$$

Under the identifications (4.4) and (4.5), the linear map ζ is identified with the homomorphism

$$\langle h\rangle_g : (M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 \to (M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0$$

induced from $h : M_{\mathbf{b}} \to M_{\mathbf{a}}$.

Proposition 4.6. *Let (g, h) be a point of U^{ci}, and P the point $\sigma(g, h) = (\langle g\rangle, \langle h(g)\rangle)$ of $F_{\mathbf{b},\mathbf{a}}$. Then we have the following diagram of fiber product:*

$$\begin{array}{ccc} T_P F_{\mathbf{b},\mathbf{a}} & \xrightarrow{(d\rho)_P} & T_{\langle h(g)\rangle} H_{n,\mathbf{a}} \\ {\scriptstyle (d\rho')_P}\downarrow & \square & \downarrow{\scriptstyle \zeta'} \\ T_{\langle g\rangle} H_{n,\mathbf{b}} & \xrightarrow[\zeta]{} & H^0(Z_{\langle g\rangle}, \mathcal{N}_{X_{\langle h(g)\rangle}/\mathbb{P}^n} | Z_{\langle g\rangle}). \end{array} \tag{4.7}$$

Proof. By the identifications (4.4) and (4.6), any vectors of $T_{\langle g\rangle} H_{n,\mathbf{b}}$ and $T_{\langle h(g)\rangle} H_{n,\mathbf{a}}$ are given as elements

$$\bar{g}' := g' \bmod (J_g M_{\mathbf{b}})_0 \quad \text{and} \quad \bar{f}' := f' \bmod (J_{h(g)} M_{\mathbf{a}})_0$$

of $(M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0$ and $(M_{\mathbf{a}}/J_{h(g)} M_{\mathbf{a}})_0$ by some $g' \in (M_{\mathbf{b}})_0$ and $f' \in (M_{\mathbf{a}})_0$, respectively. Let ε be a dual number: $\varepsilon^2 = 0$. The vectors $\bar{g}'$ and $\bar{f}'$ correspond to the infinitesimal displacements

$$Z_{\langle g+\varepsilon g'\rangle} \to \operatorname{Spec} \mathbb{C}[\varepsilon] \quad \text{and} \quad X_{\langle h(g)+\varepsilon f'\rangle} \to \operatorname{Spec} \mathbb{C}[\varepsilon]$$

of $Z_{\langle g\rangle}$ and $X_{\langle h(g)\rangle}$ defined by the homogeneous ideals $J_g + \varepsilon J_{g'}$ and $J_{h(g)} + \varepsilon J_{f'}$ of $R \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$, respectively. Then the vector $(\bar{g}', \bar{f}')$, regarded as a tangent vector to $H_{n,\mathbf{b}} \times H_{n,\mathbf{a}}$ at $(\langle g\rangle, \langle h(g)\rangle)$, is tangent to $F_{\mathbf{b},\mathbf{a}}$ if and only if $Z_{\langle g+\varepsilon g'\rangle}$ is contained in $X_{\langle h(g)+\varepsilon f'\rangle}$ as a subscheme; that is, there exist elements $h_1, h_2 \in \operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ such that $h_1 + \varepsilon h_2 \in \operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0 \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$ satisfies the following:

$$(h_1 + \varepsilon h_2)(g + \varepsilon g') = h(g) + \varepsilon f'. \tag{4.8}$$

Suppose that $h_1 + \varepsilon h_2$ satisfies (4.8). Because $h_1(g) = h(g)$, each row vector of the matrix $h_1 - h$ is contained in the syzygy of the regular sequence $(g_1, \dots, g_s)$, and hence every component of $h - h_1$ is contained in J_g. Therefore the two linear maps $\langle h\rangle_g$ and $\langle h_1\rangle_g$ from $(M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0$ to $(M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0$ are the same. The equality $h_1(g') + h_2(g) = f'$ then tells us that $f' \bmod (J_g M_{\mathbf{a}})_0$ is equal to $\langle h\rangle_g(\bar{g}')$, because $h_2(g) \in (J_g M_{\mathbf{a}})_0$. Hence $(\bar{g}', \bar{f}')$ is contained in the fiber product of ζ and ζ'. Conversely, if $(\bar{g}', \bar{f}')$ is contained in the fiber product of ζ and ζ', then it is easy to find $h_2 \in \operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ satisfying $(h + \varepsilon h_2)(g + \varepsilon g') = h(g) + \varepsilon f'$. □

Since $F_{\mathbf{b},\mathbf{a}}$ is reduced by Corollary 4.5 (1), we obtain the following:

Corollary 4.7. *Let (g, h) be an arbitrary point of U^{ci}.*

(1) *The dimension of $F_{\mathbf{b},\mathbf{a}}$ is equal to*

$$\begin{aligned} &\dim(M_{\mathbf{a}}/J_{h(g)} M_{\mathbf{a}})_0 + \dim(M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 - \dim(M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0 = \\ &\qquad \dim H_{n,\mathbf{a}} + \dim H_{n,\mathbf{b}} - \dim(M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0. \end{aligned} \tag{4.9}$$

(2) *Let P be the point $\sigma(g, h)$ of $F_{\mathbf{b},\mathbf{a}}$. Then the dimension of the cokernel of $(d\rho)_P : T_P F_{\mathbf{b},\mathbf{a}} \to T_{\langle h(g)\rangle} H_{n,\mathbf{a}}$ is equal to*

$$\dim \operatorname{Coker} \zeta = \dim \operatorname{Coker} \langle h\rangle_g = \dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + h(M_{\mathbf{b}})))_0. \qquad \square$$

Proposition 4.8. *Let (g, h) be a point of U^{ci}, and let p be a point of $Z_{\langle g\rangle}$. We put $Q := (p, \langle g\rangle, \langle h(g)\rangle)$, which is a point of $\widetilde{\mathcal{Z}}$. Let I_p denote the homogeneous ideal of R defining the point p. Then the dimension of the kernel of $(d\alpha)_Q : T_Q \widetilde{\mathcal{Z}} \to T_{\alpha(Q)} \mathcal{X}$ is equal to*

$$\dim F_{\mathbf{b},\mathbf{a}} - \dim H_{n,\mathbf{a}} - s + \dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_p h(M_{\mathbf{b}})))_0. \tag{4.10}$$

Proof. Since $\widetilde{\mathcal{Z}}$ is a closed subscheme of $H_{n,\mathbf{b}} \times \mathcal{X}$ with $\rho' \circ \pi$ and α being the projections, the kernel of $(d\alpha)_Q$ is mapped isomorphically to a subspace of $T_{\langle g\rangle} H_{n,\mathbf{b}}$ by the linear map $d(\rho' \circ \pi)_Q$. We will show that this subspace

$$(d(\rho' \circ \pi)_Q)\,(\mathrm{Ker}(d\alpha)_Q) \quad \subset \quad T_{\langle g\rangle} H_{n,\mathbf{b}} \tag{4.11}$$

coincides with the subspace

$$(I_p M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 \;\cap\; \mathrm{Ker}\langle h\rangle_g \quad \subset \quad (M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 \tag{4.12}$$

under the identification (4.4). Let g' be an element of $(M_{\mathbf{b}})_0$, and $\bar{g}'$ the element $g' \bmod (J_g M_{\mathbf{b}})_0$ of $(M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0$, giving the corresponding displacement $Z_{\langle g+\varepsilon g'\rangle} \to \mathrm{Spec}\,\mathbb{C}[\varepsilon]$ of $Z_{\langle g\rangle}$. The subspace (4.11) consists of vectors corresponding to infinitesimal displacements with p in $Z_{\langle g+\varepsilon g'\rangle}$ and with $Z_{\langle g+\varepsilon g'\rangle}$ remaining in $X_{\langle h(g)\rangle}$. The displacement $Z_{\langle g+\varepsilon g'\rangle}$ contains p if and only if $J_{g'} \subset I_p$ holds, which is equivalent to $\bar{g}' \in (I_p M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0$. On the other hand, by Proposition 4.6, the displacement $Z_{\langle g+\varepsilon g'\rangle}$ remains in $X_{\langle h(g)\rangle}$ if and only if the corresponding vector of $T_{\langle g\rangle} H_{n,\mathbf{b}}$ is contained in $\mathrm{Ker}\,\zeta$. Since ζ is identified with $\langle h\rangle_g$, this holds if and only if $\bar{g}' \in \mathrm{Ker}\langle h\rangle_g$. Therefore (4.11) coincides with (4.12) by (4.4). The cokernel of the homomorphism

$$(I_p M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 \;\hookrightarrow\; (M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 \xrightarrow{\langle h\rangle_g} (M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0$$

is $(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_p h(M_{\mathbf{b}})))_0$. On the other hand, $\dim(M_{\mathbf{b}}/I_p M_{\mathbf{b}})_0$ is equal to s. These lead us to the conclusion that $\dim \mathrm{Ker}(d\alpha)_Q$ is equal to

$$\dim(M_{\mathbf{b}}/J_g M_{\mathbf{b}})_0 - s - \dim(M_{\mathbf{a}}/J_g M_{\mathbf{a}})_0 + \dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_p h(M_{\mathbf{b}})))_0,$$

which coincides with (4.10) by Corollary 4.7 (1). □

In the sequel, we use the following notation. For positive integers d and e, let $\mathrm{Mat}(d,e)$ denote the vector space of all $d \times e$ matrices with entries in $\mathbb{C}$, and $D(d,e)$ the Zariski closed subset of $\mathrm{Mat}(d,e)$ consisting of matrices whose rank is less than $\min(d,e)$. It is easy to see that $D(d,e)$ is irreducible. We set

$$o := [1 : 0 : \cdots : 0] \in \mathbb{P}^n.$$

For a homogeneous polynomial $a \in R$, we put

$$a(o) := \text{the coefficient of } x_0^{\deg a} \text{ in } a.$$

Let I_o be the homogeneous ideal of R defining o in $\mathbb{P}^n$:

$$I_o := \langle x_1, \dots, x_n\rangle \subset R.$$

We define linear maps $\lambda_i : (I_o M_{\mathbf{a}})_0 \to \mathbb{C}^n$ $(i = 1, \dots, r)$ and $\mu_j : (I_o M_{\mathbf{b}})_0 \to \mathbb{C}^n$ $(j = 1, \dots, s)$ by

$$\lambda_i(f) := \left(\frac{\partial f_i}{\partial x_1}(o), \dots, \frac{\partial f_i}{\partial x_n}(o)\right) \quad \text{and} \quad \mu_j(g) := \left(\frac{\partial g_j}{\partial x_1}(o), \dots, \frac{\partial g_j}{\partial x_n}(o)\right).$$

Let $\lambda : (I_o M_{\mathbf{a}})_0 \to \mathrm{Mat}(r,n)$ and $\mu : (I_o M_{\mathbf{b}})_0 \to \mathrm{Mat}(s,n)$ be linear maps defined by

$$\lambda(f) := \begin{pmatrix} \lambda_1(f) \\ \vdots \\ \lambda_r(f) \end{pmatrix} \quad \text{and} \quad \mu(g) := \begin{pmatrix} \mu_1(g) \\ \vdots \\ \mu_s(g) \end{pmatrix}.$$

Both of λ and μ are surjective. We define a linear map $\eta : \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0 \to \mathrm{Mat}(r,s)$ by

$$\eta(h) := (h_{ij}(o)),$$

when h is expressed by an $r \times s$ matrix (h_{ij}) with $h_{ij} \in R_{a_i - b_j}$. Note that, if g is an element of $(I_o M_{\mathbf{b}})_0$, then, for any $h \in \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$, we have $h(g) \in (I_o M_{\mathbf{a}})_0$ and $\lambda(h(g)) = \eta(h) \cdot \mu(g)$.

We define an R-submodule $N_{\mathbf{a}}$ of $M_{\mathbf{a}}$ by

$$N_{\mathbf{a}} := \bigoplus_{i=1}^{r-1} R(a_i) \oplus I_o(a_r).$$

Note that $\mathrm{Ker}\,\lambda_r = (I_o N_{\mathbf{a}})_0$ holds in $(I_o M_{\mathbf{a}})_0$. Note also that an element h of $\mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ is contained in $\mathrm{Hom}(M_{\mathbf{b}}, N_{\mathbf{a}})_0$ if and only if the r-th row vector of $\eta(h)$ is the zero vector. We put

$$\begin{aligned} (I_o M_{\mathbf{b}})_0^{ci} &:= (I_o M_{\mathbf{b}})_0 \cap (M_{\mathbf{b}})_0^{ci}, \\ (I_o M_{\mathbf{a}})_0^{ci} &:= (I_o M_{\mathbf{a}})_0 \cap (M_{\mathbf{a}})_0^{ci}, \\ (I_o N_{\mathbf{a}})_0^{ci} &:= (I_o N_{\mathbf{a}})_0 \cap (M_{\mathbf{a}})_0^{ci}. \end{aligned}$$

For $f \in (M_{\mathbf{a}})_0^{ci}$ and $g \in (M_{\mathbf{b}})_0^{ci}$, we have the following:

$$(o, \langle f \rangle) \in \mathcal{X} \iff f \in (I_o M_{\mathbf{a}})_0^{ci}, \quad (o, \langle g \rangle) \in \mathcal{Z} \iff g \in (I_o M_{\mathbf{b}})_0^{ci}. \tag{4.13}$$

Let Γ be the Zariski closed subset of $\mathcal{X}$ consisting of critical points of $\phi : \mathcal{X} \to H_{n,\mathbf{a}}$, and Γ' the Zariski closed subset of $\mathcal{Z}$ consisting of critical points of $\phi' : \mathcal{Z} \to H_{n,\mathbf{b}}$. We put

$$\Gamma_o := \tau^{-1}(o) \cap \Gamma, \quad \Gamma'_o := \tau'^{\,-1}(o) \cap \Gamma'.$$

For $f \in (I_o M_{\mathbf{a}})_0^{ci}$ and $g \in (I_o M_{\mathbf{b}})_0^{ci}$, we have the following:

$$\begin{aligned} &(o, \langle f \rangle) \in \Gamma_o \iff \lambda(f) \in D(r,n), \\ &(o, \langle g \rangle) \in \Gamma'_o \iff \mu(g) \in D(s,n). \end{aligned} \tag{4.14}$$

If $f \in (I_o N_{\mathbf{a}})_0^{ci}$, then $\lambda(f) \in D(r,n)$. Hence we can define a morphism $\gamma : (I_o N_{\mathbf{a}})_0^{ci} \to \Gamma_o$ by

$$\gamma(f) := (o, \langle f \rangle).$$

Proposition 4.9. *The Zariski closed subset Γ_o of $\mathcal{X}$ is irreducible, and the morphism $\gamma : (I_o N_{\mathbf{a}})_0^{ci} \to \Gamma_o$ is dominant.*

Proof. By (4.13) and (4.14), the map $f \mapsto (o, \langle f \rangle)$ gives a surjective morphism from $\lambda^{-1}(D(r,n)) \cap (I_o M_{\mathbf{a}})_0^{ci}$ to Γ_o. Because λ is a surjective linear map and $D(r,n)$ is irreducible, $\lambda^{-1}(D(r,n))$ is also irreducible. Since $\lambda^{-1}(D(r,n)) \cap (I_o M_{\mathbf{a}})_0^{ci}$ is Zariski open in $\lambda^{-1}(D(r,n))$, Γ_o is also irreducible. Let f be a general element of $\lambda^{-1}(D(r,n))$. Then $\lambda(f)$ is of rank $r-1$, and the vector $\lambda_r(f)$ can be written as a linear combination of $\lambda_1(f), \dots, \lambda_{r-1}(f)$. Since $a_r \geq a_i$ for $i < r$, there exist homogeneous polynomials $c_1, \dots, c_{r-1}$ with $c_i \in R_{a_r - a_i}$ such that, if we put

$$f'_r := f_r - c_1 f_1 - \cdots - c_{r-1} f_{r-1} \quad \text{and} \quad f' := (f_1, \dots, f_{r-1}, f'_r)^T,$$

then $\lambda_r(f') = 0$ holds, which means $f' \in (I_o N_{\mathbf{a}})_0$. From $J_f = J_{f'}$, we conclude that $(o, \langle f \rangle) = (o, \langle f' \rangle)$ belongs to the image of γ. Since $(o, \langle f \rangle)$ is a general point of Γ_o, the morphism γ is dominant. □

Remark 4.10. The irreducibility of Γ and that of $S_{n,\mathbf{a}} = \phi(\Gamma)$ follow from Proposition 4.9 and the action of $PGL(n+1)$ on the diagram (4.3).

Corollary 4.11. *Suppose that $(o, \langle f\rangle)$ is a general point of Γ_o. Then the singular locus of $X_{\langle f\rangle}$ consists of only one point o, which is a hypersurface singularity of $X_{\langle f\rangle}$ with non-degenerate Hessian.* □

We put

$$\Xi := \alpha^{-1}(\Gamma) \setminus (\alpha^{-1}(\Gamma) \cap \beta^{-1}(\Gamma')) \quad \text{and} \quad \Xi_o := (\tau \circ \alpha)^{-1}(o) \cap \Xi,$$

which are locally closed subsets of $\widetilde{\mathcal{Z}}$ (possibly empty). A point $(o, \langle g\rangle, \langle f\rangle)$ of $(\tau \circ \alpha)^{-1}(o) \subset \widetilde{\mathcal{Z}}$ is contained in Ξ_o if and only if $X_{\langle f\rangle}$ is singular at o and $Z_{\langle g\rangle}$ is smooth at o. The morphism $\alpha : \widetilde{\mathcal{Z}} \to \mathcal{X}$ induces a morphism $\alpha|\Xi_o : \Xi_o \to \Gamma_o$.

Remark 4.12. Invoking the action of $PGL(n+1)$ on the diagram (4.3), we can paraphrase the second condition of Main Theorem into the condition that $\alpha|\Xi_o : \Xi_o \to \Gamma_o$ is dominant.

We define a linear subspace V of $U = (M_{\mathbf{b}})_0 \times \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ by

$$V := (I_o M_{\mathbf{b}})_0 \times \mathrm{Hom}(M_{\mathbf{b}}, N_{\mathbf{a}})_0.$$

We then put $V^{ci} := V \cap U^{ci}$ and

$$V^{\natural} := \{ (g,h) \in V^{ci} \mid \mu(g) \notin D(s,n)\} = \{ (g,h) \in V^{ci} \mid Z_{\langle g\rangle} \text{ is smooth at } o\}.$$

By definition, $V^{\natural}$ is Zariski open in the vector space V, but may possibly be empty. Recall that $\nu : U \to (M_{\mathbf{a}})_0$ is the morphism defined by $\nu(g,h) = h(g)$. We have a morphism

$$\nu|V : V \to (I_o N_{\mathbf{a}})_0 \quad \text{and} \quad \nu|V^{\natural} : V^{\natural} \to (I_o N_{\mathbf{a}})_0^{ci},$$

which are the restrictions of ν to V and $V^{\natural}$, respectively. By definition again, if $(g,h) \in V^{\natural}$, then $(o, \langle g\rangle, \langle h(g)\rangle) \in \Xi_o$. Let $\xi : V^{\natural} \to \Xi_o$ be the morphism defined by

$$\xi(g,h) := (o, \langle g\rangle, \langle h(g)\rangle).$$

Then we obtain the following commutative diagram:

$$\begin{array}{ccc} V^{\natural} & \xrightarrow{\nu|V^{\natural}} & (I_o N_{\mathbf{a}})_0^{ci} \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \Xi_o & \xrightarrow[\alpha|\Xi_o]{} & \Gamma_o. \end{array} \tag{4.15}$$

Proposition 4.13. *The morphism $\alpha|\Xi_o : \Xi_o \to \Gamma_o$ is dominant if and only if $\nu|V^{\natural} : V^{\natural} \to (I_o N_{\mathbf{a}})_0^{ci}$ is dominant.*

Proof. Since γ is dominant by Proposition 4.9, the commutativity of the digram (4.15) implies that, if $\nu|V^{\natural}$ is dominant, then so is $\alpha|\Xi_o$. Suppose conversely that $\alpha|\Xi_o$ is dominant. Let f be a general point of $(I_o N_{\mathbf{a}})_0^{ci}$. Since γ is dominant, $(o, \langle f\rangle)$ is a general point of Γ_o, and hence $(o, \langle f\rangle)$ is in the image of $\alpha|\Xi_o$. Thus there exists an element $g \in (I_o M_{\mathbf{b}})_0^{ci}$ such that $(o, \langle g\rangle, \langle f\rangle) \in \Xi_o$, which implies that $\mu(g)$ is *not* contained in $D(s,n)$, and that there exists an element $h \in \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ that satisfies $h(g) = f$. From $\lambda_r(f) = 0$ and $\eta(h) \cdot \mu(g) = \lambda(f)$, the linear independence of the row vectors of $\mu(g)$ implies that the r-th row vector of $\eta(h)$ is a zero vector. Therefore h is in fact an element of $\mathrm{Hom}(M_{\mathbf{b}}, N_{\mathbf{a}})_0$, which means $(g,h) \in V^{\natural}$. Hence the general point $f = h(g)$ of $(I_o N_{\mathbf{a}})_0^{ci}$ is contained in the image of $\nu|V^{\natural}$. □

Proposition 4.14. *Suppose that* $\alpha|\Xi_o : \Xi_o \to \Gamma_o$ *is dominant. Then there exists a unique irreducible component* Ξ'_o *of* Ξ_o *such that the restriction* $\alpha|\Xi'_o : \Xi'_o \to \Gamma_o$ *of* $\alpha|\Xi_o$ *to* Ξ'_o *is dominant. The closure of the image of* $\xi : V^\natural \to \Xi_o$ *in* Ξ_o *coincides with* Ξ'_o.

Proof. Since Γ_o is irreducible, there exists at least one irreducible component Ξ'_o of Ξ_o that is mapped dominantly onto Γ_o by $\alpha|\Xi_o$. Let $(o, \langle g\rangle, \langle f\rangle)$ be a general point of Ξ'_o. Then $\alpha(o, \langle g\rangle, \langle f\rangle) = (o, \langle f\rangle)$ is a general point of Γ_o. Since γ is dominant, we can assume that $(o, \langle f\rangle)$ is in the image of γ; that is, f is an element of $(I_o N_{\mathbf{a}})_0^{ci}$. Let $h \in \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ be a homomorphism satisfying $h(g) = f$. From $\mu(g) \notin D(s, n)$ and $\lambda_r(f) = 0$, we see that h actually is an element of $\mathrm{Hom}(M_{\mathbf{b}}, N_{\mathbf{a}})_0$. Hence (g, h) is a point of $V^\natural$, which is mapped to the general point $(o, \langle g\rangle, \langle f\rangle)$ of Ξ'_o by ξ. Therefore Ξ'_o is the closure of the image of $\xi : V^\natural \to \Xi_o$ in Ξ_o. Since $V^\natural$ is irreducible, the uniqueness of Ξ'_o, as well as the second assertion, is proved. □

For an element (g, h) of U, we define a linear map $\delta_{(g,h)} : U \to (M_{\mathbf{a}})_0$ by

$$\delta_{(g,h)}(G, H) := H(g) + h(G) \quad (\, G \in (M_{\mathbf{b}})_0,\ H \in \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0\,).$$

Under the natural isomorphisms $T_{(g,h)}U \cong U$ and $T_{\nu(g,h)}(M_{\mathbf{a}})_0 \cong (M_{\mathbf{a}})_0$, the linear map $\delta_{(g,h)}$ is equal to

$$(d\nu)_{(g,h)} : T_{(g,h)}U \to T_{\nu(g,h)}(M_{\mathbf{a}})_0.$$

By definition, we have

$$\delta_{(g,h)}(U) = (J_g M_{\mathbf{a}} + h(M_{\mathbf{b}}))_0, \tag{4.16}$$

$$\delta_{(g,h)}(V) = (J_g N_{\mathbf{a}} + I_o h(M_{\mathbf{b}}))_0, \quad \text{and} \tag{4.17}$$

$$(g, h) \in V \implies \delta_{(g,h)}(U) \subseteq (N_{\mathbf{a}})_0,\ \ \delta_{(g,h)}(V) \subseteq (I_o N_{\mathbf{a}})_0. \tag{4.18}$$

Proposition 4.15. *Suppose* $a_r \geq b_s$. *Then the following conditions on* $(n, \mathbf{a}, \mathbf{b})$ *are equivalent to each other*:

(i) *The morphism* $\alpha|\Xi_o : \Xi_o \to \Gamma_o$ *is dominant.*

(ii) *If* $(g, h) \in V$ *is general, then* $\delta_{(g,h)}(V)$ *coincides with* $(I_o N_{\mathbf{a}})_0$.

(iii) *If* $(g, h) \in V$ *is general, then the following holds*:

$$\dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}})))_0 = n + r - s.$$

(iv) *There exists at least one* $(g, h) \in V$ *such that*

$$\dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}})))_0 \leq n + r - s.$$

Proof. First we show the following:

Claim (1) For any $(g, h) \in V$, $\dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}})))_0$ is larger than or equal to $n + r - s$. (2) If $(g, h) \in V$ is chosen generally, then $\dim(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}}))_0$ is equal to $\dim(J_g N_{\mathbf{a}} + I_o h(M_{\mathbf{b}}))_0 + s$.

Let (g, h) be an arbitrary element of V. Then $(I_o h(M_{\mathbf{b}}))_0$ is contained in $(I_o N_{\mathbf{a}})_0 = \mathrm{Ker}\,\lambda_r$. On the other hand, if $f \in (J_g M_{\mathbf{a}})_0$, then the r-th component f_r of f is written as $g_1 k_1 + \cdots + g_s k_s$ with $k_j \in R_{a_r - b_j}$, and $\lambda_r(f)$ is equal to

$$k_1(o)\mu_1(g) + \cdots + k_s(o)\mu_s(g). \tag{4.19}$$

Hence the image of $(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}}))_0$ by λ_r is spanned by $\mu_1(g), \ldots, \mu_s(g)$, and therefore is of dimension $\leq s$. On the other hand, $\mathrm{Ker}\,\lambda_r = (I_o N_{\mathbf{a}})_0$ is of codimension $n + r$ in $(M_{\mathbf{a}})_0$. Hence we obtain

$$\dim(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}}))_0 \leq \dim \mathrm{Ker}\,\lambda_r + s = \dim(M_{\mathbf{a}})_0 - n - r + s,$$

which implies Claim (1).

Let (g,h) be a general element of V. Because g is general in $(I_oM_{\mathbf{b}})_0$, the vectors $\mu_1(g),\dots,\mu_s(g)$ are linearly independent. Let f be an element of $(J_gM_{\mathbf{a}})_0$. By the assumption $a_r \ge b_s$, the degrees $a_r - b_j$ of the polynomials k_j in the expression $f_r = g_1k_1 + \cdots + g_sk_s$ are non-negative for all $j \le s$. Therefore the coefficients $k_j(o)$ in (4.19) can take any values. Hence the image of $(J_gM_{\mathbf{a}} + I_oh(M_{\mathbf{b}}))_0$ by λ_r is of dimension exactly s. Moreover, if $f \in \operatorname{Ker}\lambda_r$, then $k_1(o) = \cdots = k_s(o) = 0$ holds. Hence we have $\operatorname{Ker}\lambda_r \subseteq (J_gN_{\mathbf{a}})_0$. Because $(I_oh(M_{\mathbf{b}}))_0 \subseteq \operatorname{Ker}\lambda_r$, we have

$$(J_gM_{\mathbf{a}} + I_oh(M_{\mathbf{b}}))_0 \cap \operatorname{Ker}\lambda_r = (J_gN_{\mathbf{a}} + I_oh(M_{\mathbf{b}}))_0.$$

Therefore Claim (2) is proved.

Since $\dim(M_{\mathbf{a}}/(J_gM_{\mathbf{a}} + I_oh(M_{\mathbf{b}})))_0$ is an upper semi-continuous function of (g,h), Claim (1) implies that the conditions (iii) and (iv) are equivalent. By (4.17) and (4.18), the following inequality holds for any $(g,h) \in V$:

$$\begin{aligned}\dim(M_{\mathbf{a}})_0/\delta_{(g,h)}(V) &= \dim(M_{\mathbf{a}}/(J_gN_{\mathbf{a}} + I_oh(M_{\mathbf{b}})))_0 \\ &\ge \dim(M_{\mathbf{a}}/I_oN_{\mathbf{a}})_0 = n + r.\end{aligned} \tag{4.20}$$

The condition (ii) is satisfied if and only if the *equality* in (4.20) holds for a general $(g,h) \in V$. The equivalence of the conditions (ii) and (iii) now follows from Claim (2).

By Proposition 4.13, the condition (i) is equivalent to the following:

(i)′ The morphism $\nu|V^\natural : V^\natural \to (I_oN_{\mathbf{a}})_0^{ci}$ is dominant.

On the other hand, since $\delta_{(g,h)}$ is equal to $(d\nu)_{(g,h)} : T_{(g,h)}U \to T_{\nu(g,h)}(M_{\mathbf{a}})_0$ via the natural identifications $T_{(g,h)}U \cong U$ and $T_{\nu(g,h)}(M_{\mathbf{a}})_0 \cong (M_{\mathbf{a}})_0$, the condition (ii) is equivalent to the following:

(ii)′ The morphism $\nu|V : V \to (I_oN_{\mathbf{a}})_0$ is dominant.

Since $(I_oN_{\mathbf{a}})_0^{ci}$ is Zariski open dense in $(I_oN_{\mathbf{a}})_0$, the implication (i) ⇒ (ii) is obvious. Since $V^\natural$ is Zariski open in V, the implication (ii) ⇒ (i) follows if we show that $V^\natural$ is non-empty under the condition (ii). Suppose that the condition (ii) is fulfilled. Let (g,h) be a general element of V. Since g is general in $(I_oM_{\mathbf{b}})_0$, the ideal J_g defines a complete intersection of multi-degree $\mathbf{b}$ passing through o, and $\mu(g)$ is of rank s. By (ii)′, $h(g)$ is a general element of $(I_oN_{\mathbf{a}})_0$, and hence $J_{h(g)}$ defines a complete intersection of multi-degree $\mathbf{a}$ passing through o and singular at o. Thus we have $(g,h) \in V^\natural$. □

5. Proof of Main Theorem

First we prepare two easy lemmas.

Let L_1 and L_2 be finite-dimensional vector spaces, and let $\operatorname{Hom}(L_1,L_2)$ be the vector space of linear maps from L_1 to L_2. For $\varphi \in \operatorname{Hom}(L_1,L_2)$, we have a canonical identification

$$T_\varphi \operatorname{Hom}(L_1,L_2) \cong \operatorname{Hom}(L_1,L_2). \tag{5.1}$$

Let S_k be the closed subscheme of $\operatorname{Hom}(L_1,L_2)$ defined as common zeros of all $(k+1)$-minors of the matrices expressing the linear maps in terms of certain bases of L_1 and L_2.

Lemma 5.1. *Let φ_0 be a point of $S_k \setminus S_{k-1}$. An element φ of* $\operatorname{Hom}(L_1,L_2)$ *is contained in the subspace $T_{\varphi_0}S_k$ of $T_{\varphi_0}\operatorname{Hom}(L_1,L_2)$ under the identification* (5.1) *if and only if $\varphi(\operatorname{Ker}\varphi_0)$ is contained in* $\operatorname{Im}\varphi_0$.

Proof. We can choose bases of L_1 and L_2 in such a way that φ_0 is expressed by the matrix $\left(\begin{array}{c|c} I_k & O \\ \hline O & O \end{array}\right)$. Suppose that φ is expressed by the matrix $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ under these bases. Then φ is contained in $T_{\varphi_0} S_k$ under the identification (5.1) if and only if the matrix $\left(\begin{array}{c|c} I_k + \varepsilon A & \varepsilon B \\ \hline \varepsilon C & \varepsilon D \end{array}\right)$ is of rank k, where ε is the dual number; $\varepsilon^2 = 0$. This matrix is of rank k if and only if $D = 0$, which is equivalent to $\varphi(\operatorname{Ker} \varphi_0) \subseteq \operatorname{Im} \varphi_0$. □

Let X and Y be connected complex manifolds, Z an irreducible locally closed analytic subspace of Y, $\psi : X \to Y$ a holomorphic map, and p a point of $\psi^{-1}(Z)$.

Lemma 5.2. *Suppose that Z is smooth at $\psi(p)$, and that we have*

$$T_{\psi(p)} Z \cap \operatorname{Im}(d\psi)_p = 0 \quad \text{and} \quad T_{\psi(p)} Z + \operatorname{Im}(d\psi)_p = T_{\psi(p)} Y. \tag{5.2}$$

Then $\psi^{-1}(Z)$ is smooth at p. Moreover, the dimension of $\psi^{-1}(Z)$ at p is equal to $\dim X - \dim Y + \dim Z$.

Proof. By (5.2), we have $T_p \psi^{-1}(Z) = (d\psi)_p^{-1}(T_{\psi(p)} Z) = \operatorname{Ker}(d\psi)_p$, and hence $\dim T_p \psi^{-1}(Z)$ is equal to $\dim T_p X - \dim \operatorname{Im}(d\psi)_p$, which is then equal to $\dim X - \dim Y + \dim Z$. On the other hand, the codimension of $\psi^{-1}(Z)$ in X at p is less than or equal to the codimension of Z in Y at $\psi(p)$. Combining these facts, we get the hoped-for results. □

From now on, we assume that $(n, \mathbf{a}, \mathbf{b})$ satisfies the conditions required in Main Theorem. In particular, the morphism $\alpha | \Xi_o : \Xi_o \to \Gamma_o$ is dominant. Let Ξ'_o be the unique irreducible component of Ξ_o that is mapped dominantly onto Γ_o by $\alpha | \Xi_o$ (Proposition 4.14).

Proposition 5.3. *Let $Q = (o, \langle g \rangle, \langle f \rangle)$ be a general point of Ξ'_o. Then the following hold*:

(1) *The morphism $\rho : F_{\mathbf{b},\mathbf{a}} \to H_{n,\mathbf{a}}$ is dominant.*

(2) *The kernel of $(d\alpha)_Q : T_Q \widetilde{\mathcal{Z}} \to T_{\alpha(Q)} \mathcal{X}$ is of dimension equal to*

$$\dim F_{\mathbf{b},\mathbf{a}} - \dim H_{n,\mathbf{a}} - m + 2l.$$

(3) *The image of $(d\rho)_{\pi(Q)} : T_{\pi(Q)} F_{\mathbf{b},\mathbf{a}} \to T_{\langle f \rangle} H_{n,\mathbf{a}}$ is of codimension 1.*

Proof. First of all, note that $F_{\mathbf{b},\mathbf{a}}$ is non-empty because Ξ_o is non-empty. Note also that $V^\natural$ is non-empty by Proposition 4.13, and hence is Zariski open dense in V. By Proposition 4.14, the general point Q of Ξ'_o is the image of a general point of $V^\natural$ by $\xi : V^\natural \to \Xi_o$. Therefore we can choose a general point (g, h) of V first, and then put $Q := \xi(g, h) = (o, \langle g \rangle, \langle h(g) \rangle)$.

We start with the proof of (3). By Proposition 4.15, we have

$$\delta_{(g,h)}(V) = (J_g N_{\mathbf{a}} + I_o h(M_{\mathbf{b}}))_0 = (I_o N_{\mathbf{a}})_0. \tag{5.3}$$

In particular, we have

$$\dim(M_{\mathbf{a}} / (J_g N_{\mathbf{a}} + I_o h(M_{\mathbf{b}})))_0 = n + r. \tag{5.4}$$

By Corollary 4.7 (2), to arrive at $\dim \operatorname{Coker}(d\rho)_{\pi(Q)} = 1$, all we have to show is

$$\dim(M_{\mathbf{a}} / (J_g M_{\mathbf{a}} + \ h(M_{\mathbf{b}})))_0 = 1,$$

which is equivalent to

$$\dim \delta_{(g,h)}(U) / \delta_{(g,h)}(V) = n + r - 1, \tag{5.5}$$

because of (4.16), (4.17) and (5.4). We define a linear map $\tilde{\lambda}_r : (M_{\mathbf{a}})_0 \to \mathbb{C}^r \times \mathbb{C}^n$ by

$$\tilde{\lambda}_r(f) := \Big((f_1(o), \dots, f_r(o)), \Big(\frac{\partial f_r}{\partial x_1}(o), \dots, \frac{\partial f_r}{\partial x_n}(o)\Big)\Big).$$

Then $\delta_{(g,h)}(V) = (I_o N_{\mathbf{a}})_0 = \operatorname{Ker} \tilde{\lambda}_r$ holds from (5.3). Moreover, we have $\delta_{(g,h)}(U) \subseteq (N_{\mathbf{a}})_0$ by (4.18), and $\dim \tilde{\lambda}_r((N_{\mathbf{a}})_0) = n + r - 1$. Therefore (5.5) and the following two conditions are equivalent to each other:

$$\delta_{(g,h)}(U) = (N_{\mathbf{a}})_0, \tag{5.6}$$

$$\dim \tilde{\lambda}_r(\delta_{(g,h)}(U)) \geq n + r - 1. \tag{5.7}$$

We will prove (5.5) by showing that the inequality (5.7) holds. For $\nu = 1, \dots, s$, we define $\gamma^{(\nu)} = (\gamma_1^{(\nu)}, \dots, \gamma_s^{(\nu)})^T \in (M_{\mathbf{b}})_0$ and $\eta^{(\nu)} = (\eta_{ij}^{(\nu)}) \in \operatorname{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ by

$$\gamma_j^{(\nu)} := \begin{cases} 0 & \text{if } j \neq \nu \\ x_0^{b_j} & \text{if } j = \nu \end{cases} \quad \text{and} \quad \eta_{ij}^{(\nu)} := \begin{cases} 0 & \text{if } (i,j) \neq (r,\nu) \\ x_0^{a_r - b_\nu} & \text{if } (i,j) = (r,\nu). \end{cases}$$

Note that $a_r \geq b_\nu$ by (2.2). We then define $v^{(\nu)}, w^{(\nu)} \in \delta_{(g,h)}(U)$ by

$$\begin{aligned} v^{(\nu)} &:= \delta_{(g,h)}(\gamma^{(\nu)}, 0) = h(\gamma^{(\nu)}), \\ w^{(\nu)} &:= \delta_{(g,h)}(0, \eta^{(\nu)}) = \eta^{(\nu)}(g) = (0, \dots, 0, g_\nu x_0^{a_r - b_\nu})^T. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{\lambda}_r(v^{(\nu)}) &= \Big((h_{1\nu}(o), \dots, h_{r\nu}(o)), \Big(\frac{\partial h_{r\nu}}{\partial x_1}(o), \dots, \frac{\partial h_{r\nu}}{\partial x_n}(o)\Big)\Big), \\ \tilde{\lambda}_r(w^{(\nu)}) &= \Big((0, \dots, 0), \Big(\frac{\partial g_\nu}{\partial x_1}(o), \dots, \frac{\partial g_\nu}{\partial x_n}(o)\Big)\Big). \end{aligned} \tag{5.8}$$

In order to prove (5.7), it is enough to show that the vectors $\tilde{\lambda}_r(v^{(\nu)})$ and $\tilde{\lambda}_r(w^{(\nu)})$ span a hyperplane in $\mathbb{C}^r \times \mathbb{C}^n$. Since (g,h) is general in V, the coefficients $h_{i\nu}(o)$, $\partial h_{r\nu}/\partial x_j(o)$ and $\partial g_\nu/\partial x_j(o)$ of the homogeneous polynomials g_ν and $h_{i\nu}$ that appear in (5.8) are general except for the following restrictions:

$$\begin{aligned} & h_{r\nu}(o) = 0 \ \ (1 \leq \nu \leq s), \qquad h_{i\nu}(o) = 0 \ \text{ if } a_i < b_\nu \quad \text{and} \\ & \frac{\partial h_{r\nu}}{\partial x_j}(o) = 0 \ \ (1 \leq j \leq n) \quad \text{if } a_r = b_\nu. \end{aligned}$$

Let Λ be the $2s \times (n+r)$ matrix whose row vectors are $\tilde{\lambda}_r(v^{(\nu)})$ and $\tilde{\lambda}_r(w^{(\nu)})$. Then Λ is of the shape depicted in Figure 5.1, in which the entries in the submatrices marked with $*$ are general, and the (ν, i)-component in the submatrix marked with $\sharp$ is general except for the restriction that it must be zero if $a_i < b_\nu$. Since the rank of a matrix is a lower semi-continuous function of entries, in order to prove that Λ is of rank $n+r-1$, it is enough to show that there exists at least one matrix of rank $n+r-1$ with the shape Figure 5.1. The condition (2.2) implies that $r - t_a \leq s - t_b$ holds, and that the (i,i)-component of the submatrix $\sharp$ is subject to no restrictions for $i = 1, \dots, r - t_a$. The condition (2.3) implies $n + r < 2s$ and $n + r + t_b - t_a \leq 2s$. Therefore we can define a $2s \times (n+r)$ matrix C of the shape Figure 5.1 by Table 5.1, where c_i is the i-th column vector of C and e_μ is the column vector of dimension $2s$ whose ν-th component is $\delta_{\mu\nu}$ (Kronecker's delta symbol). It is easy to see that C is of rank $n + r - 1$. Hence (5.7), and also (5.5) and (5.6), are proved.

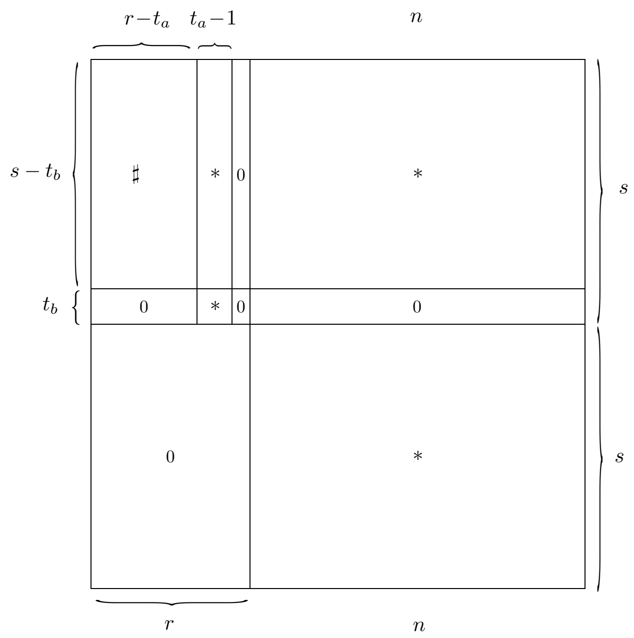

Figure 5.1. The shape of a $2s \times (n+r)$ matrix

Next we prove (1). Since both of $F_{\mathbf{b},\mathbf{a}}$ and $H_{n,\mathbf{a}}$ are smooth and irreducible, it is enough to show that the morphism $\rho : F_{\mathbf{b},\mathbf{a}} \to H_{n,\mathbf{a}}$ is a submersion at a general point of $F_{\mathbf{b},\mathbf{a}}$. By Corollary 4.7 (2), it is therefore enough to prove that the following equality holds for a general $(\tilde{g}, \tilde{h}) \in U$:

$$\dim(M_{\mathbf{a}}/(J_{\tilde{g}}M_{\mathbf{a}} + \tilde{h}(M_{\mathbf{b}})))_0 = 0. \tag{5.9}$$

Since the left-hand side of (5.9) is an upper semi-continuous function of $(\tilde{g}, \tilde{h}) \in U$, it suffices to show that there exists at least one $(\tilde{g}, \tilde{h}) \in U$ for which (5.9) holds. We will find $(\tilde{g}, \tilde{h})$ satisfying (5.9) in a small neighborhood of the chosen point (g, h) in U. From (5.6), we have

$$\dim(M_{\mathbf{a}}/(J_{\tilde{g}}M_{\mathbf{a}} + \tilde{h}(M_{\mathbf{b}})))_0 \leq \dim(M_{\mathbf{a}}/N_{\mathbf{a}})_0 = 1$$

When $t_a - 1 \geq t_b$, $c_i := \begin{cases} e_i & \text{if } 1 \leq i \leq r - t_a \\ e_{s-r+1+i} & \text{if } r - t_a < i < r \\ 0 & \text{if } i = r \\ e_{i-t_a} & \text{if } r < i \leq s+1 \\ e_{i-1} & \text{if } s+1 < i \leq n+r. \end{cases}$	When $t_a - 1 < t_b$, $c_i := \begin{cases} e_i & \text{if } 1 \leq i \leq r - t_a \\ e_{s-r+1+i} & \text{if } r - t_a < i < r \\ 0 & \text{if } i = r \\ e_{i-t_a} & \text{if } r < i \leq s + t_a - t_b \\ e_{i+t_b-t_a} & \text{if } s + t_a - t_b < i \leq n+r. \end{cases}$

Table 5.1. Definition of C

for any $(\tilde{g}, \tilde{h})$ in a small neighborhood of (g, h) in U. We suppose the following:

$$\begin{aligned}\delta_{(\tilde{g},\tilde{h})}(U) = (J_{\tilde{g}} M_{\mathbf{a}} + \tilde{h}(M_{\mathbf{b}}))_0 &\text{ is of codimension 1 in } (M_{\mathbf{a}})_0 \\ &\text{for any } (\tilde{g}, \tilde{h}) \text{ in a small neighborhood of } (g, h) \text{ in } U,\end{aligned} \tag{5.10}$$

and will derive a contradiction. For a sequence $c = (c_1, \dots, c_s)$ of complex numbers, we define $\eta^c = (\eta^c_{ij}) \in \mathrm{Hom}(M_{\mathbf{b}}, M_{\mathbf{a}})_0$ by

$$\eta^c_{ij} := \begin{cases} 0 & \text{if } i < r \\ c_j x_0^{a_r - b_j} & \text{if } i = r, \end{cases}$$

and consider the infinitesimal deformation $(g, h) + \varepsilon(0, \eta^c)$ of (g, h) in U, where ε is the dual number. Here we use the condition $a_r \geq b_j$ again. By (5.6), Lemma 5.1 and the assumption (5.10), we have

$$\delta_{(0,\eta^c)}(\mathrm{Ker}\,\delta_{(g,h)}) \subseteq \mathrm{Im}\,\delta_{(g,h)} = (N_{\mathbf{a}})_0$$

for any c, which means that, if $(G, H) \in \mathrm{Ker}\,\delta_{(g,h)}$, then $\eta^c(G) \in (N_{\mathbf{a}})_0$ for any c. Hence we have

$$(G, H) \in \mathrm{Ker}\,\delta_{(g,h)} \quad \Longrightarrow \quad G \in (I_o M_{\mathbf{b}})_0. \tag{5.11}$$

Because (2.3) implies $2s > n + r$, there exists a non-trivial linear relation

$$\sum_{\nu=1}^{s} \alpha_\nu \tilde{\lambda}_r(v^{(\nu)}) + \sum_{\nu=1}^{s} \beta_\nu \tilde{\lambda}_r(w^{(\nu)}) = 0 \qquad (\alpha_\nu, \beta_\nu \in \mathbb{C})$$

among the vectors (5.8) in $\mathbb{C}^r \times \mathbb{C}^n$. Since g is general in $(I_o M_{\mathbf{b}})_0$ and $s < n$, the vectors $\tilde{\lambda}_r(w^{(\nu)})$ $(\nu = 1, \dots, s)$ are linearly independent, and hence at least one of $\alpha_1, \dots, \alpha_s$ is non-zero. We put

$$(G_1, H_1) := \left(\sum_{\nu=1}^{s} \alpha_\nu \gamma^{(\nu)}, \sum_{\nu=1}^{s} \beta_\nu \eta^{(\nu)} \right) \in U.$$

Then we have

$$\delta_{(g,h)}(G_1, H_1) = \sum_{\nu=1}^{s} \alpha_\nu v^{(\nu)} + \sum_{\nu=1}^{s} \beta_\nu w^{(\nu)} \in \mathrm{Ker}\,\tilde{\lambda}_r = (I_o N_{\mathbf{a}})_0 = \delta_{(g,h)}(V),$$

where the last equality follows from (5.3). Hence there exists $(G_2, H_2) \in V$ such that $(G_1 - G_2, H_1 - H_2) \in \mathrm{Ker}\,\delta_{(g,h)}$. On the other hand, since $G_2 \in (I_o M_{\mathbf{b}})_0$ and at least one of $\alpha_1, \dots, \alpha_s$ is non-zero, we have $G_1 - G_2 \notin (I_o M_{\mathbf{b}})_0$, which contradicts to (5.11). Hence there must exist a point $(\tilde{g}, \tilde{h}) \in U$ in an arbitrary small neighborhood of (g, h) such that (5.9) holds. Therefore ρ is dominant.

Finally we calculate $\dim \mathrm{Ker}(d\alpha)_Q$. By Proposition 4.8, we see that $\dim \mathrm{Ker}(d\alpha)_Q$ is equal to

$$\dim F_{\mathbf{b},\mathbf{a}} - \dim H_{n,\mathbf{a}} - s + \dim(M_{\mathbf{a}}/(J_g M_{\mathbf{a}} + I_o h(M_{\mathbf{b}})))_0.$$

The fourth term is equal to $n+r-s$ by Proposition 4.15. Since $n+r-2s = -m+2l$, we complete the proof of the assertion (2). □

Now we are ready to the proof of Main Theorem.

Proof of Main Theorem. For a locally closed analytic subspace A of $H_{n,\mathbf{a}}$, we denote by

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}_A & \stackrel{\alpha_A}{\longrightarrow} & \mathcal{X}_A \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \phi_A} \\ F_A & \underset{\rho_A}{\longrightarrow} & A \end{array}$$

the pull-back of the right square of the diagram (4.3) by $A \hookrightarrow H_{n,\mathbf{a}}$.

There exists a Zariski open dense subset $\mathcal{U}$ of $H_{n,\mathbf{a}}$ such that

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}_{\mathcal{U}} & \stackrel{\alpha_{\mathcal{U}}}{\longrightarrow} & \mathcal{X}_{\mathcal{U}} \\ {\scriptstyle \pi_{\mathcal{U}}}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{U}}} \\ F_{\mathcal{U}} & \underset{\rho_{\mathcal{U}}}{\longrightarrow} & \mathcal{U} \end{array}$$

is locally trivial over $\mathcal{U}$ in the category of topological spaces and continuous maps, that $\phi_{\mathcal{U}}$ is smooth, and that $\rho_{\mathcal{U}}$ is smooth or $F_{\mathcal{U}}$ is empty. It is enough to show $F_{\mathbf{b}}(X_b) \neq \emptyset$ and $\operatorname{Im} \psi_{\mathbf{b}}(X_b) \supseteq V_m(X_b, \mathbb{Z})$ for at least one point b of $\mathcal{U}$, where X_b denotes the complete intersection corresponding to a point b of $\mathcal{U}$.

By Proposition 5.3 (1), $F_{\mathbf{b}}(X_b)$ is non-empty for any $b \in \mathcal{U}$.

By the assumption of Main Theorem, the morphism $\alpha|\Xi_o : \Xi_o \to \Gamma_o$ is dominant, and hence, by Proposition 4.14, there exists a unique irreducible component Ξ'_o of Ξ_o that is mapped dominantly onto Γ_o by $\alpha|\Xi_o$. Let $Q := (o, \langle g\rangle, \langle f\rangle)$ be a general point of Ξ'_o. Then $\alpha(Q) = (o, \langle f\rangle)$ is a general point of Γ_o. By Corollary 4.11, the point o is the only singular point of $X_{\langle f\rangle}$, and it is a hypersurface singularity with non-degenerate Hessian. In particular, the image of $(d\phi)_{\alpha(Q)} : T_{\alpha(Q)}\mathcal{X} \to T_{\langle f\rangle} H_{n,\mathbf{a}}$ is of codimension 1 in $T_{\langle f\rangle} H_{n,\mathbf{a}}$. On the other hand, by Proposition 5.3 (3), the image of $(d\rho)_{\pi(Q)} : T_{\pi(Q)} F_{\mathbf{b},\mathbf{a}} \to T_{\langle f\rangle} H_{n,\mathbf{a}}$ is also of codimension 1 in $T_{\langle f\rangle} H_{n,\mathbf{a}}$. Hence there exists a smooth curve C in $H_{n,\mathbf{a}}$ passing through $\langle f\rangle$ that satisfies

$$\operatorname{Im}(d\phi)_{\alpha(Q)} \cap T_{\langle f\rangle} C = 0, \quad \operatorname{Im}(d\rho)_{\pi(Q)} \cap T_{\langle f\rangle} C = 0, \tag{5.12}$$

and $C \cap \mathcal{U} \neq \emptyset$. We choose a sufficiently small open unit disk Δ in C with the center $\langle f\rangle$, and consider the following diagrams:

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}_C & \stackrel{\alpha_C}{\longrightarrow} & \mathcal{X}_C \\ {\scriptstyle \pi_C}\downarrow & & \downarrow{\scriptstyle \phi_C} \\ F_C & \underset{\rho_C}{\longrightarrow} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathcal{Z}}_\Delta & \stackrel{\alpha_\Delta}{\longrightarrow} & \mathcal{X}_\Delta \\ {\scriptstyle \pi_\Delta}\downarrow & & \downarrow{\scriptstyle \phi_\Delta} \\ F_\Delta & \underset{\rho_\Delta}{\longrightarrow} & \Delta. \end{array} \tag{5.13}$$

We can assume that $\Delta^\times := \Delta \setminus \{\langle f\rangle\}$ is contained in $\mathcal{U}$. By Lemma 5.2, the analytic space $\mathcal{X}_\Delta$ is smooth of dimension $m+1$. Moreover, the holomorphic map $\phi_\Delta : \mathcal{X}_\Delta \to \Delta$ has only one critical point, which is the point $(o, \langle f\rangle)$ on the central fiber $X_{\langle f\rangle}$, and at which the Hessian of ϕ_Δ is non-degenerate. We select a point b of $\mathcal{U}$ from $\Delta^\times$. Then we have a vanishing cycle $[\Sigma_b] \in H_m(X_b, \mathbb{Z})$, unique up to sign, associated to the non-degenerate critical point $(o, \langle f\rangle)$ of ϕ_Δ. It is known that $V_m(X_b, \mathbb{Z})$ is generated by $[\Sigma_b]$ as a module over the group ring $\mathbb{Z}[\pi_1(\mathcal{U}, b)]$. (See [9].) On the other hand, the image of the cylinder homomorphism $\psi_{\mathbf{b}}(X_b)$ is $\pi_1(\mathcal{U}, b)$-invariant. Therefore it is enough to show that the image of $\psi_{\mathbf{b}}(X_b)$ contains $[\Sigma_b]$.

We put $O := \pi(Q) \in F_{\mathbf{b},\mathbf{a}}$. By Lemma 5.2 and (5.12), the scheme F_C in the left diagram of (5.13) is smooth at O, and

$$\dim_O F_C = \dim F_{\mathbf{b},\mathbf{a}} - \dim H_{n,\mathbf{a}} + 1. \tag{5.14}$$

From the construction of $\widetilde{\mathcal{Z}}_C$, we see that $\mathrm{Ker}(d\alpha)_Q$ is contained in the subspace $T_Q\widetilde{\mathcal{Z}}_C$ of $T_Q\widetilde{\mathcal{Z}}$, and that $\mathrm{Ker}(d\alpha)_Q$ coincides with $\mathrm{Ker}(d\alpha_C)_Q$. Hence, by Proposition 5.3 (2), we have

$$\dim \mathrm{Ker}(d\alpha_C)_Q = \dim F_{\mathbf{b},\mathbf{a}} - \dim H_{n,\mathbf{a}} - m + 2l. \tag{5.15}$$

Since $\widetilde{\mathcal{Z}}_C$ is a closed analytic subspace of $F_C \times \mathcal{X}_C$ with π_C and α_C being projections, we have

$$\mathrm{Ker}(d\pi_C)_Q \cap \mathrm{Ker}(d\alpha_C)_Q = 0 \tag{5.16}$$

in $T_Q\widetilde{\mathcal{Z}}_C$. In particular, the linear map $(d\pi_C)_Q : T_Q\widetilde{\mathcal{Z}}_C \to T_O F_C$ maps $\mathrm{Ker}(d\alpha_C)_Q$ isomorphically to a linear subspace of $T_O F_C$. By the dimension counting (5.14) and (5.15), this subspace

$$(d\pi_C)_Q(\mathrm{Ker}(d\alpha_C)_Q) \subset T_O F_C$$

is of codimension $m - 2l + 1$. Hence there exists a closed subvariety F'_C of F_C with dimension $m - 2l + 1$ that passes through O, is smooth at O, and satisfies

$$T_O F'_C \cap (d\pi_C)_Q(\mathrm{Ker}(d\alpha_C)_Q) = 0. \tag{5.17}$$

We put

$$F'_\Delta := F'_C \cap F_\Delta, \quad \widetilde{\mathcal{Z}}'_C := \pi_C^{-1}(F'_C) \quad \text{and} \quad \widetilde{\mathcal{Z}}'_\Delta := \pi_\Delta^{-1}(F'_\Delta),$$

and let

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}'_C & \xrightarrow{\alpha'_C} & \mathcal{X}_C \\ {\scriptstyle \pi'_C}\downarrow & & \downarrow{\scriptstyle \phi_C} \\ F'_C & \underset{\rho'_C}{\longrightarrow} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathcal{Z}}'_\Delta & \xrightarrow{\alpha'_\Delta} & \mathcal{X}_\Delta \\ {\scriptstyle \pi'_\Delta}\downarrow & & \downarrow{\scriptstyle \phi_\Delta} \\ F'_\Delta & \underset{\rho'_\Delta}{\longrightarrow} & \Delta \end{array} \tag{5.18}$$

be the restriction of the diagrams (5.13). The right diagram of (5.18) is the pull-back of the left diagram of (5.18) by $\Delta \hookrightarrow C$.

Since the fiber of π passing through Q is smooth at Q by the definition of Ξ_o, the holomorphic map π'_Δ is also smooth at Q. Moreover, from (5.16) and (5.17), we have

$$\begin{aligned} \mathrm{Ker}(d\alpha'_\Delta)_Q &= T_Q\widetilde{\mathcal{Z}}'_\Delta \cap \mathrm{Ker}(d\alpha_\Delta)_Q \\ &= (d\pi_C)_Q^{-1}(T_O F'_C) \cap \mathrm{Ker}(d\alpha_C)_Q \;=\; 0. \end{aligned}$$

Therefore α'_Δ is an immersion at Q. We have $\dim F'_\Delta = m - 2l + 1$. Note that $H_m(X_b, \mathbb{Z})$ is torsion free. Hence the right diagram of (5.18) satisfies all the conditions required in Theorem 3.1 (2). We put

$$F'_\Delta(X_b) := \rho'^{-1}_\Delta(b), \quad Z'_\Delta(X_b) := \pi'^{-1}_\Delta(F'_\Delta(X_b)),$$

and consider the family

$$\begin{array}{ccc} Z'_\Delta(X_b) & \longrightarrow & X_b \\ \downarrow & & \\ F'_\Delta(X_b) & & \end{array} \tag{5.19}$$

of l-dimensional closed analytic subspaces of X_b. By Theorem 3.1, the image of the cylinder homomorphism

$$\psi'_{\mathbf{b}}(X_b) \,:\, H_{m-2l}(F'_{\Delta}(X_b), \mathbb{Z}) \;\to\; H_m(X_b, \mathbb{Z})$$

associated with the family (5.19) contains the vanishing cycle $[\Sigma_b] \in H_m(X_b, \mathbb{Z})$. By the construction, $\psi'_{\mathbf{b}}(X_b)$ is the composite of the homomorphism

$$H_{m-2l}(F'_{\Delta}(X_b), \mathbb{Z}) \to H_{m-2l}(F_{\mathbf{b}}(X_b), \mathbb{Z})$$

induced from the inclusion $F'_{\Delta}(X_b) \hookrightarrow F_{\mathbf{b}}(X_b)$ and the original cylinder homomorphism $\psi_{\mathbf{b}}(X_b)$. Hence the image of $\psi_{\mathbf{b}}(X_b)$ contains $[\Sigma_b]$. □

We put $F'_C(X_b) := {\rho'_C}^{-1}(b)$, and let $F''_C(X_b)$ be the union of irreducible components of $F'_C(X_b)$ with dimension $m-2l$. Then $F''_C(X_b)$ contains an $(m-2l)$-dimensional sphere representing the vanishing cycle $[\sigma_b] \in H_{m-2l}(F'_{\Delta}(X_b), \mathbb{Z})$ associated to the non-degenerate critical point O of ρ'_{Δ}. Let T be the Zariski closure of $\alpha'_C({\pi'_C}^{-1}(F''_C(X_b)))$ in X_b. Then T is of dimension $m-l$, and $[\Sigma_b] = \pm\psi'_{\mathbf{b}}(X_b)([\sigma_b])$ is represented by a topological cycle whose support is contained in T. Therefore we obtain the following:

Corollary 5.4. *Suppose that* $(n, \mathbf{a}, \mathbf{b})$ *satisfies the conditions of Main Theorem. Let* X *be a general complete intersection of multi-degree* $\mathbf{a}$ *in* $\mathbb{P}^n$*. Then every vanishing cycle of* X *is represented by a topological cycle whose support is contained in a Zariski closed subset of* X *with codimension* l*.* □

6. Gröbner bases method

Suppose we are given a triple $(n, \mathbf{a}, \mathbf{b})$ that satisfies the conditions (2.2) and (2.3) of Main Theorem. We will describe a method to determine whether this triple satisfies the second condition of Main Theorem.

First we choose a prime integer p, and put

$$R^{(p)} := \mathbb{F}_p[x_0, \dots, x_n].$$

We define graded $R^{(p)}$-modules $M^{(p)}_{\mathbf{a}}$, $M^{(p)}_{\mathbf{b}}$, $N^{(p)}_{\mathbf{a}}$, and ideals $I^{(p)}_o$, $J^{(p)}_g$ of $R^{(p)}$ in the same way as in §4 except for the coefficient field. We generate an element $g = (g_1, \dots, g_s)^T$ of $(I^{(p)}_o M^{(p)}_{\mathbf{b}})_0$ and a homomorphism $h = (h_{ij}) \in \mathrm{Hom}(M^{(p)}_{\mathbf{b}}, N^{(p)}_{\mathbf{a}})_0$ in a random way. Then we can calculate

$$\dim_{\mathbb{F}_p}(M^{(p)}_{\mathbf{a}}/(J^{(p)}_g M^{(p)}_{\mathbf{a}} + I^{(p)}_o h(M^{(p)}_{\mathbf{b}})))_0 \tag{6.1}$$

by means of Gröbner bases. If this dimension is $\leq n+r-s$, then the condition (iv) of Proposition 4.15 is fulfilled, because this condition is an open condition. Hence the morphism $\alpha|\Xi_o : \Xi_o \to \Gamma_o$ is dominant.

7. Application of a theorem of Debarre and Manivel

From now on, we use the following terminology. A *sequence* always means a finite non-decreasing sequence of positive integers. For a sequence $\mathbf{a}$, let $\min(\mathbf{a})$ and $\max(\mathbf{a})$ be the first and the last elements of $\mathbf{a}$, respectively, and let $|\mathbf{a}|$ denote the length of $\mathbf{a}$. Let $\mathbf{a}'$ be another sequence. We denote by $\mathbf{a} \uplus \mathbf{a}'$ the sequence of length $|\mathbf{a}|+|\mathbf{a}'|$ obtained by re-arranging the conjunction $(\mathbf{a},\mathbf{a}')$ into the non-decreasing order. For an integer $a\geq 2$, we define $(a)!$ to be the sequence $(2,\dots,a)$ of length $a-1$, and for a sequence $\mathbf{a}=(a_1,\dots,a_r)$ with $\min(\mathbf{a})\geq 2$, we put

$$\mathbf{a}! := (a_1)! \uplus \cdots \uplus (a_r)!\ .$$

We sometimes write a sequence by indicating the number of repetition of each integer in the sequence by a superscript. For example, we have $(2,3,3,4)! = (2,2,2,2,3,3,3,4) = (2^4,3^3,4)$.

Let n and ℓ be positive integers, and $\mathbf{a}=(a_1,\dots,a_r)$ a sequence. According to [5], we put

$$\delta(n,\mathbf{a},\ell) := (\ell+1)(n-\ell) - \sum_{i=1}^{r} \binom{a_i+\ell}{\ell},$$

and $\delta_-(n,\mathbf{a},\ell) := \min\{\delta(n,\mathbf{a},\ell), n-2\ell-|\mathbf{a}|\}$.

Theorem 7.1 ([5], Théorème 2.1)**.** *A general complete intersection of multi-degree* $\mathbf{a}$ *in* $\mathbb{P}^n$ *contains an* ℓ*-dimensional linear subspace if and only if* $\delta_-(n,\mathbf{a},\ell)\geq 0$. □

Theorem 7.2. *Let* $\mathbf{a}=(a_1,\dots,a_r)$ *be a sequence satisfying* $\min(\mathbf{a})\geq 2$ *and* $\sum_{i=1}^r a_i \leq n$*. Let* $\mathbf{a}'$ *be a sub-sequence of* $\mathbf{a}$ *such that* $\max(\mathbf{a}')=\max(\mathbf{a})$*, and let* $\mathbf{a}''$ *be the complement to* $\mathbf{a}'$ *in* $\mathbf{a}$*. (When* $\mathbf{a}'=\mathbf{a}$*,* $\mathbf{a}''$ *is the empty sequence.) Suppose that a positive integer* λ *satisfies the following:*

$$(7.1)\qquad \delta_-(n-|\mathbf{a}'|,\mathbf{a}'!,\lambda-1)\geq 0,\quad |\mathbf{a}''|<\lambda \quad \text{and} \quad n-r>2(\lambda-|\mathbf{a}''|).$$

We put $\mathbf{b}:=(1^{n-\lambda})\uplus\mathbf{a}''$*. Then* $F_{\mathbf{b}}(X)$ *is non-empty for a general complete intersection* X *of multi-degree* $\mathbf{a}$ *in* $\mathbb{P}^n$*, and the image of the cylinder homomorphism* $\psi_{\mathbf{b}}(X)$ *contains* $V_m(X,\mathbb{Z})$*.*

Proof. Note that $l=n-|\mathbf{b}|$ is equal to $\lambda-|\mathbf{a}''|$. Since $\max(\mathbf{a}')=\max(\mathbf{a})$, we can assume that a_r is a member of $\mathbf{a}'$. Let $f=(f_1,\dots,f_r)^T$ be a general element of $(I_oN_{\mathbf{a}})_0$, and let Y_i be the hypersurface of degree a_i defined by $f_i=0$. We put

$$X' := \bigcap_{a_i\in\mathbf{a}'} Y_i \quad\text{and}\quad X'' := \bigcap_{a_i\in\mathbf{a}''} Y_i.$$

By Proposition 4.9, X' is a general member of the family of complete intersections of multi-degree $\mathbf{a}'$ possessing a singular point at o. By means of the projection with the center o, we see that X' contains a linear subspace of dimension $\ell>0$ that passes through o if and only if a general complete intersection of multi-degree $\mathbf{a}'!$ in $\mathbb{P}^{n-|\mathbf{a}'|}$ contains an $(\ell-1)$-dimensional linear subspace. By Theorem 7.1, the first condition of (7.1) implies that X' contains a linear subspace Λ of dimension λ passing through o. In particular, we have $\lambda<n-|\mathbf{a}'|$. Using this inequality and the second and the third conditions of (7.1), we can easily check that $(n,\mathbf{a},\mathbf{b})$ satisfies the conditions (2.2) and (2.3) in Main Theorem.

We put $Z:=\Lambda\cap X''$. Then Z is a complete intersection of multi-degree $\mathbf{b}$ contained in $X_{\langle f\rangle}=X'\cap X''$ and passing through o. Moreover, since the polynomials f_i $(a_i\in\mathbf{a}'')$ are general with respect to Λ, Z is smooth. Thus the second condition of Main Theorem is also satisfied. □

n	$\mathbf{a}$
6	(3)
7	(3)
8	$(2,3),(3),(4)$
9	$(2,3),(3),(3^2),(4)$
10	$(2,3),(3),(3^2),(4)$
11	$(2^2,3),(2,4),(3),(3^2),(4),(5)$
12	$(2^2,3),(2,3),(2,3^2),(2,4),(3),(3^2),(3^3),(5)$
13	$(2^3,3),(2^2,4),(2,3),(2,3^2),(2,4),(3),(3^2),(3^3),(3,4),(5)$
14	$(2^3,3),(2^2,4),(2,5),(3),(3^2),(3^3),(3,4),(4),(4^2),(5)$
15	$(2^2,3),(2^2,3^2),(2^2,4),(2,3),(2,3^2),(2,3^3),(2,3,4),(2,5),(3),(3^2),$ $(3^3),(3^4),(4),(4^2),(6)$
16	$(2^4,3),(2^3,4),(2^2,5),(2,3),(2,3^2),(2,3^3),(2,3,4),(2,5),(3),(3^2),$ $(3^3),(3^4),(3,5),(6)$
17	$(2^3,4),(2^2,5),(2,4),(2,4^2),(2,6),(3),(3^2),(3^3),(3^4),(3^2,4),(3,5),(6)$
18	$(2^5,3),(2^4,4),(2^2,3,4),(2^2,5),(2,3,5),(2,6),(4,5)$
19	$(2^4,4),(2^3,5),(2,3),(2,3^2),(2,3^3),(2,3^4),(2,3,5),(2,6),(3),(3^2),$ $(3^3),(3^4),(3^5),(3,6),(7)$
20	$(2^3,3,4),(2^3,5),(2^2,6),(3),(3^2),(3^3),(3^4),(3^5),(3^2,5),(3,6),(7)$
21	$(2^5,4),(2^4,5),(2^2,3,5),(2^2,6),(2,7)$
22	$(2^4,5),(2^3,6),(2,3,6),(2,7)$
23	$(2^6,4),(2^3,3,5),(2^3,6),(3),(3^2),(3^3),(3^4),(3^5),(3^6),(3,7),(8)$
24	$(2^5,5),(2^2,3,6),(2^2,7)$
25	$(2^4,6)$
26	$(2^6,5),(2^3,7)$
27	$(2^5,6)$

TABLE 8.1. The 148 pairs

8. The generalized Hodge conjecture for complete intersections

Suppose we are given a pair $(n,\mathbf{a})$ satisfying $\min(\mathbf{a}) \geq 2$ and $\sum a_i \leq n$. We put $k := [(n-\sum a_i)/\max(\mathbf{a})]+1$. The Hodge structure of the middle cohomology group $H^m(X)$ of a general complete intersection X of multi-degree $\mathbf{a}$ in $\mathbb{P}^n$ satisfies (1.1). We will investigate the consequence of the generalized Hodge conjecture that there should exist a Zariski closed subset T of X with codimension k such that every element of $H_m(X,\mathbb{Q})$ is represented by a topological cycle whose support is contained in T. Note that $H_m(X,\mathbb{Q})$ is generated by vanishing cycles and, if m is even, the homology class of an intersection of X and a linear subspace of $\mathbb{P}^n$. Hence, by Corollary 5.4, this consequence is verified if we can find $\mathbf{b}$ with the following properties:

$$l := n - |\mathbf{b}| = k \quad \text{and}$$
$$(n,\mathbf{a},\mathbf{b}) \text{ satisfies the assumptions of Main Theorem.} \tag{8.1}$$

n	$\mathbf{a}$	$\mathbf{b}$
10	$(2^2,3)$	$(1^7,2)$
11	$(2,3)$	$(1^7,2)$
	$(2,3^2)$	$(1^7,2,3)$
12	$(2^3,3)$	$(1^7,2^3)$
13	$(2^2,3)$	$(1^8,2^2)$
	$(2^2,3^2)$	$(1^8,2^2,3)$
14	$(2^2,3)$	$(1^{10},2)$
	$(2^2,3^2)$	$(1^9,2^2,3)$
	$(2,3)$	$(1^9,2)$
	$(2,3^2)$	$(1^9,2,3)$
	$(2,3^3)$	$(1^9,2,3^2)$

n	$\mathbf{a}$	$\mathbf{b}$
15	$(2^4,3)$	$(1^9,2^4)$
16	$(2^3,3)$	$(1^{10},2^3)$
	$(2^3,3^2)$	$(1^{10},2^3,3)$
17	$(2^3,3)$	$(1^{13},2)$
	$(2^3,3^2)$	$(1^{11},2^3,3)$
	$(2^2,3)$	$(1^{11},2^2)$
	$(2^2,3^2)$	$(1^{11},2^2,3)$
	$(2^2,3^3)$	$(1^{11},2^2,3^2)$
18	$(2^2,3)$	$(1^{13},2)$
	$(2^2,3^2)$	$(1^{13},2,3)$
	$(2^2,3^3)$	$(1^{13},2,3^2)$

n	$\mathbf{a}$	$\mathbf{b}$
18	$(2,3)$	$(1^{12},2)$
	$(2,3^2)$	$(1^{12},2,3)$
	$(2,3^3)$	$(1^{12},2,3^2)$
	$(2,3^4)$	$(1^{12},2,3^3)$
19	$(2^4,3)$	$(1^{12},2^4)$
	$(2^4,3^2)$	$(1^{12},2^4,3)$
20	$(2^3,3)$	$(1^{13},2^3)$
	$(2^3,3^2)$	$(1^{13},2^3,3)$
	$(2^3,3^3)$	$(1^{13},2^3,3^2)$

TABLE 8.2. Examples of triples obtained by Gröbner bases method

In the following, we assume $m > 2k$. This inequality $m > 2k$ fails to hold if and only if $m \leq 2$ or $\mathbf{a} = (2)$ or ($\mathbf{a} = (2,2)$ and m even). In these cases, the Hodge conjecture has been already proved.

Proposition 8.1 ([14], [16]). (1) *If* $k = 1$, *then* $\mathbf{b} = (1^{n-1})$ *satisfies* (8.1).
(2) *If* $\mathbf{a} = (2^r)$, *then* $\mathbf{b} = (1^{n-[n/2]}, 2^{r-1})$ *satisfies* (8.1).

Proof. Put $\mathbf{a}' = \mathbf{a}$ in the case (1) and $\mathbf{a}' = (2)$ in the case (2), and apply Theorem 7.2. □

In these cases, the consequence of the generalized Hodge conjecture is verified in any dimension.

We have made an exhaustive search in $n \leq 40$, and found 148 pairs $(n, \mathbf{a})$ that are not covered by Proposition 8.1, but for which Theorem 7.2 yields $\mathbf{b}$ satisfying (8.1) by taking an appropriate sub-sequence $\mathbf{a}'$. We list up these $(n, \mathbf{a})$ in Table 8.1. No such $(n, \mathbf{a})$ are found in $n > 27$. Even if $(n, \mathbf{a})$ does not appear in Table 8.1, the calculation of the dimension (6.1) by Gröbner bases sometimes gives us $\mathbf{b}$ with (8.1). Examples of these $(n, \mathbf{a}, \mathbf{b})$ in $n \leq 20$ are given in Table 8.2. From these results, we can find $\mathbf{b}$ with (8.1) for any $(n, \mathbf{a})$ with $n \leq 9$. When $n = 10$, $\mathbf{a} = (2,4)$ and $\mathbf{a} = (5)$ appear in neither Tables 8.1 nor 8.2.

As a closing remark, let us return to the classical example of cubic threefolds ([2]). Our method shows that, not only the family of lines $\mathbf{b} = (1^3)$, but also the family of curves with $\mathbf{b} = (1^2, 2)$ or $(1, 2^2)$ or (2^3) give a surjective cylinder homomorphism on the middle homology group $H_3(X, \mathbb{Z})$ of a general cubic threefold X.

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