THE GRAPHS OF HOFFMAN-SINGLETON, HIGMAN-SIMS, AND MCLAUGHLIN, AND THE HERMITIAN CURVE OF DEGREE 6 IN CHARACTERISTIC 5: COMPUTATIONAL DATA

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This note is an explanation of the computational data in the web page

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/HSgraphs.html`

These data are used in the proof of the main results in the paper

"The graphs of Hoffman-Singleton, Higman-Sims, and McLaughlin, and the Hermitian curve of degree 6 in characteristic 5",

which will be referred as [P] in the sequel. In the following, we use the same notation as in [P].

1. Geometric construction

The following are used to verify the results in Section 1.1 of [P].

- The list `P` is the list of Weierstrass points of Γ_5; that is the list of $\mathbb{F}_{25}$-rational points of Γ_5. We have $|\mathtt{P}| = 126$. A member $[a, b, c]$ of `P` is the point $(a : b : c) \in \mathbb{P}^2$, where $a, b, c \in \mathbb{F}_{25} = \mathbb{F}_5(\alpha)$ and $\alpha := \sqrt{2}$. Let p_i denote the ith point in `P`.
- The list `S` is the list of collinear six points in `P`; that is, the list of special secant lines $\mathcal{S}_5$ of Γ_5. We have $|\mathtt{S}| = 525$. A member $[i_1, \dots, i_6]$ of `S` indicates the set of collinear six points $\{p_{i_1}, \dots, p_{i_6}\}$. We denote by L_j the special secant line corresponding to the jth member $[i_1, \dots, i_6]$ of `S`. We have
$$L_j \cap \Gamma_5 = \{p_{i_1}, \dots, p_{i_6}\}.$$
- The list `Q` is the list of co-conical set of six points in `P`; that is, the list $\mathcal{Q}_5$ of conics totally tangent to Γ_5. We have $|\mathtt{Q}| = 3150$. A member $[i_1, \dots, i_6]$ of `Q` indicates the set of co-conical set of six points $\{p_{i_1}, \dots, p_{i_6}\}$. We denote by Q_ν the conic in $\mathcal{Q}_5$ corresponding to the νth member $[i_1, \dots, i_6]$ of `Q`. We have
$$Q_\nu \cap \Gamma_5 = \{p_{i_1}, \dots, p_{i_6}\}.$$
- The list `SQ` is the list of special secant lines of conics in $\mathcal{Q}_5$. If the νth member of `SQ` is $[j_1, \dots, j_{15}]$, then we have
$$\mathcal{S}(Q_\nu) = \{L_{j_1}, \dots, L_{j_{15}}\}.$$
- The list `EQ` is the list of defining equations of conics in $\mathcal{Q}_5$. If the νth member of `EQ` is $[a, b, c, d, e, f]$, then Q_ν is defined by
$$a\,x^2 + b\,y^2 + c\,z^2 + d\,xy + e\,yz + f\,zx = 0,$$
where $a, b, \dots, f \in \mathbb{F}_{25}$.

- The matrix `M0` is the 3150×3150 matrix whose (i, j) entry is equal to $|Q_i \cap Q_j \cap \Gamma_5|$.
- The matrix `M1` is the 3150×3150 matrix whose (i, j) entry is equal to $|\mathcal{S}(Q_i) \cap \mathcal{S}(Q_j)|$.
- The matrix `M2` is the 3150×3150 matrix whose (i, j) entry is

$$\begin{cases} 1 & \text{if } |Q_i \cap Q_j| = 4, \\ 0 & \text{otherwise.} \end{cases}$$

- The matrix `AG` is the 3150×3150 matrix whose (i, j) entry is

$$\begin{cases} 1 & \text{if } Q_i \text{ and } Q_j \text{ are adjacent in } G, \\ 0 & \text{otherwise.} \end{cases}$$

- The list `D` is the list of connected components of G. We have $|\mathtt{D}| = 150$. A member $[\nu_1, \dots, \nu_{21}]$ of `D` indicates the connected component of G whose set of vertices is

$$\{Q_{\nu_1}, \dots, Q_{\nu_{21}}\}.$$

  Let D_k denote the connected component of G corresponding to the kth member of the list `D`.
- The matrix `tmat` is the 3150×150 matrix whose (ν, k) entry is

$$\begin{cases} 0 & \text{if } Q_\nu \in D_k, \\ \mathtt{aa} & \text{if } t(Q_\nu, D_k) = \alpha, \\ \mathtt{bb} & \text{if } t(Q_\nu, D_k) = \beta, \\ \mathtt{cc} & \text{if } t(Q_\nu, D_k) = \gamma. \end{cases}$$

- The matrix `TT` is the 150×150 matrix whose (j, k) entry is

$$\begin{cases} 0 & \text{if } j = k, \\ \mathtt{bb21} & \text{if } T(D_j, D_k) = \beta^{21}, \\ \mathtt{cc21} & \text{if } T(D_j, D_k) = \gamma^{21}, \\ \mathtt{aa15cc6} & \text{if } T(D_j, D_k) = \alpha^{15}\gamma^6, \\ \mathtt{aa3cc18} & \text{if } T(D_j, D_k) = \alpha^3\gamma^{18}. \end{cases}$$

- The matrix `AH` is the 150×150 matrix whose (j, k) entry is

$$\begin{cases} 1 & \text{if } D_j \text{ and } D_k \text{ are adjacent in } H, \\ 0 & \text{otherwise.} \end{cases}$$

- The list `C` is the list $[\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3]$ of the lists of vertices of connected components of H. The νth member $[k_1, \dots, k_{50}]$ of `C` indicates

$$\mathcal{C}_\nu = \{D_{k_1}, \dots, D_{k_{50}}\}.$$

- The list `AHfSg` is the adjacency matrices of the three graphs $H|\mathcal{C}_1$, $H|\mathcal{C}_2$, $H|\mathcal{C}_3$. If the νth member of `C` is $[k_1, \dots, k_{50}]$, then the νth member of `AHfSg` is the 50×50 matrix whose (i, j) entry is

$$\begin{cases} 1 & \text{if } D_{k_i} \text{ and } D_{k_j} \text{ are adjacent in } H|\mathcal{C}_\nu, \\ 0 & \text{otherwise.} \end{cases}$$

- The matrix `AHH` is the 150×150 matrix whose (j,k) entry is
$$\begin{cases} 1 & \text{if } D_j \text{ and } D_k \text{ are adjacent in } H', \\ 0 & \text{otherwise.} \end{cases}$$
- The list `AHgSm` is the adjacency matrices of the three graphs $H|(\mathcal{C}_2 \cup \mathcal{C}_3)$, $H|(\mathcal{C}_3 \cup \mathcal{C}_1)$, $H|(\mathcal{C}_1 \cup \mathcal{C}_2)$. Let $[\nu_1, \nu_2]$ be one of $[2,3], [3,1], [1,2]$, and let μ be the element of $\{1,2,3\}$ such that $\mu \neq \nu_1$ and $\mu \neq \nu_2$. Suppose that the ν_1th member of `C` is $[k_1, \dots, k_{50}]$ and the ν_2th member of `C` is $[k_{50+1}, \dots, k_{50+50}]$. Then the μth member of `AHgSm` is the 100×100 matrix whose (i,j) entry is
$$\begin{cases} 1 & \text{if } D_{k_i} \text{ and } D_{k_j} \text{ are adjacent in } H'|(\mathcal{C}_{\nu_1} \cup \mathcal{C}_{\nu_2}), \\ 0 & \text{otherwise.} \end{cases}$$
- The list `gg` is the list of three maps g_1, g_2, g_3 defined in Proposition 1.11 of [P]. Let $[\nu_1, \nu_2]$ be one of $[2,3], [3,1], [1,2]$, and let μ be the element of $\{1,2,3\}$ such that $\mu \neq \nu_1$ and $\mu \neq \nu_2$. Suppose that the ν_1th member of `C` is $[k_1, \dots, k_{50}]$ and the ν_2th member of `C` is $[k_{50+1}, \dots, k_{50+50}]$. Then the μth member of `gg` is the list
$$[g_\mu(D_{k_1}), \dots, g_\mu(D_{k_{50+50}})]$$
of 100 members, each of which indicates a 15-coclique in H. The ith member $[j_1, \dots, j_{15}]$ of the μth member of `gg` indicates that
$$g_\mu(D_{k_i}) = \{D_{j_1}, \dots, D_{j_{15}}\}.$$
- The list `E1` is the list of edges of $H|\mathcal{C}_1$. We have $|\mathtt{E1}| = 175$. A member $[j,k]$ of `E1` indicates the edge $\{D_j, D_k\}$ of $H|\mathcal{C}_1$.
- The matrix `AE1` is the adjacency matrix of $(\mathcal{E}_1, \sim)$. The vertices $\mathcal{E}_1$ are sorted by the list `E1`.
- The matrix `AMcL` is the adjacency matrix of H''. Let `C2` and `C3` be the second and the third member of `C`. Suppose that
$$\begin{aligned} \mathtt{E1} &= [[j_1, k_1], \dots, [j_{175}, k_{175}]], \\ \mathtt{C2} &= [i_1, \dots\dots\dots, i_{50}], \\ \mathtt{C3} &= [i_{50+1}, \dots, i_{50+50}]. \end{aligned}$$
Then the vertices of H'' are sorted as
$$\{D_{j_1}, D_{k_1}\}, \ \dots, \ \{D_{j_{175}}, D_{k_{175}}\}, \ \ D_{i_1}, \ \ \dots, \ \ D_{i_{100}},$$
and `AMcL` is constructed according to this order of vertices of H''.

The following are data defined in Section 2 of [P].

- The list `K` is the list of 6-cliques in G. We have $|\mathtt{K}| = 1050$, and a member $[\nu_1, \dots, \nu_6]$ of `K` indicates $\{Q_{\nu_1}, \dots, Q_{\nu_6}\}$.
- The list `KD` is the list of the lists of 6-cliques in each connected component of G. If the kth member of `KD` is
$$[[\nu_1^{(1)}, \dots, \nu_6^{(1)}], \dots, [\nu_1^{(7)}, \dots, \nu_6^{(7)}]],$$
then the seven 6-cliques in the kth connected component D_k is
$$\{Q_{\nu_1^{(1)}}, \dots, Q_{\nu_6^{(1)}}\}, \ \dots, \ \{Q_{\nu_1^{(7)}}, \dots, Q_{\nu_6^{(7)}}\}.$$

- The list PK is the list $\mathcal{PK}$. Each member $[[i_1,\dots,i_6],[j_1,\dots,j_6]]$ indicates a pair $\{K,K'\}$ such that
$$K=\{Q_{i_1},\dots,Q_{i_6}\},\quad K'=\{Q_{j_1},\dots,Q_{j_6}\}.$$
- The list SD is the list of $\mathcal{S}_D$. The jth member $[i_1,\dots,i_{105}]$ of SD indicates
$$\mathcal{S}_{D_j}=\{L_{i_1},\dots,L_{i_{105}}\}.$$
- The list CC is the list $[\widetilde{\mathcal{C}}_1,\widetilde{\mathcal{C}}_2,\widetilde{\mathcal{C}}_3]$. A member $[\nu_1,\dots,\nu_{1050}]$ of CC indicates the subset $\{Q_{\nu_1},\dots,Q_{\nu_{1050}}\}$ of $\mathcal{Q}_5$.
- The list RQ is the list of $\mathcal{R}(Q_\nu)$. The νth member $[i_1,\dots,i_{45}]$ of RQ indicates
$$\mathcal{R}(Q_\nu)=\{Q_{i_1},\dots,Q_{i_{45}}\}.$$
- The list RD is the list of $\mathcal{R}_D$. The jth member $[i_1,\dots,i_{735}]$ of RD indicates
$$\mathcal{R}_{D_j}=\{Q_{i_1},\dots,Q_{i_{735}}\}.$$
- The list ff is the list of all $[j,k,\mu,\nu]$, where $T(D_j,D_k)=\beta^{21}$, $Q_\mu\in D_j$, $Q_\nu\in D_k$ and $f_{D_j,D_k}(Q_\mu)=Q_\nu$.
- The list nn is the list of vectors $n(Q_i)$ in Table 4.3 of [P]. The first member $n(Q_1)$ is $[\infty,\infty,\infty,\infty]$.

2. Group-theoretic construction

We express each element of $\mathrm{PGU}_3(\mathbb{F}_{25})$ as a 3×3 matrix with components in $\mathbb{F}_{25}$ acting on $\mathbb{P}^2$ by the left multiplication. Since $|\mathrm{PGU}_3(\mathbb{F}_{25})|$ is large, we present $\mathrm{PGU}_3(\mathbb{F}_{25})$ in the following way. Note that $\mathrm{PGU}_3(\mathbb{F}_{25})$ acts on the set P transitively.

- We denote by GS the stabilizer of the first member $p_1:=[0,1,2]$ of P:
$$\texttt{GS}:=\{\ g\in\mathrm{PGU}_3(\mathbb{F}_{25})\ \mid\ g(p_1)=p_1\ \}.$$
- We denote by GT the list of representatives of the cosets $\mathrm{PGU}_3(\mathbb{F}_{25})/\texttt{GS}$ such that the ith element of GT maps p_1 to the ith point of P.

Hence each element of $\mathrm{PGU}_3(\mathbb{F}_{25})$ is uniquely written as $\tau\sigma$, where $\sigma\in$ GS and $\tau\in$ GT. By [i, j], we denote the element $\tau_j\sigma_i$ of $\mathrm{PGU}_3(\mathbb{F}_{25})$, where σ_i is the ith member of GS and τ_j is the jth member of GT. A permutation on the list P is written as
$$[\nu_1,\dots,\nu_{126}],$$
which indicates the permutation
$$\text{the } k\text{th point of P}\ \mapsto\ \text{the } \nu_k\text{th point of P}.$$
Each permutation on Q or D is expressed in the same way by a list of 3150 or 150 indices, respectively.

- GSonP is the list of permutations on P induced by the elements of GS. The ith member of GSonP is the permutation induced by the ith element of GS.
- GTonP is the list of permutations on P induced by the elements of GT.
- GSonQ is the list of permutations on Q induced by the elements of GS.
- GTonQ is the list of permutations on Q induced by the elements of GT.
- GSonD is the list of permutations on D induced by the elements of GS.
- GTonD is the list of permutations on D induced by the elements of GT.
- stabQ1 is the stabilizer subgroup $\mathrm{stab}(Q_1)$ of the first member Q_1 of Q written as the list of [i, j]'s.

- `stabD1` is the stabilizer subgroup $\mathrm{stab}(D_1)$ of the first member D_1 of `D` written as the list of [`i`, `j`]'s.
- `stabQ1Qs` is the list of $\mathrm{stab}(Q_1, Q_\nu)$ for $Q_\nu \in$ `Q`. The νth member of `stabQ1Qs` is $[i_1, \dots, i_N]$ if $\mathrm{stab}(Q_1, Q_\nu)$ consists of the i_kth elements ($k = 1, \dots, N$) of the list `stabQ1`.
- `stabD1Ds` is the list of $\mathrm{stab}(D_1, D_\mu)$ for $D_\mu \in$ `D`. The μth member of `stabD1Ds` is $[j_1, \dots, j_M]$ if $\mathrm{stab}(D_1, D_\mu)$ consists of the j_kth elements ($k = 1, \dots, M$) of the list `stabD1`.
- `stabQ1onP` is the permutation representation of $\mathrm{stab}(Q_1)$ on `P`. The ith member of `stabQ1onP` is the permutation on `P` induced by the ith member of `stabQ1`. Using `stabQ1Qs` and `stabQ1onP`, we can determine the isomorphism classes of the groups $\mathrm{stab}(Q_1, Q_\nu)$.
- `stabQ1Qsgr` is the list of isomorphism classes of the groups $\mathrm{stab}(Q_1, Q_\nu)$, where

  `0` : the trivial group, `C2` $:= \mathbb{Z}/2\mathbb{Z}$, `C3` $:= \mathbb{Z}/3\mathbb{Z}$, `C22` $:= (\mathbb{Z}/2\mathbb{Z})^2$,

  `D8` $:= \mathfrak{D}_8$, `D10` $:= \mathfrak{D}_{10}$, `D12` $:= \mathfrak{D}_{12}$,

  `A4` $:= \mathfrak{A}_4$, `S5` $:= \mathfrak{S}_5$ (only when $\nu = 1$).
- `Qdatas` is the the list of $(a(Q_\nu), s(Q_\nu), n(Q_\nu), \mathrm{stab}(Q_1, Q_\nu))$. This is produced from the first row of `M0`, the first row of `M1`, the list of vectors `nn` and `stabQ1Qsgr`.
- `stabQ1onQ` is the permutation representation of $\mathrm{stab}(Q_1)$ on `Q`. The ith member of `stabQ1onQ` is the permutation on `Q` induced by the ith member of `stabQ1`.
- `Qreps` is the list of representatives of the cosets $\mathrm{PGU}_3(\mathbb{F}_{25})/$`stabQ1` (written as the list of [`i`, `j`]'s) such that the ith element of `Qreps` maps Q_1 to the ith conic of `Q`. Using `stabQ1onQ` and `Qreps`, we can calculate $\mathrm{stab}(Q_i)$ for any $Q_i \in$ `Q`.
- `orbsQ` is the list of orbits of the action of $\mathrm{stab}(Q_1)$ on `Q`. An orbit $\{Q_{i_1}, \dots, Q_{i_N}\}$ is written as $[i_1, \dots, i_N]$.

The triangular graph $T(7)$ is defined as the graph whose set of vertices is the set of unordered pairs of distinct elements of $\{1, 2, 3, 4, 5, 6, 7\}$ and whose set of edges is the set of pairs $\{\{i, j\}, \{i', j'\}\}$ such that $\{i, j\} \cap \{i', j'\} \neq \emptyset$.

- The first member of `D` is

$$\texttt{D1} = [1, 309, 434, 1454, 1535, 1628, 2063, 2120, 2187, 2445, \\ 2489, 2511, 2556, 2592, 2615, 2708, 2790, 3082, 3086, 3116, 3122].$$

- The list

$$\texttt{kappa} = [[1, 2], [1, 3], [2, 3], [4, 5], [3, 4], [3, 5], [6, 7], [3, 6], [3, 7], [5, 7], \\ [4, 6], [1, 5], [2, 6], [2, 7], [1, 4], [5, 6], [4, 7], [2, 4], [2, 5], [1, 6], [1, 7]]$$

  indicates the isomorphism of the graphs $\kappa : D_1 \to T(7)$ such that the vertex Q_{i_ν} of D_1 corresponding to the νth index i_ν of `D1` is mapped to the νth pair of `kappa` by κ.
- `A7` is the list of elements of $\mathfrak{A}_7$. Each member is written as $[i_1, \dots, i_7]$, which is the permutation that maps ν to i_ν for $\nu = 1, \dots, 7$.
- `SGa`, `SGb`, `SGb2`, `SGc`, `SGd` are the lists of elements of the subgroups Σ_a, Σ_b, Σ'_b, Σ_c and Σ_d of $\mathfrak{A}_7$ defined in Section 1.2 of [P].

- The list

$$\texttt{total0} = [[[1,2],[3,4],[5,6]],[[1,3],[2,5],[4,6]],[[1,4],[2,6],[3,5]], [[1,5],[2,4],[3,6]],[[1,6],[2,3],[4,5]]]$$

  is the total t_0 used in the definition of Σ_d.
- `stabD1onD1` is the permutation representation of $\mathrm{stab}(D_1)$ on `D1`. The ith member $[j_1, \dots, j_{21}]$ of `stabD1onD1` is the permutation

  the kth member of `D1` $\mapsto$ the j_kth member of `D1` $\quad (k = 1, \dots, 21)$

  induced by the ith element of `stabD1`.
- `stabD1on7` is the permutation representation of $\mathrm{stab}(D_1)$ on $\{1, \dots, 7\}$ via `kappa`. The ith member of `stabD1on7` is the permutation of $\{1, \dots, 7\}$ induced by the ith element of `stabD1`. Using `stabD1Ds` and `stabD1on7`, we can determine the image of $\mathrm{stab}(D_1, D_\nu)$ in `A7`.
- `conjstabD1Ds` is the list of $[\gamma_\nu, \texttt{group\_name}]$ for $\nu = 1, \dots, 150$ such that γ_ν is an element of `A7` satisfying

$$\gamma_\nu^{-1} \cdot \mathrm{stab}(D_1, D_\nu) \cdot \gamma_\nu = \texttt{SG\_group\_name},$$

  where `group_name` is `Aseven` (only when $\nu = 1$), `a`, `b`, `b2`, `c` or `d`.
- `genMs` is the list of the matrices $g_2, \dots, g_6$ given at the last section of [P].

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